Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS
MATH 1110

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Applications of Double Integrals

Density and Mass

Consider a lamina with variable density. Suppose the lamina occupies a region of the - plane and its **density** (in units of mass per unit area) at a point in *D* is given by , where is a continuous function on . This means that

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$$

where Δm and Δ A are the mass and area of a small rectangle that contains *x, y* and the limit is taken as the dimensions of the rectangle approach 0. (See Figure 1.)

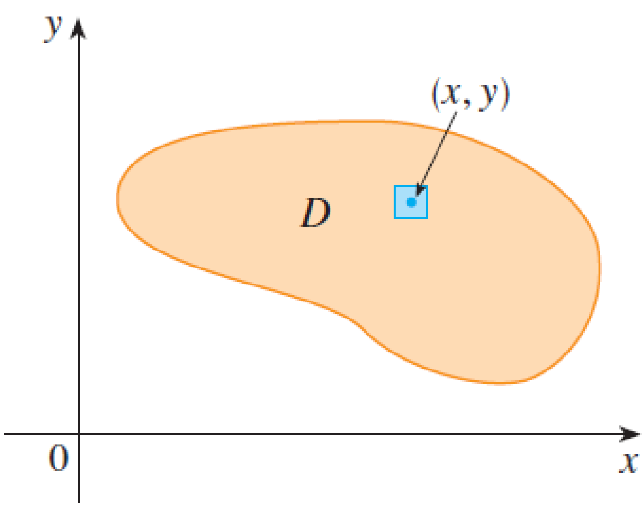
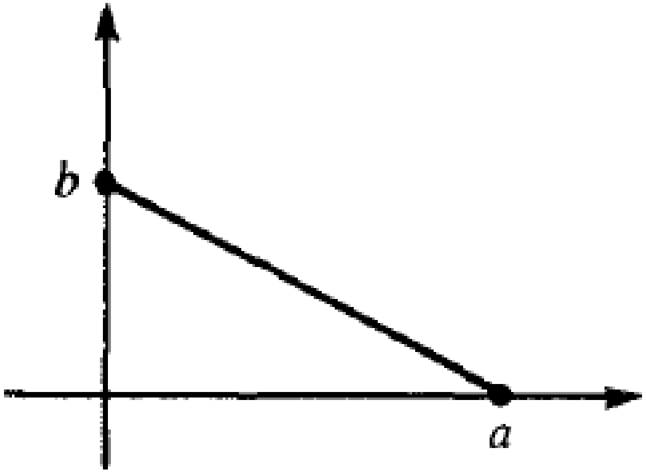


FIGURE 1

$$m = \iint_D \rho(x, y)\, dA$$

EXAMPLE 1:

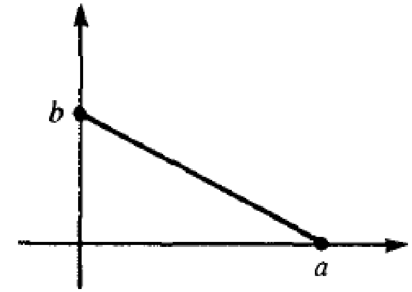
Find the mass of a plate in the form of a right triangle with legs *a* and *b,* if the density (mass per unit area) is numerically equal to the sum of the distances from the legs.



The equation of a straight line is usually written this way:

$y = mx + b$

SOLUTION 1:



The density $\delta(x, y) = x + y$.

$$m = \iint_D \rho(x, y)\, dA$$

$$= \int_0^a \int_0^{b-(b/a)x} (x + y)\, dy\, dx$$

$$\int_0^a (xy + \tfrac{1}{2} y^2) \Big]_0^{b-(b/a)x}\, dx$$

$$= \frac{b}{a} \int_0^a (a - x)[x + \tfrac{1}{2}(b/a)(a - x)]\, dx$$

$$= \frac{b}{a} \int_0^a \left[ax - x^2 + \tfrac{1}{2}(b/a)(a-x)^2\right] dx =$$

$$(b/a)\left[\tfrac{1}{2}ax^2 - \tfrac{1}{3}x^3 - \tfrac{1}{2}(b/a)\cdot\tfrac{1}{3}(a-x)^3\right]_0^a$$

$$= (b/a)\{(\tfrac{1}{2}a^3 - \tfrac{1}{3}a^3) - [-\tfrac{1}{2}(b/a)\cdot\tfrac{1}{3}a^3]\} =$$

$$(b/a)(\tfrac{1}{6}a^3 + \tfrac{1}{6}ba^2) = \tfrac{1}{6}ba(a+b).$$

EXAMPLE 2:

Find the mass of a circular plate $\mathcal{R}$ of radius a whose density is numerically equal to the distance from the center.

Let the circle be $\quad r = a.$

Then $\quad M = \iint_{\mathcal{R}} r\,dA = \int_0^{2\pi} \int_0^a r \cdot r\,dr\,d\theta$

$= \int_0^{2\pi} \frac{1}{3} r^3 \Big]_0^a \,d\theta = \int_0^{2\pi} \frac{1}{3} a^3\,d\theta =$

$\frac{1}{3} a^3 \cdot 2\pi = \frac{2}{3}\pi a^3.$

EXAMPLE 3:

Find the mass of a solid right circular cylinder $\mathscr{R}$ of height h and radius of base b, if the density (mass per unit volume) is numerically equal to the square of the distance from the axis of the cylinder.

$$M = \iiint r^2 \, dV =$$

$$\int_0^{2\pi} \int_0^b \int_0^h r^2 \cdot r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^b r^3 h \, dr \, d\theta = \int_0^{2\pi} \tfrac{1}{4} h r^4 \Big]_0^b \, d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{4} h b^4 \, d\theta$$

$$= \tfrac{1}{4} h b^4 \cdot 2\pi = \tfrac{1}{2} \pi h b^4.$$