Second Fundamental Form
Math 473
Introduction to Differential Geometry
Lecture 26

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Example (Orthogonal Sections):

Let Π be a plane containing the surface normal N. Let γ be the curve that we obtain in the intersection of the surface with the plane Π.

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Let $X : U \to \mathbb{R}^3$ be a regular injective surface patch. Let $N : U \to \mathbb{R}^3$ be a unit normal on the surface X. Let $\gamma(t) = X(u(t), v(t))$ be a curve on the surface X.

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Recall:

Remember how we computed $\gamma' = u' \cdot X_u + v' \cdot X_v$ and

$$\begin{aligned} |\gamma'|^2 = \gamma' \bullet \gamma' &= (u' \cdot X_u + v' \cdot X_v) \bullet (u' \cdot X_u + v' \cdot X_v) \\ &= (u')^2 \cdot (X_u \bullet X_u) + 2u'v' \cdot (X_u \bullet X_v) + (v')^2 \cdot (X_v \bullet X_v) \end{aligned}$$

and therefore defined the coefficients of the first fundamental form as

$$E = X_u \bullet X_u, \quad F = X_u \bullet X_v, \quad G = X_v \bullet X_v.$$

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Notation: $X_u = \frac{\partial X}{\partial u}$, $X_v = \frac{\partial X}{\partial v}$, $X_{uu} = \frac{\partial}{\partial u}\frac{\partial}{\partial u}X$, $X_{uv} = \frac{\partial}{\partial u}\frac{\partial}{\partial v}X$, $X_{vu} = \frac{\partial}{\partial v}\frac{\partial}{\partial u}X$, $X_{vv} = \frac{\partial}{\partial v}\frac{\partial}{\partial v}X$.
Note that $X_{uv} = \frac{\partial}{\partial u}\frac{\partial}{\partial v}X = \frac{\partial}{\partial v}\frac{\partial}{\partial u}X = X_{vu}$.

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$$\begin{aligned}
\tilde{T}' = \gamma'' =& (u'X_u + v'X_v)' \\
=& u''X_u + u'(u'(X_u)_u + v'(X_u)_v) + v''X_v + v'(u'(X_v)_u + v'(X_v)_v) \\
=& u''X_u + v''X_v + u'u'X_{uu} + u'v'X_{uv} + v'u'X_{vu} + v'v'X_{vv} \\
=& u''X_u + v''X_v + (u')^2X_{uu} + 2u'v'X_{uv} + (v')^2X_{vv}
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and

$$\begin{aligned}
\tilde{T}' \bullet \tilde{N} =& (u''X_u + v''X_v + (u')^2X_{uu} + 2u'v'X_{uv} + (v')^2X_{vv}) \bullet N \\
=& u'' \cdot (X_u \bullet N) + v'' \cdot (X_v \bullet N) \\
& + (u')^2 \cdot (X_{uu} \bullet N) + 2u'v' \cdot (X_{uv} \bullet N) + (v')^2 \cdot (X_{vv} \bullet N).
\end{aligned}$$

Using the fact that the normal N is perpendicular to the tangent plane (or using the fact that N is a multiple of $X_u \times X_v$), we see that $X_u \bullet N = X_v \bullet N = 0$, hence

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$$\tilde{T}' \bullet \tilde{N} = (u')^2 \cdot (X_{uu} \bullet N) + 2u'v' \cdot (X_{uv} \bullet N) + (v')^2 \cdot (X_{vv} \bullet N).$$

Definition (1):
The **coefficients of the second fundamental form** of the surface patch $X : U \to \mathbb{R}^3$ are

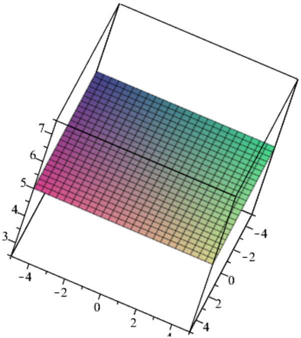
$$e = X_{uu} \bullet N, \quad f = X_{uv} \bullet N = X_{vu} \bullet N, \quad g = X_{vv} \bullet N.$$

The **second fundamental form** of X is

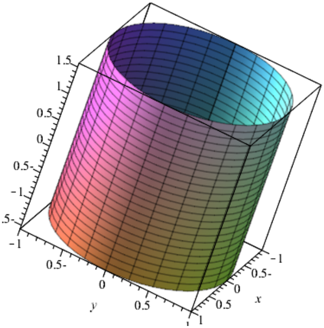
$$\mathrm{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

Examples

Example (1):
Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $X(u, v) = (u, v, 5)$. Compute the second fundamental form of the surface X.



Example (2):
Let $X:\mathbb{R}^2\to\mathbb{R}^3$ be given by $X(u,v)=(a\cos u, a\sin u, bv)$, where a, b are constants. Compute the coefficients of the second fundamental form of the surface X.



Proposition (1):
Let $\gamma(t) = X(u(t), v(t))$ be a curve on the surface X. Then the normal curvature of γ is

$$\kappa_n = \frac{(u')^2 \cdot e + 2u'v' \cdot f + (v')^2 \cdot g}{(u')^2 \cdot E + 2u'v' \cdot F + (v')^2 \cdot G} = \frac{\begin{pmatrix} u' & v' \end{pmatrix} \cdot \begin{pmatrix} e & f \\ f & g \end{pmatrix} \cdot \begin{pmatrix} u' \\ v' \end{pmatrix}}{\begin{pmatrix} u' & v' \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix} \cdot \begin{pmatrix} u' \\ v' \end{pmatrix}}.$$

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Proof:

Proposition (2):

Let $X : U \to \mathbb{R}^3$ be a regular surface. If the second fundamental form of the surface X vanishes i.e. $\mathrm{II} = 0$, then the surface X is part of a plane.

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Proof:

Thanks for listening.