Applications of Differential Geometry: Cartography (Map-making)
Math 473
Introduction to Differential Geometry
Lecture 23

Dr. Nasser Bin Turki

King Saud University
Department of Mathematics

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For the geographic map to be useful, we would like this mapping to be bijective, continuous, smooth. We would also like the mapping to preserve some important metric information, for example to preserve distances, lengths, angles, areas.

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Proposition (1):
A surface patch $X : U \to \mathbb{R}^3$ is a true map, i.e. it preserves the lengths of all curves if and only if it has the first fundamental form $I(u, v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for all $(u, v) \in U$.

Remark: Examples of surface patches with the first fundamental form $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are the cylinder and the cone.

Remark: Examples of surface patches with the first fundamental form $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are the cylinder and the cone.We know that we can make (a part of) the cylinder and the cone from a flat piece of paper without stretching or contracting, hence we have true maps of the surface of the cylinder and the cone respectively.

Proposition (2):
There is no true map of the Earth, i.e. there is no mapping from (a part of) the plane to (an open part of) a sphere which preserves the lengths of all curves.

Remark: A surface patch $X : U \to \mathbb{R}^3$ changes all lengths of curves by the same constant factor $\lambda > 0$ if and only if it has the first fundamental form $I(u,v) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for all $(u,v) \in U$.

Remark: Even though there is no true map of the Earth, there are many maps with various useful properties, for example preserving angles or areas.

Definition (2): We say that a surface patch $X : U \to \mathbb{R}^3$ is **conformal** if it preserves the angles between all curves,

Definition (2): We say that a surface patch $X : U \to \mathbb{R}^3$ is **conformal** if it preserves the angles between all curves,i.e. if the angle between two curves in U is the same as the angle between the corresponding curves on the surface.

Remark: A surface patch $X : U \to \mathbb{R}^3$ is conformal if and only if the coefficients of the first fundamental form satisfy the equations $F(u,v) = 0$ and $E(u,v) = G(u,v)$ for all $(u,v) \in U$. For example the stereographic projection is a conformal surface patch.

Thanks for listening.