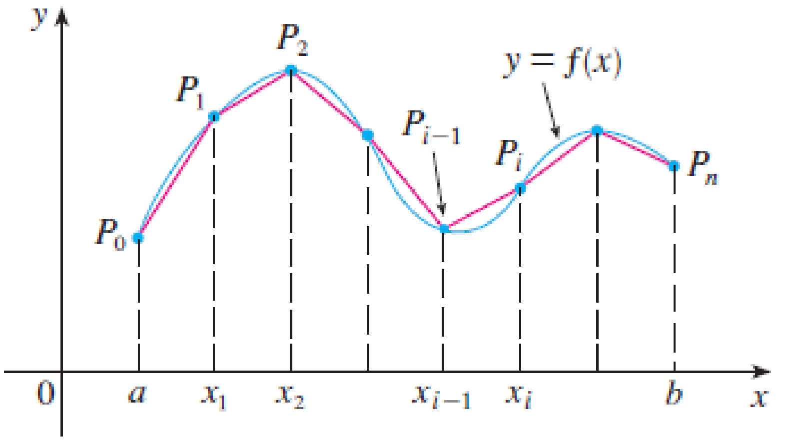
Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS
MATH 1110

Instructor:
Dr. Mohamed El-Shazly
Assistant Prof. of Mechanical Design and Tribology
mohamed.elshazly@ams-sae.com
Office: S053

Line Integral

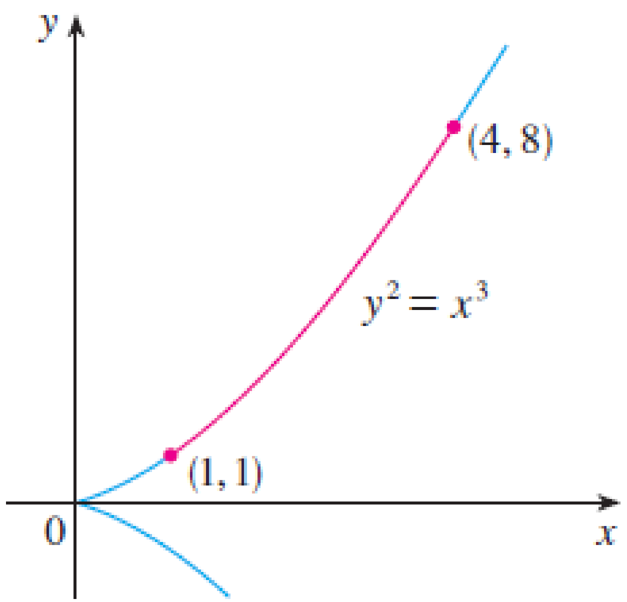
Arc Length



$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

EXAMPLE 1 Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.



SOLUTION For the top half of the curve we have

$$y = x^{3/2} \qquad \frac{dy}{dx} = \tfrac{3}{2}x^{1/2}$$

and so the arc length formula gives

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{1 + \tfrac{9}{4}x}\, dx$$

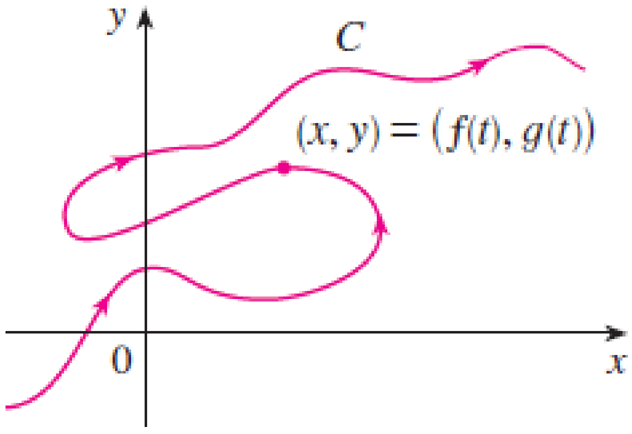
If we substitute $u = 1 + \tfrac{9}{4}x$, then $du = \tfrac{9}{4}\, dx$. When $x = 1$, $u = \tfrac{13}{4}$; when $x = 4$, $u = 10$.

Therefore

$$L = \tfrac{4}{9}\int_{13/4}^{10} \sqrt{u}\, du = \tfrac{4}{9} \cdot \tfrac{2}{3}u^{3/2}\Big]_{13/4}^{10}$$

$$= \tfrac{8}{27}\left[10^{3/2} - \left(\tfrac{13}{4}\right)^{3/2}\right] = \tfrac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$$

Curves Defined by Parametric Equations



- Imagine that a particle moves along the curve *C* shown in Figure 1. It is impossible to describe *C* by an equation of the form y = f(x) because *C* fails the Vertical Line Test.
- But the *x*- and *y*-coordinates of the particle are functions of time and so we can write x = f(t) and y = g(t) .
- Such a pair of equations is often a convenient way of describing a curve and gives rise to the following definition.
- Suppose that x and y are both given as functions of a third variable (called a **parameter**) by the equation

$$x = f(t) \qquad y = g(t)$$

Calculus with Parametric Curve

Arc Length

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Using Parametric equation x = f(t) and y = g(t), $\alpha \le t \le \beta$, where $dx/dt = f'(t) > 0$.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 2

Find the Length of one arch of the cycloid x = r (θ- sinθ) , y = r (1-cosθ)

Soultion

$$\frac{dx}{d\theta} = r(1 - \cos\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = r\sin\theta$$

we have

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - \cos\theta)^2 + r^2\sin^2\theta}\; d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\; d\theta$$

$$= r\int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\; d\theta$$

To evaluate this integral we use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ with $\theta = 2x$, which gives $1 - \cos\theta = 2\sin^2(\theta/2)$. Since $0 \leqslant \theta \leqslant 2\pi$, we have $0 \leqslant \theta/2 \leqslant \pi$ and so $\sin(\theta/2) \geqslant 0$. Therefore

$$\sqrt{2(1 - \cos\theta)} = \sqrt{4\sin^2(\theta/2)} = 2\left|\sin(\theta/2)\right| = 2\sin(\theta/2)$$

and so

$$L = 2r\int_0^{2\pi} \sin(\theta/2)\, d\theta = 2r\Big[-2\cos(\theta/2)\Big]_0^{2\pi}$$

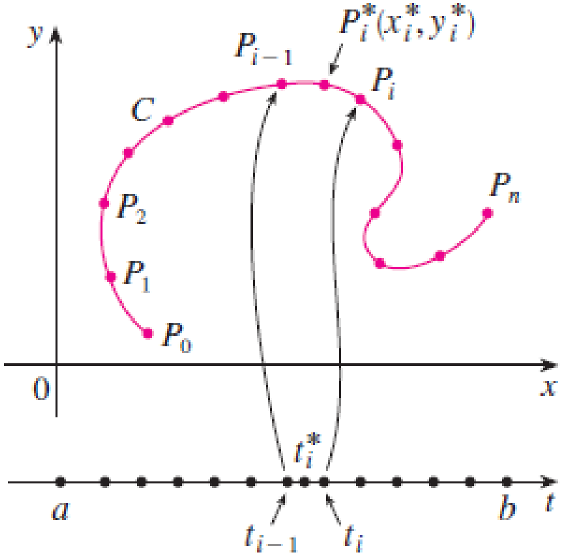
$$= 2r[2 + 2] = 8r$$

Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval [a,b], we integrate over a curve C .

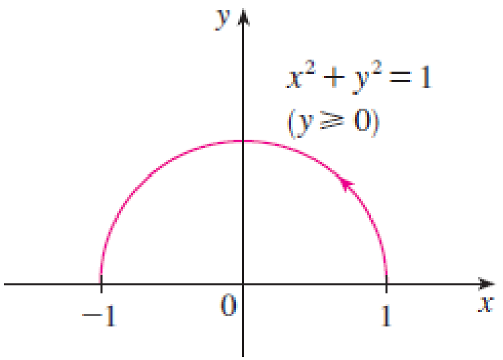
Such integrals are called *line integrals*, although “curve integrals” would be better terminology.

They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.



$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

EXAMPLE 1 Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.



Recall that the unit circle can be parametrized by means of the equations

$$x = \cos t \qquad y = \sin t$$

and the upper half of the circle is described by the parameter interval $0 \leqslant t \leqslant \pi$. (See Figure 3.) Therefore Formula 3 gives

$$\begin{aligned}
\int_C (2 + x^2 y)\, ds &= \int_0^{\pi} (2 + \cos^2 t \sin t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^{\pi} (2 + \cos^2 t \sin t) \sqrt{\sin^2 t + \cos^2 t}\, dt \\
&= \int_0^{\pi} (2 + \cos^2 t \sin t)\, dt = \left[2t - \frac{\cos^3 t}{3} \right]_0^{\pi} \\
&= 2\pi + \tfrac{2}{3}
\end{aligned}$$

EXAMPLE 2 Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from (0, 0) to (1, 1) followed by the vertical line segment C_2 from (1, 1) to (1, 2).

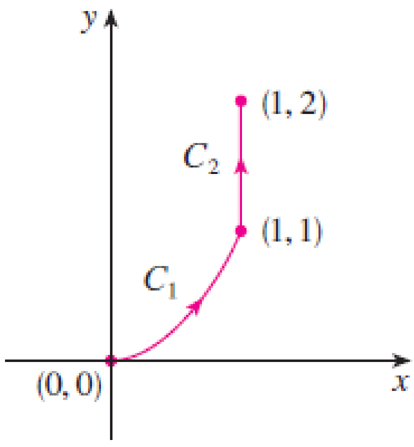


FIGURE 5

SOLUTION The curve C is shown in Figure 5. C_1 is the graph of a function of x, so we can choose x as the parameter and the equations for C_1 become

$$x = x \qquad y = x^2 \qquad 0 \leqslant x \leqslant 1$$

Therefore

$$\int_{C_1} 2x\, ds = \int_0^1 2x \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 2x\sqrt{1 + 4x^2}\, dx$$

$$= \tfrac{1}{4} \cdot \tfrac{2}{3}(1 + 4x^2)^{3/2}\Big]_0^1 = \frac{5\sqrt{5} - 1}{6}$$

On C_2 we choose y as the parameter, so the equations of C_2 are

$$x = 1 \qquad y = y \qquad 1 \leqslant y \leqslant 2$$

and

$$\int_{C_2} 2x\, ds = \int_1^2 2(1) \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\, dy = \int_1^2 2\, dy = 2$$

Thus

$$\int_C 2x\, ds = \int_{C_1} 2x\, ds + \int_{C_2} 2x\, ds = \frac{5\sqrt{5} - 1}{6} + 2$$

Example 2

Evaluate $\int_C xy^4\,ds$ where C is the right half of the circle, $x^2 + y^2 = 16$ rotated in the counter clockwise direction.

Solution

We first need a parameterization of the circle. This is given by,

$$x = 4\cos t \qquad y = 4\sin t$$

We now need a range of t's that will give the right half of the circle. The following range of t's will do this.

$$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

Now, we need the derivatives of the parametric equations and let's compute ds.

$$\frac{dx}{dt} = -4\sin t \qquad \frac{dy}{dt} = 4\cos t$$

$$ds = \sqrt{16\sin^2 t + 16\cos^2 t}\,dt = 4\,dt$$

The line integral is then,

$$\int_C xy^4 \, ds = \int_{-\pi/2}^{\pi/2} 4\cos t \left(4\sin t\right)^4 (4)\, dt$$

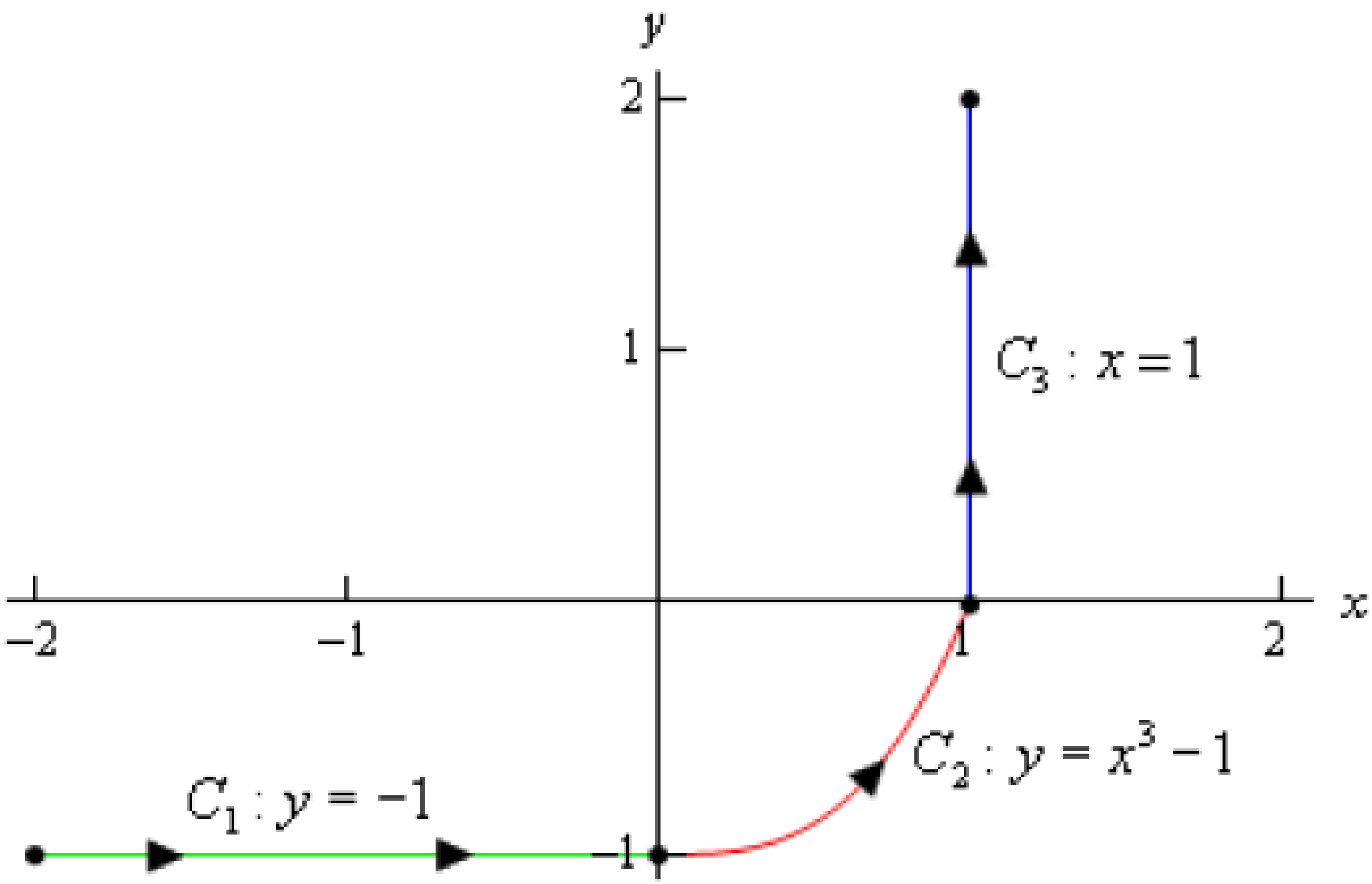
$$= 4096 \int_{-\pi/2}^{\pi/2} \cos t \, \sin^4 t \, dt$$

$$= \frac{4096}{5} \sin^5 t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{8192}{5}$$

Example 3

Evaluate $\int_C 4x^3\, ds$ where C is the curve shown below.



$$C_1 : \; x = t, \; y = -1, \qquad -2 \le t \le 0$$

$$C_2 : \; x = t, \; y = t^3 - 1, \qquad 0 \le t \le 1$$

$$C_3 : \; x = 1, \; y = t, \qquad 0 \le t \le 2$$

Now let's do the line integral over each of these curves.

$$\int_{C_1} 4x^3 \, ds = \int_{-2}^{0} 4t^3 \sqrt{(1)^2 + (0)^2} \, dt = \int_{-2}^{0} 4t^3 \, dt = t^4 \Big|_{-2}^{0} = -16$$

$$\int_{C_2} 4x^3 \, ds = \int_0^1 4t^3 \sqrt{(1)^2 + \left(3t^2\right)^2} \, dt$$

$$= \int_0^1 4t^3 \sqrt{1 + 9t^4} \, dt$$

$$= \frac{1}{9}\left(\frac{2}{3}\right)\left(1 + 9t^4\right)^{\frac{3}{2}} \Big|_0^1 = \frac{2}{27}\left(10^{\frac{3}{2}} - 1\right) = 2.268$$

$$\int_{C_3} 4x^3\, ds = \int_0^2 4(1)^3 \sqrt{(0)^2 + (1)^2}\, dt = \int_0^2 4\, dt = 8$$

Finally, the line integral that we were asked to compute is,

$$\begin{aligned}\int_C 4x^3\, ds &= \int_{C_1} 4x^3\, ds + \int_{C_2} 4x^3\, ds + \int_{C_3} 4x^3\, ds \\ &= -16 + 2.268 + 8 \\ &= -5.732\end{aligned}$$