NUMERICAL INTEGRATION

Sometimes we face definite integrals that cannot be solved even if the integrands are continuous functions such as $\sqrt{1+x^3}$ and e^{x^2}. In our discussion in this course so far, we are not able to evaluate such integrals. We exploit this to show the reader a new technique to approximate the definite integrals.

1. The Trapezoidal Rule :

It is used to approximate $\int_a^b f(x)\ dx$ with a regular partition of the interval $[a,b]$, where $\Delta x = \frac{b-a}{n}$, by using the formula

$$\int_a^b f(x)\ dx \approx \frac{b-a}{2n}\left[f\left(x_0\right)+2f\left(x_1\right)+\cdots+2f\left(x_{n-1}\right)+f\left(x_n\right)\right]$$

Example : Approximate the integral $\int_0^1 \sqrt{x+x^2}\ dx$ using Trapezoidal rule with $n=4$.

Answer : $[a,b]=[0,1]$, $f(x)=\sqrt{x+x^2}$ and $\Delta x = \frac{1-0}{4} = 0.25$

n	x_n	$f\left(x_n\right)$	m	$mf\left(x_n\right)$
0	0	0	1	0
1	0.25	0.559017	2	1.11803
2	0.5	0.86625	2	1.73205
3	0.75	1.14564	2	2.29129
4	1	1.41421	1	1.41421
				6.55559

$$\int_0^1 \sqrt{x+x^2}\ dx \approx \frac{1-0}{2(4)}\left[f(0)+2f(0.25)+2f(0.5)+2f(0.75)+f(1)\right]$$

$$\int_0^1 \sqrt{x+x^2}\ dx \approx \frac{1}{8}\left[6.55559\right] \approx 0.819448\ .$$

Exercise : Approximate the integral $\int_2^2 \frac{1}{x-1}\ dx$ using Trapezoidal rule with $n=4$.

2. Simpson's Rule :

It is used to approximate $\int_a^b f(x)\ dx$ with a regular partition of the interval $[a,b]$, where $\Delta x = \frac{b-a}{n}$, and n is <u>even</u> , by using the formula

$$\begin{aligned}\int_a^b f(x)\ dx \quad &\approx \quad \frac{b-a}{3n}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots\right.\\ &\left.+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]\end{aligned}$$

Example : Approximate the integral $\int_0^{10} \sqrt{10x - x^2}\ dx$ using Simpson's rule with $n = 4$.
Answer : $[a, b] = [0, 10]$, $f(x) = \sqrt{10x - x^2}$ and $\Delta x = \frac{10-0}{4} = 2.5$

n	x_n	$f(x_n)$	m	$mf(x_n)$
0	0	0	1	0
1	2.5	4.33013	4	17.3204
2	5	5	2	10
3	7.5	4.33013	4	17.3204
4	10	0	1	0
				44.6408

$$\int_0^{10} \sqrt{10x - x^2}\ dx \approx \frac{10-0}{3(4)} \left[f(0) + 4f(2.5) + 2f(5) + 4f(7.5) + f(10)\right]$$
$$\int_0^{1} \sqrt{10x - x^2}\ dx \approx \frac{10}{12} [44.6408] \approx 37.2007\ .$$

Exercise : Approximate the integral $\int_0^{2} \frac{x}{x+1}\ dx$ using Simpson's rule with $n = 4$.