Surfaces
Math 473
Introduction to Differential Geometry
Lecture 20

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Recall

Let $\alpha : I \to \mathbb{R}^3$ be a regular space curve. The **tangent surface** of the curve α is the union of all tangent lines. A parametrisation of the tangent surface is given by

$$X : I \times \mathbb{R} \to \mathbb{R}^3, \quad X(u,v) = \alpha(u) + v \cdot \alpha'(t).$$

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Remark(1)

The tangent surface $X(u,v)$ is a regular surface. **Why?**

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In order to obtain a better understanding of this behaviour, we are going to study curves on surfaces in more detail.

Definition (1):
Let $X : U \to \mathbb{R}^3$ be a surface patch. We say that a parametrised space curve $\alpha : I \to \mathbb{R}^3$ is a **curve on the surface** X if for any $t \in I$ there exists $(u(t), v(t)) \in U$ such that $\alpha(t) = X(u(t), v(t))$.

Proposition (1):

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If the surface patch X is injective,then for any curve $\alpha : I \to \mathbb{R}^3$ on the surface X there exists a plane curve $(u(t), v(t))$ in U such that $\alpha(t) = X(u(t), v(t))$ for all $t \in I$ and such a plane curve $(u(t), v(t))$ is unique.

Examples:

Consider the cylinder $X : \mathbb{R}^2 \to \mathbb{R}^3$, $X(u, v) = (\cos u, \sin u, v)$.

Examples of curves on the cylinder:

1. $X(t, 0) = (\cos t, \sin t, 0)$ is a horizontal circle.
2. $X(0, t) = (1, 0, t)$ is a vertical line.
3. $X(t, t) = (\cos t, \sin t, t)$ is a helix.
4. $X(t, 2t) = (\cos t, \sin t, 2t)$ is another helix.

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Proposition (2):
Let $X : U \to \mathbb{R}^3$ be an injective surface patch. Let $(u(t), v(t))$ be a plane curve $I \to U$. Let $\alpha : I \to \mathbb{R}^3$ be the corresponding curve $\alpha(t) = X(u(t), v(t))$ on the surface X. Then the velocity of the curve α is

$$\alpha' = u' X_u + v' X_v,$$

thus the velocity α' is a linear combination of the vectors X_u and X_v.

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Proof:
Hint We use the chain rule to differentiate $\alpha(t) = X(u(t), v(t))$.

Proposition (3):
Let $X : U \to \mathbb{R}^3$ be an injective surface patch. Let $(u, v) \in U$. Then any linear combination of the vectors X_u and X_v is the velocity α' of some curve $\alpha : I \to \mathbb{R}^3$ on the surface X.

Tangent Vectors

Let $X : U \to \mathbb{R}^3$ be an injective regular surface patch. Let (u, v) be a point in U and $X(u, v)$ the corresponding point on the surface X.

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Definition (2): Tangent Vector
A **tangent vector** to the surface X at the point $X(u, v)$ is a linear combination of the vectors $X_u(u, v)$ and $X_v(u, v)$, i.e. a velocity of some curve on X through $X(u, v)$.

Tangent Plane

Let $X : U \to \mathbb{R}^3$ be an injective regular surface . Let (u, v) be a point in U and $P = X(u, v)$ the corresponding point on the surface X.

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Definition (3):

The **tangent plane** to the surface X at the point $P = X(u, v)$ is the set of all tangent vectors of X at the point $P = X(u, v)$, i.e. it is the set of all linear combinations of $X_u(u, v)$ and $X_v(u, v)$.

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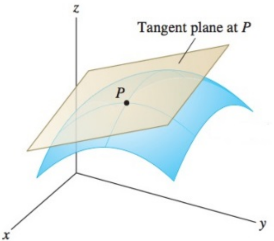

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Remark (2)

The tangent plane $T_P X$ is perpendicular to the vector $X_u \times X_v(u, v)$.

Example (1)
Let $U = \{(u,v) \in \mathbb{R}^2 \;:\; u^2 + v^2 < 1\}$. Consider the surface patch $X : U \to \mathbb{R}^3$ given by

$$X(u,v) = (u, v, \sqrt{1 - u^2 - v^2}).$$

Determine whether X is regular surface patch? Find the equation of the tangent plane $T_P X$, where $P = X(\frac{1}{2}, \frac{1}{2})$ and for $P = X(0,0)$?

Thanks for listening.