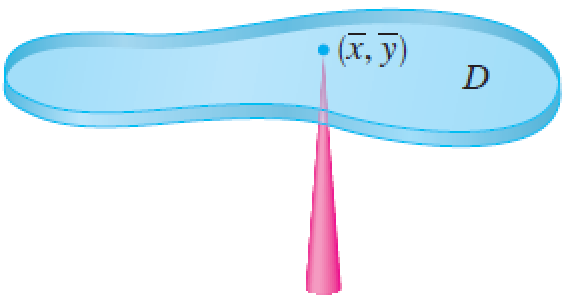
Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS
MATH 1110

Instructor:
Dr. Mohamed El-Shazly
Associate Prof. of Mechanical Design and Tribology
melshazly@ksu.edu.sa
Office: F056

Moments and Centers of Mass

$$M_x = \iint_D y\rho(x, y)\, dA$$



$$M_y = \iint_D x\rho(x, y)\, dA$$

The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

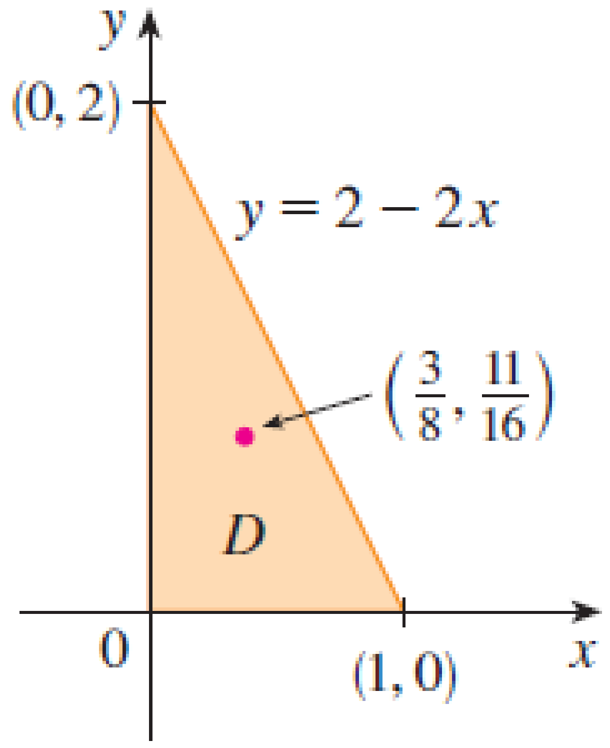
$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)\, dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)\, dA$$

where the mass m is given by

$$m = \iint_D \rho(x, y)\, dA$$

Example 1

Find the mass and center of mass of a triangular lamina with vertices (0, 0), (1, 0), and (0, 2) if the density function is $\rho(x, y) = 1 + 3x + y$.



SOLUTION

$$m = \iint_D \rho(x, y)\, dA = \int_0^1 \int_0^{2-2x} (1 + 3x + y)\, dy\, dx$$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= 4 \int_0^1 (1 - x^2)\, dx = 4 \left[x - \frac{x^3}{3} \right]_0^1 = \frac{8}{3}$$

$$\bar{x} = \frac{1}{m} \iint_D x\rho(x, y)\, dA = \tfrac{3}{8} \int_0^1 \int_0^{2-2x} (x + 3x^2 + xy)\, dy\, dx$$

$$= \frac{3}{8} \int_0^1 \left[xy + 3x^2 y + x\frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= \frac{3}{2} \int_0^1 (x - x^3)\, dx = \frac{3}{2} \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{3}{8}$$

$$\bar{y} = \frac{1}{m}\iint\limits_D y\rho(x, y)\, dA = \tfrac{3}{8}\int_0^1\int_0^{2-2x} (y + 3xy + y^2)\, dy\, dx$$

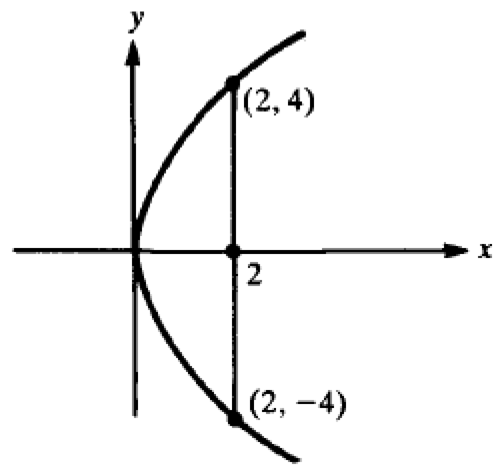
$$= \frac{3}{8}\int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3}\right]_{y=0}^{y=2-2x} dx = \tfrac{1}{4}\int_0^1 (7 - 9x - 3x^2 + 5x^3)\, dx$$

$$= \frac{1}{4}\left[7x - 9\frac{x^2}{2} - x^3 + 5\frac{x^4}{4}\right]_0^1 = \frac{11}{16}$$

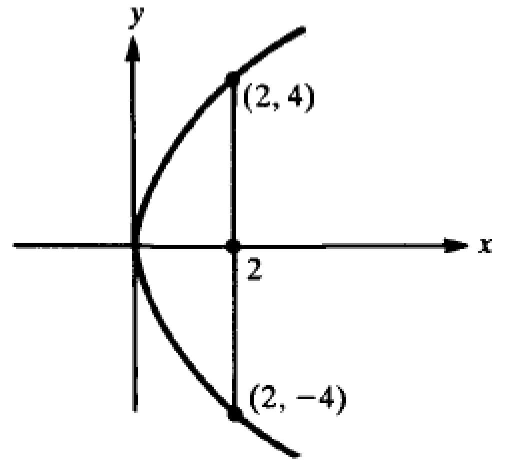
The center of mass is at the point $\left(\frac{3}{8}, \frac{11}{16}\right)$.

Example 2

Find the center of mass $(\bar{x}, \bar{y})$ of the plate cut from the parabola $y^2 = 8x$ by its latus rectum $x = 2$ if the density is numerically equal to the distance from the latus rectum.



SOLUTION



By symmetry, $\bar{y} = 0$.

The mass $M = \iint (2 - x)\, dA = \int_{-4}^{4} \int_{y^2/8}^{2} (2 - x)\, dx\, dy =$

$$\int_{-4}^{4} \left(2x - \frac{1}{2}x^2\right)\Big]_{y^2/8}^{2} dy = \int_{-4}^{4} \left[2 - \left(\frac{y^2}{4} - \frac{y^4}{128}\right)\right] dy =$$

$$\left(2y - \frac{1}{12}y^3 + \frac{1}{128}\frac{y^5}{5}\right)\Big]_{-4}^{4} = 8\left(2 - \frac{16}{12} + \frac{1}{128}\cdot\frac{256}{5}\right) =$$

$$\frac{128}{15}$$

The moment about the y-axis is given by

$$M_y = \int_{-4}^{4} \int_{y^2/8}^{2} x(2 - x)\, dx\, dy$$

$$= \int_{-4}^{4} \left(x^2 - \frac{1}{3}x^3\right)\Big]_{y^2/8}^{2} dy =$$

$$\int_{-4}^{4} \left[\frac{4}{3} - \left(\frac{y^4}{64} - \frac{y^6}{3\cdot 512}\right)\right] dy = \left(\frac{4y}{3} - \frac{1}{64}\frac{y^5}{5} + \frac{1}{3\cdot 512}\frac{y^7}{7}\right)\Big]_{-4}^{4} =$$

$$8\left(\frac{4}{3} - \frac{4}{5} + \frac{8}{21}\right) = \frac{256}{35}.$$ Hence, $\bar{x} = \dfrac{M_y}{M} =$

$$\frac{\frac{256}{35}}{\frac{128}{15}} = \frac{6}{7}.$$ Thus, the center of mass is $\left(\frac{6}{7}, 0\right)$.

Example 3

The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

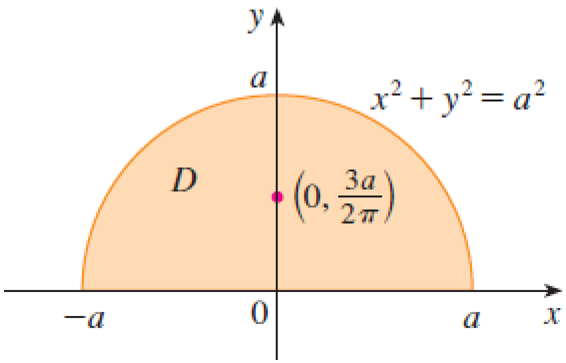


FIGURE 6

SOLUTION Let's place the lamina as the upper half of the circle $x^2 + y^2 = a^2$. (See Figure 6.) Then the distance from a point (x, y) to the center of the circle (the origin) is $\sqrt{x^2 + y^2}$. Therefore the density function is

$$\rho(x, y) = K\sqrt{x^2 + y^2}$$

where K is some constant. Both the density function and the shape of the lamina suggest that we convert to polar coordinates. Then $\sqrt{x^2 + y^2} = r$ and the region D is given by $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant \pi$. Thus the mass of the lamina is

$$\begin{aligned} m &= \iint_D \rho(x, y)\, dA = \iint_D K\sqrt{x^2 + y^2}\, dA \\ &= \int_0^{\pi} \int_0^{a} (Kr)\, r\, dr\, d\theta = K \int_0^{\pi} d\theta \int_0^{a} r^2\, dr \\ &= K\pi \frac{r^3}{3}\bigg]_0^a = \frac{K\pi a^3}{3} \end{aligned}$$

Both the lamina and the density function are symmetric with respect to the y-axis, so the center of mass must lie on the y-axis, that is, $\bar{x} = 0$. The y-coordinate is given by

$$\bar{y} = \frac{1}{m} \iint_D y\rho(x, y)\, dA = \frac{3}{K\pi a^3} \int_0^{\pi} \int_0^{a} r \sin\theta\, (Kr)\, r\, dr\, d\theta$$

$$= \frac{3}{\pi a^3} \int_0^{\pi} \sin\theta\, d\theta \int_0^{a} r^3\, dr = \frac{3}{\pi a^3} \left[-\cos\theta\right]_0^{\pi} \left[\frac{r^4}{4}\right]_0^{a}$$

$$= \frac{3}{\pi a^3} \frac{2a^4}{4} = \frac{3a}{2\pi}$$

Therefore the center of mass is located at the point $(0, 3a/(2\pi))$.