Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS
MATH 1110

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Integration by Parts

Integration by parts is based on the product formula for derivatives:

$$\frac{d}{dx}\left[f(x)\, g(x)\right] = f(x)\, g'(x) + g(x)\, f'(x)$$

We rearrange the above equation to get:

$$f(x)\, g'(x) = \frac{d}{dx}\left[f(x)\, g(x)\right] - g(x)\, f'(x)$$

And integrating both sides yields:

$$\int f(x)\, g'(x) = \int \frac{d}{dx}\left[f(x)\, g(x)\right] - \int g(x)\, f'(x)$$

Which reduces to:

$$\int f(x)\, g'(x) = f(x)\, g(x) - \int g(x)\, f'(x)$$

Integration by Parts Formula

If we let u = f (x) and v = g (x) in the preceding formula the equation transforms into a more convenient form:

Integration by Parts Formula

$$\int u\, dv = uv - \int v\, du$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

This method is useful when the integral on the left side is difficult and changing it into the integral on the right side makes it easier.

Remember, we can easily check our results by differentiating our answers to get the original integrals.

Example

$$\int u\,dv = uv - \int v\,du$$

Consider $\int x e^{x}\,dx$

Our previous method of substitution does not work. Examining the left side of the integration by parts formula yields two possibilities.

Option 1

$$\int \overbrace{x}^{u}\ \overbrace{e^{x}\,dx}^{dv}$$

Option 2

$$\int \overbrace{e^{x}}^{u}\ \overbrace{x\,dx}^{dv}$$

Let's try option 1.

Example - continued

$$\int u\,dv = uv - \int v\,du$$

$$\int x e^x\,dx$$

We have decided to let u = x and dv = e^x dx, (Note: du = dx and v = e^x), yielding

$$\int x e^x\,dx = xe^x - \int e^x\,dx$$

Which is easy to integrate

$$= x e^x - e^x + C$$

As mentioned this is easy to check by differentiating.

$$\frac{d}{dx}\left(x e^x - e^x + C\right) = x e^x + e^x - e^x = x e^x$$

Selecting u and dv

$$\int u\,dv = uv - \int v\,du$$

1. **The product u dv must equal the original integrand.**

2. **It must be possible to integrate dv by one of our known methods.**

3. For integrals involving $x^p e^{ax}$, try

   $u = x^p$ and $dv = e^{ax}\,dx$

4. For integrals involving $x^p \ln x^q$, try

   $u = (\ln x)^q$ and $dv = x^p\,dx$

Example 2

$\int x^3 \ln x \, dx$	$\int u \, dv = uv - \int v \, du$

Let $u = \ln x$ and $dv = x^3 \, dx$, yielding

$du = 1/x \, dx$ and $v = x^4/4$, and the integral is

$$\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \frac{1}{x} \, dx$$

Which when integrated is:

$$= \frac{x^4}{4} \ln x - \frac{x^4}{16} + C$$

Check by differentiating.

$$\frac{d}{dx}\left(\frac{x^4}{4} \ln x - \frac{x^4}{16} + C \right) = \frac{x^4}{4} \frac{1}{x} + x^3 \ln x - \frac{4x^3}{16} = x^3 \ln x$$

Observe:

$$\int (\ln x)^3\,dx \xrightarrow{by\ parts} \int (\ln x)^2\,dx \xrightarrow{by\ parts} \int (\ln x)\,dx$$

Reduction Formula

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$

REMARK3: sometimes The reduction formula is useful because by using it repeatedly we could eventually express our integral.

Reduction Formula

$$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx \quad (n \neq 1)$$

Example

$$\int \tan^5 x \; dx$$

$$5 \to 3 \to 1$$

Example

$$\int \tan^6 x \; dx$$

$$6 \to 4 \to 2 \to 0$$

Reduction Formula

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

Example $\int \cos^5 x \; dx$

$$5 \to 3 \to 1$$

Example $\int \cos^6 x \; dx$

$$6 \to 4 \to 2 \to 0$$

Reduction Formula

$$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx$$

Reduction Formula

$$\int \cos^n x dx \qquad \int \tan^n x dx \qquad \int x^n e^x dx$$

$$\int \sin^n x dx \qquad \int \sec^n x dx \qquad \int (\ln x)^n dx$$