مثال اوجد التكاملات التالية :

① $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \frac{2}{1}\int \sin\sqrt{x}\cdot\frac{1}{2\sqrt{x}}\,dx$

u

نفرض أن : $u=\sqrt{x}$

$\therefore \int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx \qquad \Rightarrow du=\frac{1}{2\sqrt{x}}\,dx$

$=2\int \sin\sqrt{x}\cdot\frac{1}{2\sqrt{x}}\,dx$

$=2\int \sin u\,du$

$=2(-\cos u)+C$

$=-2\cos\sqrt{x}+C.$

② $I = \int_0^4 \frac{x}{\sqrt{9+x^2}}\,dx = \int_0^4 (x^2+9)^{-\frac{1}{2}} x\,dx$

u

نفرض أن :

$u = x^2+9 \Rightarrow du = 2x\,dx$,

وكذلك عندما :

$x=0 \rightarrow u = 0+9 = 9.$

$x=4 \rightarrow u = 16+9 = 25$

$\therefore I = \frac{1}{2}\int_9^{25} u^{-\frac{1}{2}}\,du = \frac{1}{2}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_9^{25}$

$= (25)^{\frac{1}{2}} - (9)^{\frac{1}{2}}$

$= \sqrt{25} - \sqrt{9} = 5-3 = 2.$

⑤ $I=\int \frac{\sqrt{3+\tan x}}{\cos^2 x}\,dx=\int (3+\tan x)^{\frac{1}{2}}\sec^2 x\,dx$

نفرض أن :

$u=3+\tan x \Rightarrow du=\sec^2 x\,dx$

$\therefore I=\int u^{\frac{1}{2}}\,du=\frac{2}{3}u^{\frac{3}{2}}+C$

$=\frac{2}{3}(3+\tan x)^{\frac{3}{2}}+C$

ملاحظة: اذا كان:

$$\int f(x)\,dx = F(x)+C$$

فإن:

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+C$$

حيث a لا تساوي صفر

باستعمال هذه الملاحظة يمكن تعميم التكاملات في القائمة الاساسية مع الحذف التالي:

① $\int (ax+b)^r\,dx = \frac{1}{a}\frac{(ax+b)^{r+1}}{r+1}+C,\ r\neq -1.$

* $\int (5x-1)^{10}\,dx = \frac{1}{5}\frac{(5x-3)^{11}}{11}+C$

② $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+C$

* $\int \cos(2x-1)\,dx = \frac{1}{2}\sin(2x-1)+C$

③ $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+C$

④ $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)+C$

⑤ $\int \csc^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)+C$

⑥ $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)+C$

⑦ $\int \csc(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\csc(ax+b)+C$

* $\int \sec(3x)\tan(3x)\,dx = \frac{1}{3}\sec(3x)+C$

* $\int \csc^2(4x+3)\,dx = -\frac{1}{4}\cot(4x+3)+C$

⑧ $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$

* $\int \frac{1}{\sqrt{25-x^2}}\,dx = \sin^{-1}\left(\frac{x}{5}\right)+C$

$a^2=25 \therefore a=5$

⑨ $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

* $\int \frac{1}{5+x^2}\,dx = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$

$a^2=5 \therefore a=\sqrt{5}$

⑩ $\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+C$

* $\int \frac{1}{x\sqrt{x^2-16}}\,dx = \frac{1}{4}\sec^{-1}\left(\frac{x}{4}\right)+C$

مثال / اوجد التكاملات التالية:

① $I=\int \frac{dx}{\sqrt{2x+7}}=\int (2x+7)^{-\frac{1}{2}}dx=\frac{1}{2}\frac{(2x+7)^{\frac{1}{2}}}{\frac{1}{2}}+C$

② $I=\int \frac{1}{9+4x^2}dx=\int \frac{1}{4\left(\frac{9}{4}+x^2\right)}dx$

$=\frac{1}{4}\int \frac{1}{\frac{9}{4}+x^2}dx=\frac{1}{4}\cdot\frac{1}{\left(\frac{3}{2}\right)}\tan^{-1}\left(\frac{x}{\frac{3}{2}}\right)+C$

$=\frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+C$

a^2

③ $I=\int 2x^5\sqrt[3]{x^2+3}\,dx=\int (x^2+3)^{\frac{1}{3}}2x^5dx$

نفرض أن:

$u=x^2+3\Rightarrow du=2x\,dx$

$\therefore I=\int (x^2+3)^{\frac{1}{3}}\cdot x^4\cdot 2x\,dx$

u ?? du

$u=x^2+3$ نكتب

$\Rightarrow x^2=u-3$

$\therefore x^4=(u-3)^2$

$=\int u^{\frac{1}{3}}(u-3)^2du$

$=\int u^{\frac{1}{3}}(u^2-6u+9)du$

$=\int (u^{\frac{7}{3}}-6u^{\frac{4}{3}}+9u^{\frac{1}{3}})du$

$=\frac{3}{10}u^{\frac{10}{3}}-6\cdot\frac{3}{7}u^{\frac{7}{3}}+9\cdot\frac{3}{4}u^{\frac{4}{3}}+C$

$=\frac{3}{10}(x^2+3)^{\frac{10}{3}}-\frac{18}{7}(x^2+3)^{\frac{7}{3}}+\frac{27}{4}(x^2+3)^{\frac{4}{3}}+C$

$$④ \int \frac{2+\cos^3\theta}{\sin^2\theta}\,d\theta = \int \left(\frac{2}{\sin^2\theta} + \frac{\cos^3\theta}{\sin^2\theta}\right) d\theta$$

$$= 2\int \csc^2\theta\,d\theta + \int \frac{\cos^2\theta}{\sin^2\theta}\cos\theta\,d\theta$$

$$= -2\cot\theta + \int \frac{1-\sin^2\theta}{\sin^2\theta}\cos\theta\,d\theta$$

نفرض أنه

$$\hookrightarrow u = \sin\theta \Rightarrow du = \cos\theta\,d\theta$$

$$\therefore = -2\cot\theta + \int \frac{1-u^2}{u^2}\,du$$

$$= -2\cot\theta + \int \left(\frac{1}{u^2} - 1\right) du$$

$$= -2\cot\theta + \int \left(u^{-2} - 1\right) du$$

$$= -2\cot\theta + \left[\frac{u^{-1}}{-1} - u\right] + C$$

$$= -2\cot\theta + \left[\frac{-1}{\sin\theta} - \sin\theta\right] + C.$$