The Laplace Transform

- Let f be a function. Its Laplace transform (function) is denoted by the corresponding capitol letter F. Another notation is $\mathcal{L}(f)$.
- Input to the given function f is denoted by t; input to its Laplace transform F is denoted by s.
- By default, the domain of the function $f=f(t)$ is the set of all non-negative real numbers. The domain of its Laplace transform depends on f and can vary from a function to a function.

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Definition of the Laplace Transform

- The Laplace transform ***F=F(s)*** of a function ***f=f(t)*** is defined by

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty e^{-ts} f(t)\, dt.$$

- The integral is evaluated with respect to *t*, hence once the limits are substituted, what is left are in terms of *s*.

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Example: Find the Laplace transform of the constant function

$$f(t) = 1, \ 0 \le t < \infty.$$

Solution:

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-ts} f(t)\,dt = \int_0^\infty e^{-ts}(1)\,dt \\
&= \lim_{b \to +\infty} \int_0^b e^{-ts}\,dt \\
&= \lim_{b \to +\infty} \left[\frac{e^{-ts}}{-s} \right]_0^b \quad \text{provided } s \neq 0. \\
&= \lim_{b \to +\infty} \left[\frac{e^{-bs}}{-s} - \frac{e^0}{-s} \right] \\
&= \lim_{b \to +\infty} \left[\frac{e^{-bs}}{-s} - \frac{1}{-s} \right]
\end{aligned}$$

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At this stage we need to recall a limit from Cal 1:

$$e^{-x} \to \begin{cases} 0 & \text{if } x \to +\infty \\ +\infty & \text{if } x \to -\infty \end{cases}.$$

Hence,

$$\lim_{b \to +\infty} \frac{e^{-bs}}{-s} = \begin{cases} 0 & \text{if } s > 0 \\ +\infty & \text{if } s < 0 \end{cases}.$$

Thus,

$$F(s) = \frac{1}{s}, \; s > 0.$$

In this case the domain of the transform is the set of all positive real numbers.

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Table of Transforms

$f(t) = 1,\ t \geq 0$	$F(s) = \frac{1}{s},\ s \geq 0$
$f(t) = t^n,\ t \geq 0$	$F(s) = \frac{n!}{s^{n+1}},\ s \geq 0$
$f(t) = e^{at},\ t \geq 0$	$F(s) = \frac{1}{s-a},\ s > a$
$f(t) = \sin(kt),\ t \geq 0$	$F(s) = \frac{k}{s^2+k^2}$
$f(t) = \cos(kt),\ t \geq 0$	$F(s) = \frac{s}{s^2+k^2}$
$f(t) = \sinh(kt),\ t \geq 0$	$F(s) = \frac{k}{s^2-k^2},\ s > \|k\|$
$f(t) = \cosh(kt),\ t \geq 0$	$F(s) = \frac{s}{s^2-k^2},\ s > \|k\|$

The Laplace Transform is Linear

If a is a constant and f and g are functions, then

$$\mathcal{L}(af) = a\mathcal{L}(f) \tag{1}$$

$$\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g) \tag{2}$$

For example, by the above property (1)

$$\mathcal{L}(3t^5) = 3\mathcal{L}(t^5) = 3\left(\frac{5!}{s^6}\right) = \frac{360}{s^6}, \;\; s > 0.$$

As an another example, by property (2)

$$\mathcal{L}(e^{5t} + \cos(3t)) = \mathcal{L}(e^{5t}) + \mathcal{L}(\cos(3t)) = \frac{1}{s-5} + \frac{s}{s^2+9}, \;\; s > 5.$$

An example where both (1) and (2) are used,

$$\mathcal{L}(3t^7+8)=\mathcal{L}(3t^7)+\mathcal{L}(8)=3\mathcal{L}(t^7)+8\mathcal{L}(1)=3(\frac{7!}{s^8})+8(\frac{1}{s}),\ \ s>0.$$

The Laplace transform of the product of two functions

$$\mathcal{L}(fg)\neq\mathcal{L}(f)\mathcal{L}(g).$$

As an example, we determine

$$\begin{aligned}\mathcal{L}(3+e^{6t})^2 &= \mathcal{L}(3+e^{6t})(3+e^{6t})=\mathcal{L}(9+6e^{6t}+e^{12t})\\ &= \mathcal{L}(9)+\mathcal{L}(6e^{6t}+\mathcal{L}(e^{12t})\\ &= 9\mathcal{L}(1)+6\mathcal{L}(e^{6t})+\mathcal{L}(e^{12t})\\ &= \frac{9}{s}+\frac{6}{s-6}+\frac{1}{s-12}.\end{aligned}$$

The respective domains of the above three transforms are $s>0$, $s>6$, and $s>12$; equivalently, $s>12$.

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The Inverse Transform

Lea f be a function and $\mathcal{L}(f) = F$ be its Laplace transform. Then, by definition, f is the *inverse* transform of F. This is denoted by $\mathcal{L}^{-1}(F) = f.$

As an example, from the Laplace Transforms Table, we see that

$$\mathcal{L}(\sin(6t)) = \frac{6}{s^2+36}.$$

Written in the inverse transform notation

$$\mathcal{L}^{-1}\left(\frac{6}{s^2+36}\right) = \sin(6t).$$

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Recall that $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$. Hence, for example,

$$\mathcal{L}^{-1}\left(\frac{7!}{s^8}\right) = t^7.$$

Here, by examining the power s^8 we saw that n=7.

Now consider $\mathcal{L}^{-1}\left(\frac{5}{s^{11}}\right)$. Here n+1=11. Hence n=10.

Now, we need to make the numerator to be 10!.

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{5}{s^{11}}\right) &= 5\mathcal{L}^{-1}\left(\frac{1}{s^{11}}\right) \\
&= \frac{5}{10!}\mathcal{L}^{-1}\left(\frac{10!}{s^{11}}\right) \\
&= \frac{5}{10!}t^{10}.
\end{aligned}$$

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More Examples of Inverse Transforms

Consider $\mathcal{L}^{-1}\left(\frac{7s+15}{s^2+2}\right)$. The form of the denominator, s^2+k^2, is that of the Laplace transforms of sin/cos.

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{7s+15}{s^2+2}\right) &= \mathcal{L}^{-1}\left(\frac{7s}{s^2+2}\right)+\mathcal{L}^{-1}\left(\frac{15}{s^2+2}\right) \\
&= 7\mathcal{L}^{-1}\left(\frac{s}{s^2+2}\right)+15\mathcal{L}^{-1}\left(\frac{1}{s^2+2}\right) \\
&= 7\cos(\sqrt{2}t)+\frac{15}{\sqrt{2}}\mathcal{L}^{-1}\left(\frac{\sqrt{2}}{s^2+2}\right) \\
&= 7\cos(\sqrt{2}t)+\frac{15}{\sqrt{2}}\sin(\sqrt{2}t)
\end{aligned}$$

Partial Fractions

Consider the rational expressions

$$\frac{3s+5}{s^2-3s-10} = \frac{3s+5}{(s-5)(s+2)}$$

The denominator is **factored**, and the **degree** of the numerator **is at least one less** that that of the denominator, in fact, it is exactly one less than the degree of the denominator.

We can, therefore, put the rational expression in partial fractions. This means for constants A and B,

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we have the decomposition

$$\frac{3s+5}{(s-5)(s+2)} = \frac{A}{s-5} + \frac{B}{s+2}.$$

To determine A and B, first clear the denominators:

$$\frac{3s+5}{\cancel{(s-5)(s+2)}}\cancel{(s-5)(s+2)} = \frac{A}{\cancel{(s-5)}}\cancel{(s-5)}(s+2) + \frac{B}{\cancel{(s+2)}}(s-5)\cancel{(s+2)}.$$

Thus we have the polynomial equality:

$$3s+5 = A(s+2) + B(s-5) = (A+B)s + 2A - 5B.$$

By comparing the coefficients of s and constant coefficients, we get two equations in A and B.

$$\begin{aligned} A + B &= 3 \\ 2A - 5B &= 5 \end{aligned}$$

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We can solve for A and B by using Cramer's rule

$$A = \frac{\det\begin{pmatrix} 3 & 1 \\ 5 & -5 \end{pmatrix}}{\det\begin{pmatrix} 1 & 1 \\ 2 & -5 \end{pmatrix}}, \quad \text{and} \quad B = \frac{\det\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}}{\det\begin{pmatrix} 1 & 1 \\ 2 & -5 \end{pmatrix}}.$$

Now by the definition of the determinant,

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb.$$

Hence,

$$A = \frac{20}{7}, \quad \text{and} \quad B = \frac{1}{7}.$$

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We can now determine the inverse transform

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{3s+5}{(s-5)(s+2)}\right) &= \mathcal{L}^{-1}\left(\frac{A}{s-5}+\frac{B}{s+2}\right) \\
&= A\,\mathcal{L}^{-1}\left(\frac{1}{s-5}\right)+B\,\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) \\
&= \frac{20}{7}e^{5t}+\frac{1}{7}e^{-2t}.
\end{aligned}$$

This could also have been directly determined by using a formula from your Table of Laplace Transforms from the text.

This inverse transform will be used in slide #19 to solve an IVP.

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Partial Fractions: More Examples

Put $\frac{3s+4}{(s-2)(s^2+7)}$ in partial fractions.

Since s^2+7 is a quadratic, when it is put in partial fractions, its numerator must be the general polynomial of degree one.

$$\frac{3s+4}{(s-2)(s^2+7)} = \frac{A}{s-2} + \frac{Bs+D}{s^2+7}$$

$$\frac{3s+4}{\cancel{(s-2)(s^2+7)}}\cancel{(s-2)(s^2+7)} = \frac{A}{\cancel{s-2}}\cancel{(s-2)}(s^2+7) + \frac{Bs+D}{\cancel{s^2+7}}(s-2)\cancel{(s^2+7)}$$

Hence, we have the equality of polynomials:

$$3s+4 = A(s^2+7)+(Bs+D)(s-2) = (A+B)s^2+(-2B+D)s+7A-2D$$

Comparing the coefficients of s^2, s, and constant coefficients,

$$A+B=0, \quad -2B+D=3, \quad \text{and} \quad 7A-2D=4.$$

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From the first equation, we get that *B*=-*A*. Sub in the second, to get

$$2A + D = 3 \quad (1)$$

$$7A - 2D = 4 \quad (2)$$

Then,

$$A = \frac{\det\begin{pmatrix} 3 & 1 \\ 4 & -2 \end{pmatrix}}{\det\begin{pmatrix} 2 & 1 \\ 7 & -2 \end{pmatrix}} = \frac{-10}{-11}, \quad \text{and} \quad D = \frac{\det\begin{pmatrix} 2 & 3 \\ 7 & 4 \end{pmatrix}}{\det\begin{pmatrix} 2 & 1 \\ 7 & -2 \end{pmatrix}} = \frac{-13}{-11}.$$

Hence,

$$A = \frac{10}{11}, \; B = -A = -\frac{10}{11}, \; D = \frac{13}{11}.$$

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Transforms of Derivatives

Given a function $y=y(t)$, the transform of its derivative y' can be expressed in terms of the Laplace transform of y: $\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$.

The corresponding formula for y'' can be obtained by replacing y by y' (equation 1 below).

$$\begin{aligned} \mathcal{L}(y')' &= s\mathcal{L}(y') - y'(0) && (1) \\ &= s(s\mathcal{L}(y) - y(0)) - y'(0) && (2) \\ &= s^2\mathcal{L}(y) - sy(0) - y'(0). && (3) \end{aligned}$$

Hence, $\boxed{\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0).}$

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Solving IVP with Laplace Transforms

Example 1: Solve the IVP $y' - 5y = -e^{-2t},\ y(0) = 3.$

Solution: Taking the Laplace transform of both sides,

$$\mathcal{L}(y' - 5y) = \mathcal{L}(-e^{-2t}) \tag{1}$$

$$\mathcal{L}(y') - 5\mathcal{L}(y) = -\frac{1}{s+2} \tag{2}$$

$$s\mathcal{L}(y) - y(0) - 5\mathcal{L}(y) = -\frac{1}{s+2} \tag{3}$$

$$(s-5)\mathcal{L}(y) - 3 = -\frac{1}{s+2} \tag{4}$$

$$(s-5)\mathcal{L}(y) = -\frac{1}{s+2} + 3 = \frac{3s+5}{s+2} \tag{5}$$

$$\mathcal{L}(y) = \frac{3s+5}{(s-5)(s+2)} \tag{6}$$

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Hence, by the definition of the inverse transform,

$$y = \mathcal{L}^{-1}\left(\frac{3s+5}{(s-5)(s+2)}\right) = \frac{20}{7}e^{5t} + \frac{1}{7}e^{-2t}.$$

The above inverse transform was found in <u>slide # 14</u>.

Example 2: Solve the IVP:

$$y'' + 7y = 10e^{2t},\ \ y(0) = 0,\ y'(0) = 3.$$

Solution: Taking the Laplace transform of both sides,

$$\mathcal{L}(y'') + 7\mathcal{L}(y) = 10\mathcal{L}(e^{2t})$$

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + 7\mathcal{L}(y) = \frac{10}{s-2}$$

0 3

Whence, $(s^2+7)\mathcal{L}(y) - 3 = \frac{10}{s-2}$

$$(s^2+7)\mathcal{L}(y) = 3 + \frac{10}{s-2} = \frac{3(s-2)+10}{s-2}$$

$$\mathcal{L}(y) = \frac{3s+4}{(s-2)(s^2+7)}$$

Then, $y = \mathcal{L}^{-1}\left(\frac{3s+4}{(s-2)(s^2+7)}\right)$. Now using the partial fraction decomposition in slide # 15 and 16,

$$y = \frac{10}{11}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) - \frac{10}{11}\mathcal{L}^{-1}\left(\frac{s}{s^2+7}\right) + \frac{13}{11}\mathcal{L}^{-1}\left(\frac{1}{s^2+7}\right).$$

$$y = \frac{10}{11}e^{2t} - \frac{10}{11}\cos(\sqrt{7}t) + \frac{13}{11\sqrt{7}}\sin(\sqrt{7}t).$$

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Lemma: Let $\mathcal{L}(y(t)) = Y(s)$. Then $\mathcal{L}(e^{at}y(t)) = Y(s-a)$.

Example 1 $\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$. To find $\mathcal{L}(e^{at}\cos(kt))$ we replace s by s-a. Hence,

$$\mathcal{L}(e^{at}\cos(kt)) = \frac{s-a}{(s-a)^2+k^2}.$$

The inverse version is also useful:

$$\mathcal{L}^{-1}\left(\frac{s-a}{(s-a)^2+k^2}\right) = e^{at}\cos(kt).$$

Notice matching s-a on the numerator and denominator.

The corresponding sine version is

$$\mathcal{L}^{-1}\left(\frac{k}{(s-a)^2+k^2}\right) = e^{at}\sin(kt).$$

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Find: $\mathcal{L}^{-1}\left(\frac{4s+1}{s^2+10s+34}\right)$.

Here the denominator does not factor over the reals.

Hence complete the square.

$$s^2+10s+34 = \underbrace{s^2+10s+25} -25+34 = (s+5)^2+9.$$

$$\mathcal{L}^{-1}\left(\frac{4s+1}{s^2+10s+34}\right) = \mathcal{L}^{-1}\left(\frac{4s+1}{(s+5)^2+9}\right)$$

this s must now be made into (s+5).

$$= \mathcal{L}^{-1}\left(\frac{4(s+5)-20+1}{(s+5)^2+9}\right) = \mathcal{L}^{-1}\left(\frac{4(s+5)-19}{(s+5)^2+9}\right)$$

$$= 4\mathcal{L}^{-1}\left(\frac{(s+5)}{(s+5)^2+9}\right) - 19\mathcal{L}^{-1}\left(\frac{1}{(s+5)^2+9}\right)$$

$$= 4e^{-5t}\cos(3t) - \frac{19}{3}\mathcal{L}^{-1}\left(\frac{3}{(s+5)^2+9}\right)$$

$$= 4e^{-5t}\cos(3t) - \frac{19}{3}e^{-5t}\sin(3t).$$

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Lemma: Let $\mathcal{L}(f(t)) = F(s)$. Then $\mathcal{L}(tf(t)) = -F'(s)$.

Example 1: Now $\mathcal{L}(e^{at}) = \frac{1}{s-a}$, $s \neq a$.

Hence, $\mathcal{L}(te^{at}) = -\left(\frac{1}{s-a}\right)' = \frac{1}{(s-a)^2}$, $s \neq a$.

In other words, $\mathcal{L}^{-1}\left(\frac{1}{(s-a)^2}\right) = te^{at}$.

Example 2: Find $f(t) = \mathcal{L}^{-1}(\ln\left(\frac{s+2}{s-3}\right))$.

Solution: Now $\mathcal{L}(f(t)) = \ln\left(\frac{s+2}{s-3}\right)$. Then by the Lemma,

$$\mathcal{L}(tf(t)) = -\left(\ln\left(\frac{s+2}{s-3}\right)\right)' = -(\ln(s+2) - \ln(s-3))'$$

Next Slide

$$\mathcal{L}(tf(t)) = -(\frac{1}{s+2} - \frac{1}{s-3}) = \frac{1}{s-3} - \frac{1}{s+2}.$$

Hence, $$tf(t) = \mathcal{L}^{-1}(\frac{1}{s-3}) - \mathcal{L}^{-1}(\frac{1}{s+2})$$

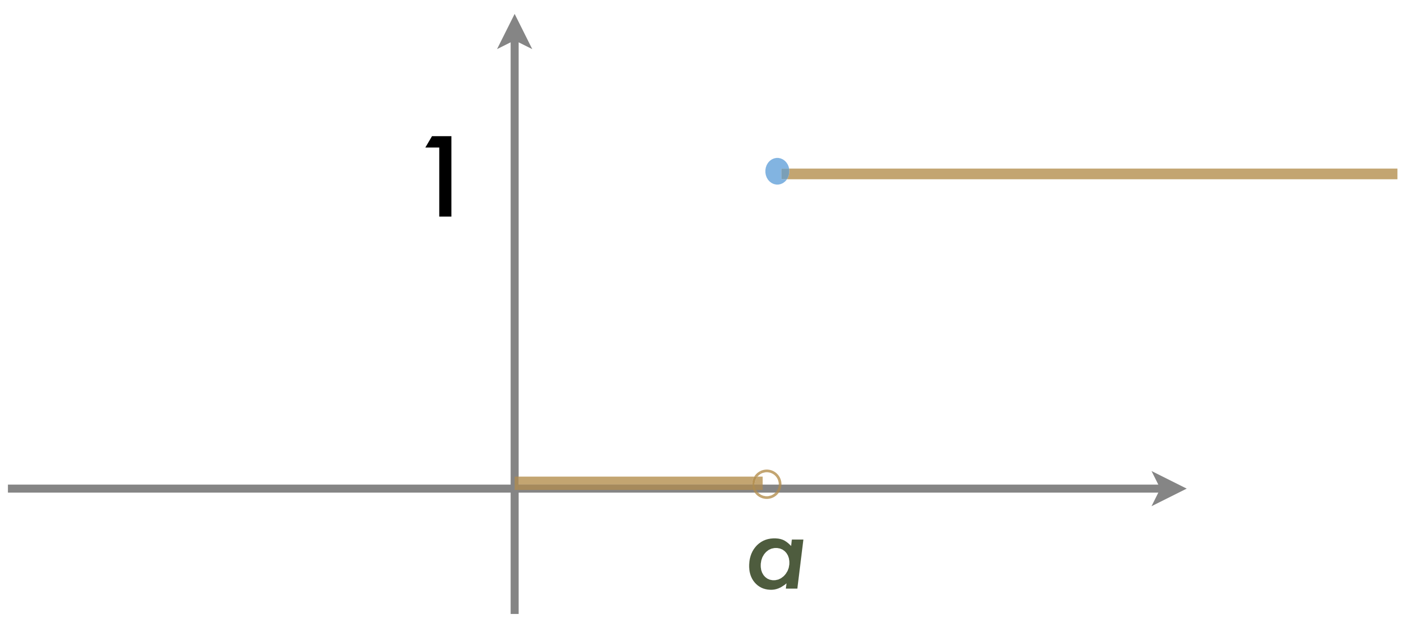
$$= e^{3t} - e^{-2t}.$$

Hence, $$f(t) = \frac{e^{3t} - e^{-2t}}{t},\ t > 0.$$

Unit Function and Piece-wise Defined Functions

Let $a \geq 0$. The Heaviside unit function $U(t-a)$ is defined by

$$U(t-a) = \begin{cases} 1 & \text{if } \; t \geq a \\ 0 & \text{if } \; 0 \leq t < a \end{cases}.$$



The unit function can be used to express piecewise functions.

Example 1: Let f be the piecewise defined function

$$f(t) = \begin{cases} 4 & \text{if } \ 0 \le t < 8 \\ 6 & \text{if } \ t \ge 8 \end{cases}.$$

Now consider the function $4 + (6-4)U(t-8)$.

If $0 \le t < 8$, then $U(t-8) = 0$. Then $4 + (6-4)U(t-8) = 4$.

If $t \ge 8$, then $U(t-8) = 1$. Then

$$4 + (6-4)U(t-8) = 4 + 6 - 4 = 6.$$

Thus, we see that

$$f(t) = 4 + 2U(t-8).$$

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Example 2: Consider the piecewise defined function

$$f(t) = \begin{cases} t & \text{if } \ 0 \le t < 2 \\ t^2 & \text{if } \ 2 \le t < 6 \\ t^3 & \text{if } \ 6 \le t \end{cases}.$$

We can express f in terms of unit functions.

$$f(t) = t + (t^2 - t)U(t-2) + (t^3 - t^2)U(t-6).$$

Notice how the coefficients of the unit functions are related to the outputs by the piece-wise defined function.

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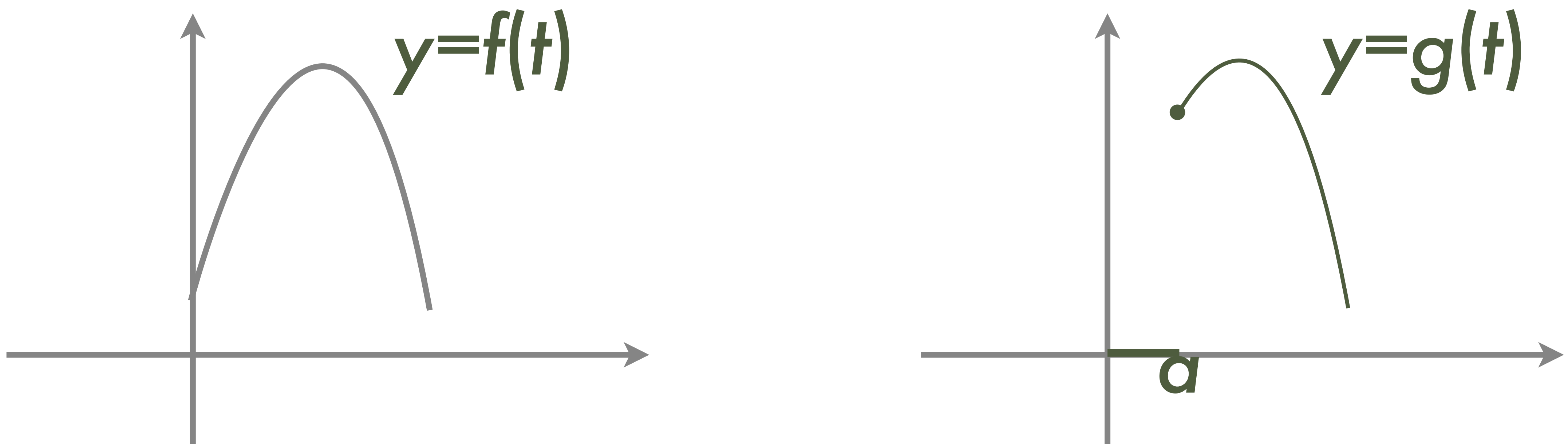
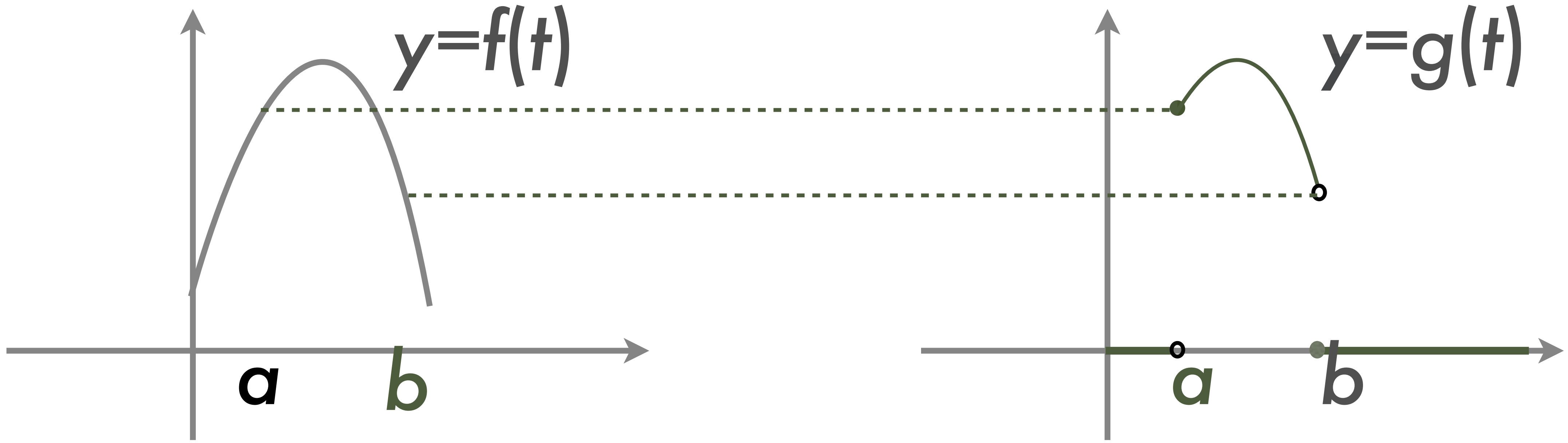
Truncating a Function



The graph of *g* has been obtained by truncating that of *f*.

$$g(t) = \begin{cases} 0 & \text{if } \; 0 \le t < a \\ f(t) & \text{if } \; t \ge a \end{cases} \longrightarrow \boxed{g(t) = f(t)U(t-a).}$$

Truncating a Function



The graph of g has been obtained by truncating that of f.

$$g(t) = \begin{cases} 0 & \text{if } 0 \le t < a \\ f(t) & \text{if } a \le t < b \\ 0 & \text{if } b \le t \end{cases}.$$

$$\boxed{g(t) = f(t)U(t-a) - f(t)U(t-b).}$$

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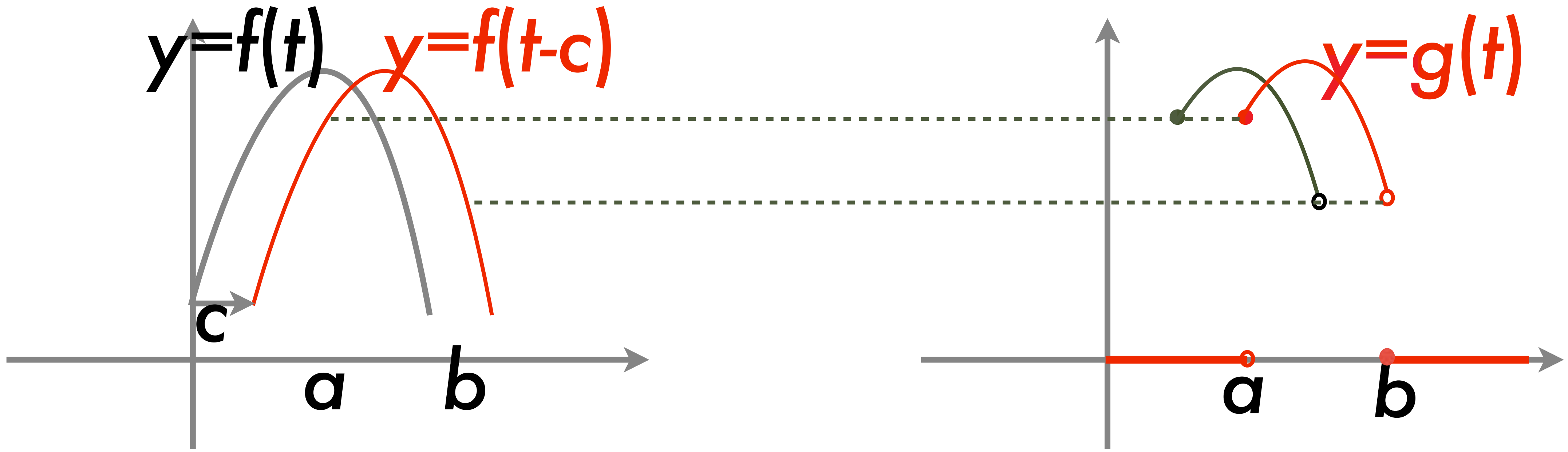
Translating and Truncating a Function



The graph of g has been obtained translating the graph of f by c units to the right and then truncating it.

$$g(t) = \begin{cases} 0 & \text{if } 0 \leq t < a \\ f(t-c) & \text{if } a \leq t < b \\ 0 & \text{if } b \leq t \end{cases}.$$

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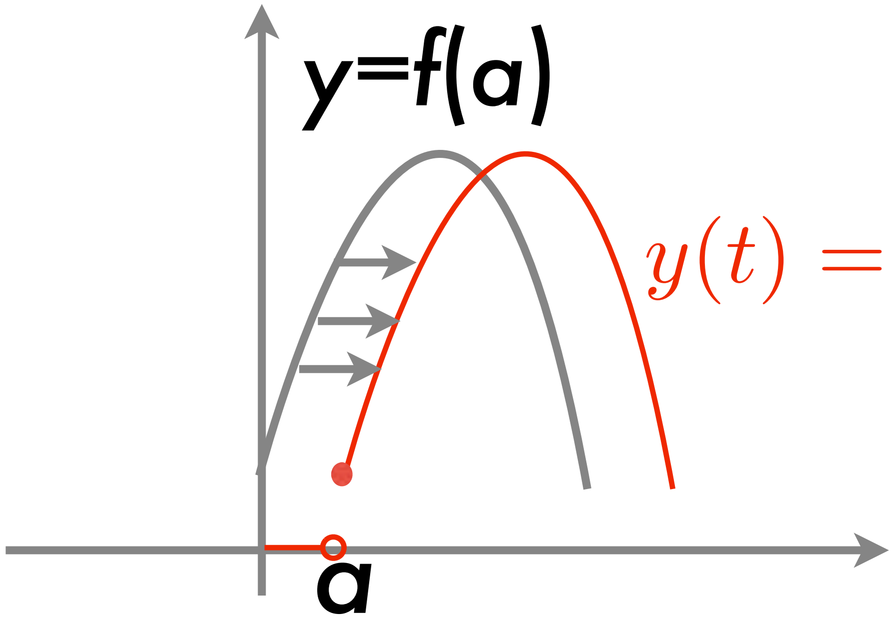

$$y(t) = f(t-a)U(t-a) = \begin{cases} 0 & \text{if } 0 \le t < a \\ f(t-a) & \text{if } a \le t \end{cases}.$$

Proposition 1 *Let $a \ge 0$ and $\mathcal{L}(f(t)) = F(s)$. Then*

$$\mathcal{L}(f(t-a)U(t-a)) = e^{-as}F(s).$$

Example 1: Find $\mathcal{L}(\sin(t-2)U(t-2))$.

Solution: Here *a* is 2 and $f(t-2) = \sin(t-2)$. We need *f*(*t*) to determine *F*(*s*). We can get it from the formula for *f*(*t*–2) by replacing *t* by *t*+2.

$f(t+2-2) = \sin(t+2-2)$, i.e., $f(t) = \sin(t)$. Hence,

$\mathcal{L}(\sin(t-2)U(t-2)) = F(s)e^{-2s} = \frac{1}{s^2+4}e^{-2s}$.

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Example 2: Determine $\mathcal{L}(t^2U(t-2))$.

Solution: Recall the formula

$$\mathcal{L}(f(t-a)U(t-a)) = e^{-as}F(s), \quad \text{where} \quad F(s) = \mathcal{L}(f).$$

In this case, $a=2$, and $f(t-a) = f(t-2) = t^2.$

to obtain $F(s)$, we first need $f(t)$. In order to do that

replace t in the formula for $f(t-2)$ by $t+2$.

$$\begin{aligned}
f(t-2) &= t^2 \\
f(t+2-2) &= (t+2)^2 = t^2 + 4t + 4 \\
f(t) &= t^2 + 4t + 4 \\
F(s) &= \mathcal{L}(t^2) + 4\mathcal{L}(t) + 4\mathcal{L}(1) \\
&= \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}, \quad s > 0.
\end{aligned}$$

Thus $\mathcal{L}(t^2U(t-2)) = \left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right) e^{-2s}, \quad s > 0.$

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Example 3: Determine $\mathcal{L}\left(\sin(\frac{\pi}{2}t)U(t-3)\right)$.

Solution: Comparing $U(t-3)$ with $U(t-a)$, we get a=3.

Hence, $f(t-3)=\sin(\frac{\pi}{2}t)$. Now to obtain $f(t)$, replace t by $t+3$ in the formula for $f(t-3)$. Then,

$$f(t+3-3)=\sin(\frac{\pi}{2}(t+3))=\sin(\frac{\pi}{2}t+\frac{\pi}{2}3).$$

Now an elementary trig identity states

$$\sin(3\frac{\pi}{2}+\theta)=-\cos(\theta).$$

Whence, $f(t)=-\cos(\frac{\pi}{2}t)$. Thus $F(s)=-\frac{s}{s^2+\frac{\pi^2}{4}}$.

$$\mathcal{L}\left(\sin(\frac{\pi}{2}t)U(t-3)\right)=e^{-3s}\frac{s}{s^2+\frac{\pi^2}{4}}.$$

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Let $\mathcal{L}^{-1}(F(s)) = f(t).$ Then

$$\boxed{\mathcal{L}^{-1}(e^{-as}F(s)) = U(t-a)f(t-a).}$$

Example 1: Recall $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+4}\right) = \cos(2t)$

Hence $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+4}e^{-7s}\right) = \cos(2(t-7))U(t-7).$

The presence of e^{-7s} caused two changes to $\cos(2t)$: the input *t* was replaced by $t-7$ and then $\cos(2(t-7))$ was multiplied by $U(t-7).$

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Example 2: Determine $\mathcal{L}^{-1}\left(\dfrac{e^{-7s}}{(s-5)^2+4}\right)$.

First consider the inverse without the factor e^{-7s}.

$$\mathcal{L}^{-1}\left(\frac{1}{(s-5)^2+4}\right) = e^{5t}\sin(2t).$$

Thus,

Replace t by $t-7$.

$$\mathcal{L}^{-1}\left(\frac{e^{-7s}}{(s-5)^2+4}\right) = \overbrace{e^{5(t-7)}\sin(2(t-7))}\,U(t-7).$$

Convolutions

Let *f* and *g* be functions. The convolution of *f* with *g* is defined by

$$\overbrace{f * g}(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Thus in a convolution integral, in general, you will see a τ factor (the *t* in the output by *f* replaced by τ), and a $t-\tau$ factor (the *t* in the output by *g* replaced by $t-\tau$).

$$\sin 3t * e^{5t} = \int_0^t \underbrace{\sin(3\tau)}_{\text{the } \tau \text{ factor}} \underbrace{e^{5(t-\tau)}}_{\text{the } t-\tau \text{ factor}}\, d\tau.$$

the τ factor | the $t-\tau$ factor

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Example 1: Let $f(t) = t^2$ and $g(t) = 2t + 3$. Find $f * g$.

Solution:

$$f * g(t) = \int_0^t f(t)g(t-\tau)\,d\tau = \int_0^t \tau^2(2(t-\tau)+3)\,d\tau.$$

$$= \int_0^t \tau^2(2t+3-\tau)\,d\tau = (2t+3)\int_0^t \tau^2\,d\tau - \int_0^t \tau^3\,d\tau$$

$$= (2t+3)\left[\frac{\tau^3}{3}\right]_0^t - \left[\frac{\tau^4}{4}\right]_0^t = (2t+3)\frac{t^3}{3} - \frac{t^4}{4}$$

$$= \frac{5t^4}{12} + t^3.$$

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Example 2:

Express the following integral as a convolution.

$$\int_0^t \tau^3 \cos(t-\tau)\, d\tau.$$

'τ' factor

Replace τ by t to get the first factor of the convolution

'$t-\tau$' factor

Replace $t-\tau$ by t to get the second factor of the convolution

$$\int_0^t \tau^3 \cos(t-\tau)\, d\tau = t^3 * \cos(t)$$

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Example 3: $\int_0^t e^{2(t-\tau)}\tau^3\,d\tau = t^3 * e^{2t}.$

Example 4:

In the following example $`t-\tau'$ factor is missing.

That is because the second factor of the convolution was a constant.

$$\int_0^t \tau^3\,d\tau = t^3 * 1.$$

Example 5: $\int_0^t e^{t-2\tau}\,d\tau = \int_0^t e^{-\tau}e^{t-\tau}\,d\tau = e^{-t} * e^t.$

We see that the convolution $e^{-t} * e^t$ is not the constant function 1. Here is an alternative view of the same integral

$$\int_0^t e^{t-2\tau}\,d\tau = \int_0^t e^t e^{-2\tau}\,d\tau = e^t\int_0^t e^{-2\tau}\,d\tau = e^t(e^{-2t} * 1).$$

However, in general $f(g*h) \neq (fg)*(fh).$

Laplace Transform of Convolutions

The Laplace transform of the product of two functions is not equal to the product of the two transforms:

$$\mathcal{L}(fg) \neq \mathcal{L}(f)\mathcal{L}(g).$$

The convolution behaves far better:

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g).$$

Example 1: Without evaluating the integral, find

$$\mathcal{L}\left(\int_0^t e^{2(t-\tau)}\tau^3\, d\tau\right).$$

By Example 3 of the previous slide

$$\mathcal{L}\left(\int_0^t e^{2(t-\tau)}\tau^3\, d\tau\right) = \mathcal{L}\left(t^3 * e^{2t}\right) = \mathcal{L}(t^3)\mathcal{L}(e^{2t}) = \frac{3!}{s^4(s-2)}, \quad s > 2.$$

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Example 2: $\mathcal{L}\left(\int_0^t e^{t+\tau}\sin(t-\tau)\,d\tau\right)$

$$= \mathcal{L}\left(\int_0^t e^t e^{\tau}\sin(t-\tau)\,d\tau\right)$$

$$= \mathcal{L}\left(e^t\int_0^t e^{\tau}\sin(t-\tau)\,d\tau\right)$$

$$= \mathcal{L}(e^t(e^t * \sin t))$$

Now $\mathcal{L}(e^t * \sin t) = \mathcal{L}(e^t)\mathcal{L}(\sin t) = \frac{1}{s-1}\frac{1}{s^2+1},\ \ s > 1.$

The effect of multiplying this input to the Laplace transform by e^t is to replace the s in the output by s-1. Hence

$$\mathcal{L}(e^t(e^t * \sin t)) = \frac{1}{(s-1)-1}\frac{1}{(s-1)^2+1} = \frac{1}{(s-2)((s-1)^2+1)},\ \ s > 2.$$

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Example 3: Find $\mathcal{L}\left(\int_0^t t\sin(\tau)\,d\tau\right)$.

Solution:

$$\mathcal{L}\left(\int_0^t t\sin(\tau)\,d\tau\right) = \mathcal{L}\left(t\int_0^t \sin(\tau)\,d\tau\right)$$
$$= \mathcal{L}\left(t(\sin t * 1)\right)$$

Recall that $\mathcal{L}(tf(t)) = -F'(s)$, where $\mathcal{L}(f(t)) = F(s)$.

Now, $\mathcal{L}\left((\sin t * 1)\right) = \mathcal{L}(\sin t)\mathcal{L}(1) = \frac{1}{s^2+1}\frac{1}{s}, \ \ s > 0.$

Hence, $\mathcal{L}\left(t(\sin t * 1)\right) = -\left(\frac{1}{s(s^2+1)}\right)', \ \ s > 0$

$$= -\frac{3s^2+1}{s^2(s^2+1)^2}, \ \ s > 0.$$

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Integro-differential Equations

Example: Solve: $f(t) = t + \int_0^t \sin(\tau) f(t-\tau)\, d\tau.$ input to f

Solution: Notice that $f(t) = t + \sin t * f(t)$. Now use the Laplace transform to convert the convolution product to regular products.

$$\mathcal{L}(f(t)) = \mathcal{L}(t) + \mathcal{L}(\sin t * f(t)) = \frac{1}{s^2} + \frac{1}{s^2+1}\mathcal{L}(f(t).$$

Hence, $(1 - \frac{1}{s^2+1})\mathcal{L}(f(t)) = \frac{1}{s^2}$

$(\frac{s^2+1-1}{s^2+1})\mathcal{L}(f(t)) = \frac{1}{s^2}$ i.e., $(\frac{s^2}{s^2+1})\mathcal{L}(f(t)) = \frac{1}{s^2}$

Thus, $\mathcal{L}(f(t)) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$. Hence,

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \frac{1}{3!}\mathcal{L}^{-1}\left(\frac{3!}{s^4}\right) = t + \frac{t^3}{6}.$$

Solve: $y'(t) = 1 - \sin t - \int_0^t y(\tau)\,d\tau,\ y(0) = 0.$ Take Laplace transforms to get $\mathcal{L}(y'(t)) = \mathcal{L}(1) - \mathcal{L}(\sin t) - \mathcal{L}\left(\int_0^t y(\tau)\,d\tau\right)$,

$$s\mathcal{L}(y(t)) - y(0) = \frac{1}{s} - \frac{1}{s^2+1} - \mathcal{L}(y(t) * 1),$$

$$s\mathcal{L}(y(t)) = \frac{1}{s} - \frac{1}{s^2+1} - \mathcal{L}(y(t))\mathcal{L}(1),$$

$$s\mathcal{L}(y(t)) = \frac{1}{s} - \frac{1}{s^2+1} - \frac{\mathcal{L}(y(t))}{s},$$

$$s\mathcal{L}(y(t)) + \frac{\mathcal{L}(y(t))}{s} = \frac{1}{s} - \frac{1}{s^2+1},$$

$$(s + \tfrac{1}{s})\mathcal{L}(y(t)) = \frac{1}{s} - \frac{1}{s^2+1},$$

$$\frac{(s^2+1)}{s}\mathcal{L}(y(t)) = \frac{1}{s} - \frac{1}{s^2+1},$$

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$$\mathcal{L}(y(t)) = \frac{s}{(s^2+1)}\frac{1}{s} - \frac{s}{(s^2+1)}\frac{1}{s^2+1},$$

$$\mathcal{L}(y(t)) = \frac{1}{(s^2+1)} - \frac{s}{(s^2+1)^2},$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{(s^2+1)}\right) - \mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right),$$

$$y(t) = \sin t - \frac{1}{2}t\sin t \quad \square$$

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Using the Dirac's Delta Function

Solve the IVP $y' + 4y = \delta(t-2),\ y(0) = 6.$

$$\mathcal{L}(y') + 4\mathcal{L}(y) = \mathcal{L}(\delta(t-2))$$

$$\overbrace{s\mathcal{L}(y) - y(0)} + 4\mathcal{L}(y) = \frac{e^{-2s}}{s}$$

$$(s+4)\mathcal{L}(y) = y(0) + \frac{e^{-2s}}{s} = 6 + \frac{e^{-2s}}{s}$$

$$\mathcal{L}(y) = \frac{6}{(s+4)} + \frac{e^{-2s}}{s(s+4)} = \frac{6}{(s+4)} + \frac{e^{-2s}}{4s} - \frac{e^{-2s}}{4(s+4)}$$

$$\boxed{y(t) = 6e^{-4t} + \frac{1}{4}U(t-2) - \frac{e^{-4(t-2)}}{4}U(t-2).}$$

Systems of Equations and Laplace Transform

Let $x = x(t)$ and $y = y(t)$ be functions of t. Suppose

$$x' + 3x + y' = 1 \quad (1)$$

$$x' - x + y' = e^t, \quad x(0) = 0, \; y(0) = 0. \quad (2)$$

By taking the Laplace Transforms, we will solve for x and y by converting the equations into two in $\mathcal{L}(x)$ and $\mathcal{L}(y)$.

From (1) $$\mathcal{L}(x') + 3\mathcal{L}(x) + \mathcal{L}(y') = \mathcal{L}(1) \quad (3)$$

From (2) $$\mathcal{L}(x') - \mathcal{L}(x) + \mathcal{L}(y') = \mathcal{L}(e^t). \quad (4)$$

From (3) $$s\mathcal{L}(x) - x(0) + 3\mathcal{L}(x) + s\mathcal{L}(y) - y(0) = \frac{1}{s}$$

From (4) $$s\mathcal{L}(x) - x(0) - \mathcal{L}(x) + s\mathcal{L}(y) - y(0) = \frac{1}{s-1}.$$

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Using $x(0) = 0$ and $y(0) = 0,$ and collecting like terms

$$(s+3)\mathcal{L}(x) + s\mathcal{L}(y) = \frac{1}{s}$$

$$(s-1)\mathcal{L}(x) + s\mathcal{L}(y) = \frac{1}{s-1}$$

Using Cramer's rule,

$$\mathcal{L}(x) = \frac{\det\begin{pmatrix} \frac{1}{s} & s \\ \frac{1}{s-1} & s \end{pmatrix}}{\det\begin{pmatrix} s+3 & s \\ s-1 & s \end{pmatrix}}, \quad \mathcal{L}(y) = \frac{\det\begin{pmatrix} s+3 & \frac{1}{s} \\ s-1 & \frac{1}{s-1} \end{pmatrix}}{\det\begin{pmatrix} s+3 & s \\ s-1 & s \end{pmatrix}}$$

$$\mathcal{L}(x) = \frac{1 - \frac{s}{s-1}}{s^2+3s-s^2+s}, \quad \mathcal{L}(y) = \frac{\frac{s+3}{s-1} - \frac{s-1}{s}}{s^2+3s-s^2+s}$$

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$$\mathcal{L}(x) = \frac{1 - \frac{s}{s-1}}{4s}, \quad \mathcal{L}(y) = \frac{\frac{s+3}{s-1} - \frac{s-1}{s}}{4s}$$

$$\mathcal{L}(x) = \frac{1}{4s} - \frac{1}{4(s-1)}, \quad \mathcal{L}(y) = \frac{s+3}{4s(s-1)} - \frac{s-1}{4s^2}$$

Hence, $x(t) = \frac{1}{4} - \frac{1}{4}e^t$. In $\mathcal{L}(y)$ put the first fraction in partial fractions and distribute the s^2 in the second to get $\mathcal{L}(y) = -\frac{3}{4s} + \frac{1}{s-1} - \left(\frac{1}{4s} - \frac{1}{4s^2}\right)$. Hence,

$\mathcal{L}(y) = -\frac{1}{s} + \frac{1}{s-1} + \frac{1}{4s^2}$. Hence, $y = -1 + e^t + \frac{1}{4}t.$

The partial fraction calculations:

$\frac{s+3}{4s(s-1)} = \frac{A}{s} + \frac{B}{s-1} \Rightarrow s + 3 = 4A(s-1) + 4Bs$. Hence,

$4A + 4B = 1$ and $-4A = 3$. Solve for A and B.

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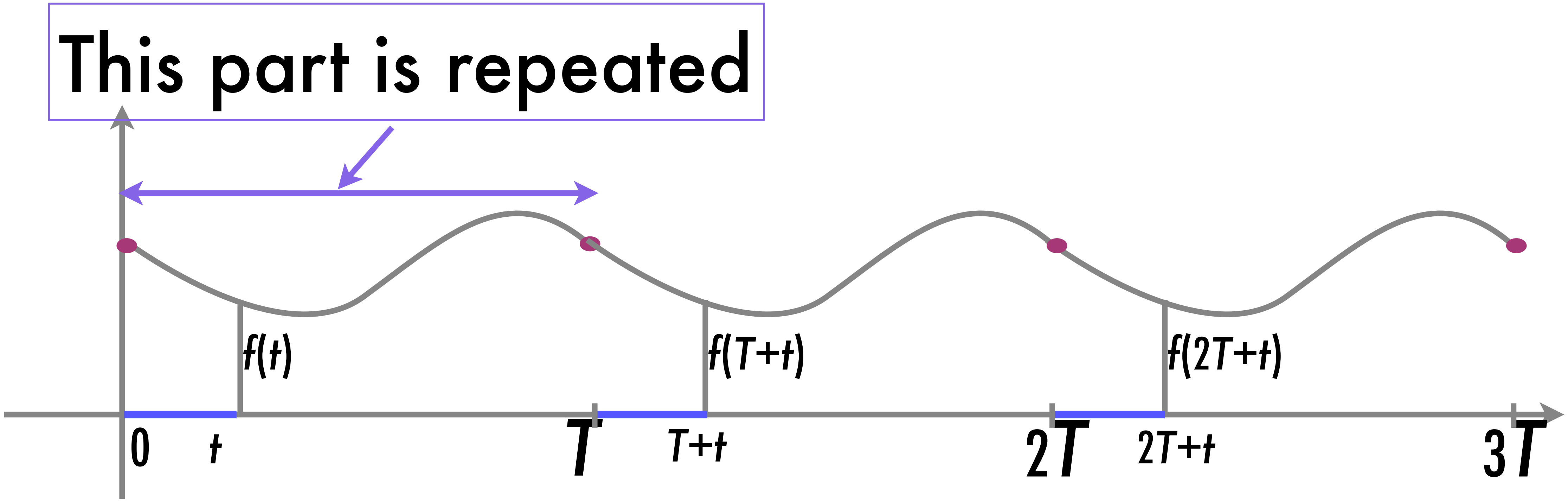
Periodic Functions



The graph is made by repeating a beginning part. The smallest $T>0$ such that $f(t+T)=f(t)$ for all t is called period of the function. The Laplace transform of f is

$$\mathcal{L}(f)(s) = \frac{1}{1-e^{-Ts}} \int_0^T e^{-ts} f(t)\, dt.$$

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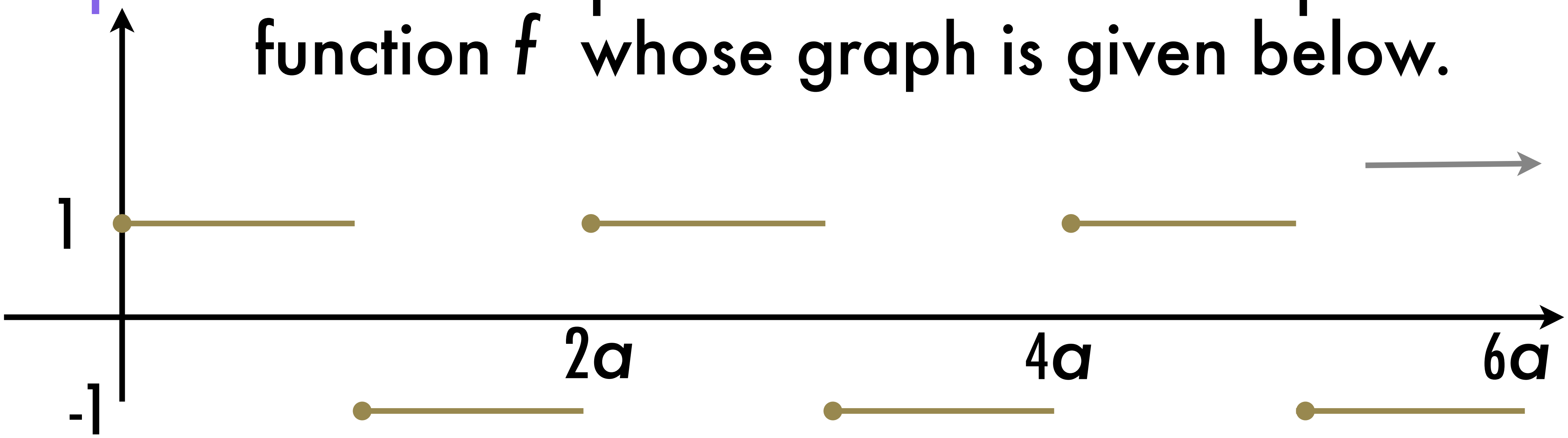
Example: Find the Laplace Transform of the periodic function f whose graph is given below.



The period of f is $2a$ and, i.e., $f(t)=f(t+2a)$, for all t and

$$f(t)=\begin{cases}1 & \text{if } \ 0\le t<a\\ -1 & \text{if } \ a\le t<2a\end{cases}.$$

Applying the formula for the transform of a periodic function, we have

$$\mathcal{L}(f(t))(s)=\frac{1}{1-e^{-2as}}\int_0^{2a} e^{-ts} f(t)\,dt,$$

$$= \frac{1}{1-e^{-2as}} \left(\int_0^a e^{-ts} f(t)\, dt + \int_a^{2a} e^{-ts} f(t)\, dt \right)$$

$$= \frac{1}{1-e^{-2as}} \left(\int_0^a e^{-ts}\, dt - \int_a^{2a} e^{-ts}\, dt \right)$$

$$= \frac{1}{1-e^{-2as}} \left(\left[\frac{e^{-ts}}{-s} \right]_0^a - \left[\frac{e^{-ts}}{-s} \right]_a^{2a} \right)$$

$$= \frac{1}{1-e^{-2as}} \left(\frac{e^{-as}}{-s} - \frac{1}{-s} - \frac{e^{-2as}}{-s} + \frac{e^{-as}}{-s} \right)$$

$$= \frac{1}{(1-e^{-2as})} \frac{1}{s} (1 - 2e^{-as} + e^{-2as})$$

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$$= \frac{1}{s}\frac{1}{(1-e^{-as})(1+e^{-as})}(1-e^{-as})(1-e^{-as})$$

$$= \frac{1}{s}\frac{(1-e^{-as})}{(1+e^{-as})}, \;\; s>0$$

2x=as

Recall

$$\tanh(x) = \frac{(e^{x}-e^{-x})}{(e^{x}+e^{-x})} = \frac{e^{-x}(e^{x}-e^{-x})}{e^{-x}(e^{x}+e^{-x})} = \frac{(1-e^{-2x})}{(1+e^{-2x})}$$

Whence,

$$\mathcal{L}(f)(s) = \frac{\tanh(\frac{as}{2})}{s}, \;\; s>0.$$

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