Bertrand Curves
Math 473
Introduction to Differential Geometry
Lecture 14

Dr. Nasser Bin Turki

King Saud University
Department of Mathematics

October 14, 2018

Bertrand Curves

Definition (1): Let $\alpha : I \mapsto \mathbb{R}^3$ be unit speed curve. Let the curvature $\kappa(t) > 0$ and the torsion $\tau(t) \neq 0$ for all $t \in I$.

Definition (1): Let $\alpha : I \mapsto \mathbb{R}^3$ be unit speed curve. Let the curvature $\kappa(t) > 0$ and the torsion $\tau(t) \neq 0$ for all $t \in I$.The Curve α is called a **Bertrand Curve** if there exists a curve $\beta : I \mapsto \mathbb{R}^3$ such that the principle normal lines of α and β at $t \in I$ are equal

$$N_\alpha(t) = \pm N_\beta(t).$$

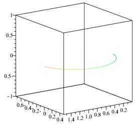
Definition (1): Let $\alpha : I \mapsto \mathbb{R}^3$ be unit speed curve. Let the curvature $\kappa(t) > 0$ and the torsion $\tau(t) \neq 0$ for all $t \in I$.The Curve α is called a **Bertrand Curve** if there exists a curve $\beta : I \mapsto \mathbb{R}^3$ such that the principle normal lines of α and β at $t \in I$ are equal

$$N_\alpha(t) = \pm N_\beta(t).$$

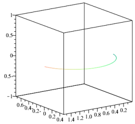
In this case β is called a Bertrand mate of α.

Example

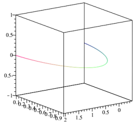
Let $\alpha(t) = \frac{1}{2}(\cos^{-1} t - t\sqrt{1 - t^2}, 1 - t^2, 0)$ be unit speed curve



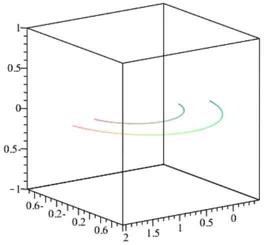
Let $\alpha(t) = \frac{1}{2}(\cos^{-1} t - t\sqrt{1-t^2}, 1 - t^2, 0)$ be unit speed curve



and $\beta(t) = \frac{1}{2}(\cos^{-1} t - t\sqrt{1-t^2}, 1 - t^2 + \sqrt{1-t^2}, 0)$.



The curve α and β are Bertrand curves and we say that β is a Bertrand mate of α.



Theorem(1):

Let $\alpha : I \mapsto \mathbb{R}^3$ be a Bertrand curve and $\beta : I \mapsto \mathbb{R}^3$ be the Bertrand mate of α. Then, the distance between corresponding points of $\alpha(t)$ and $\beta(t)$ is constant.

Theorem(1):
Let $\alpha : I \mapsto \mathbb{R}^3$ be a Bertrand curve and $\beta : I \mapsto \mathbb{R}^3$ be the Bertrand mate of α. Then, the distance between corresponding points of $\alpha(t)$ and $\beta(t)$ is constant.
(Hint: if $\alpha(t)$ is unit speed parametrisation curve, then there is a function $\lambda(t)$ such that $\alpha(t) + \lambda(t)N_\alpha(t) = \beta(t)$ gives the other curve.)

Proof:

Theorem(2):

Let $\alpha : I \mapsto \mathbb{R}^3$ be a Bertrand curve and $\beta : I \mapsto \mathbb{R}^3$ be the Bertrand mate of α. Then

Theorem(2):
Let $\alpha : I \mapsto \mathbb{R}^3$ be a Bertrand curve and $\beta : I \mapsto \mathbb{R}^3$ be the Bertrand mate of α. Then

1. $T_\alpha(t) \bullet T_\beta(t) = c_1$
2. $B_\alpha(t) \bullet T_\beta(t) = c_2$,

where c_1, c_2 are constant.

Proof:

Thanks for listening.