Sectional Curvature
Math 473
Introduction to Differential Geometry
Lecture 27

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Corollary (1):

The normal curvature of a curve on a surface only depends on the velocity of the curve, i.e. if γ_1, γ_2 are two curves through a point on the surface with the same velocity at this point then their normal curvatures at this point coincide(equal).

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Remark:
The sign of the normal curvature depends on the choice of the unit normal. If N is a unit normal on the surface then $-N$ is also a unit normal. The change of sign $N \mapsto -N$ for the unit normal causes further sign changes: $(T, U, N) \mapsto (T, -U, -N)$, $\kappa_g \mapsto -\kappa_g$, $\kappa_n \mapsto -\kappa_n$, $\kappa_t \mapsto \kappa_t$, $\mathrm{I} \mapsto \mathrm{I}$, $\mathrm{II} \mapsto -\mathrm{II}$.

Definition (1):
The **sectional curvature** of the surface X at the point $X(u_0, v_0)$ in the direction of the tangent vector $x \cdot X_u(u_0, v_0) + y \cdot X_v(u_0, v_0)$, $x, y \in \mathbb{R}$, is given by

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$$\kappa(x \cdot X_u + y \cdot X_v) = \frac{x^2 \cdot e + 2xy \cdot f + y^2 \cdot g}{x^2 \cdot E + 2xy \cdot F + y^2 \cdot G} = \frac{\begin{pmatrix} x & y \end{pmatrix} \cdot \begin{pmatrix} e & f \\ f & g \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}}{\begin{pmatrix} x & y \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}}.$$

Remark:

For a curve $\gamma(t) = X(u(t), v(t))$ on the surface X the normal curvature κ_n of γ is equal to the sectional curvature of X in the direction of the tangent vector $\gamma' = u' \cdot X_u + v' \cdot X_v$.

Example (1):
We consider the cylinder $X(u,v) = (\cos u, \sin u, v)$.
The first and the second fundamental forms are

$$\mathrm{I}(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{II}(u,v) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

What is the range of possible values of the sectional curvature

$$\kappa(x, y) := \kappa(x \cdot X_u + y \cdot X_v) = \frac{x^2 \cdot e + 2xy \cdot f + y^2 \cdot g}{x^2 \cdot E + 2xy \cdot F + y^2 \cdot G},$$

what are the maxima and the minima?

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Proof

Remark:

The matrix $\mathrm{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is invertible since

$\det \mathrm{I} = EG - F^2 = |X_u \times X_v|^2 \neq 0$ for a regular surface patch.

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Note:

The sectional curvature is the same along any line $\{r \cdot (x, y),\ \ r \neq 0\}$ since

$$\begin{aligned} p(rx, ry) =& (rx)^2 \cdot e + 2(rx)(ry) \cdot f + (ry)^2 \cdot g \\ =& r^2 \cdot (x^2 \cdot e + 2xy \cdot f + y^2 \cdot g) = r^2 \cdot p(x, y) \end{aligned}$$

and similarly $q(rx, ry) = r^2 \cdot q(x, y)$, therefore

$$\begin{aligned} \kappa(rx, ry) =& \frac{p(rx, ry)}{q(rx, ry)} \\ =& \frac{r^2 \cdot p(x, y)}{r^2 \cdot q(x, y)} = \frac{p(x, y)}{q(x, y)} = \kappa(x, y). \end{aligned}$$

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Theorem (1):

Let I resp. II be the matrices of the first resp. second fundamental form at a point p on the surface X. Then the matrix $\mathrm{I}^{-1} \cdot \mathrm{II}$ has real eigenvalues κ_1 and κ_2 which coincide with the global maximum and the global minimum of the sectional curvature over all non-zero tangent vectors. The following two cases are possible:

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1. If $\kappa_1 \neq \kappa_2$ then if (a, b) is a non-zero eigenvector of the matrix $\mathrm{I}^{-1} \cdot \mathrm{II}$ to the eigenvalue κ_i then $\kappa(a \cdot X_u + b \cdot X_v) = \kappa_i$.
2. If $\kappa_1 = \kappa_2$ then the sectional curvature is constant and equal to $\kappa_1 = \kappa_2$ and is achieved in any direction. In this case the point is called **umbilic**.

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Example (2): All points on a sphere are umbilic. On the paraboloid $X(u, v) = (u, v, u^2 + v^2)$ the point $X(0, 0) = (0, 0, 0)$ is umbilic.

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Thanks for listening.