Numerical Methods

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Aims

In this lecture, we will . . .

- Introduce the concept of interpolation
- Introduce Lagrange polynomials

Interpolation and polynomial approximation

Here, we will be investigating how to use polynomials to approximation to fit data and functions in order to interpolate between the given data.
One of the most common, and most useful, classes of functions that are used to data are the class of algebraic polynomials, i.e., the set of functions of the form

$$f(x) = p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \tag{1}$$

where n denotes the degree of the polynomial; and $a_0, a_1, \ldots, a_n$ are constants coefficients. Since there are $(n+1)$ coefficients, so $(n+1)$ data points are required to obtain unique value for the coefficients. The important property of polynomials that makes them suitable for **approximating functions** is due to the following theorem called, the *Weierstrass approximation theorem*.

Important Points

I. In this chapter we look for the approximate solution of a function at the given arbitrary point.

II. We shall use polynomial interpolation (approximation of a function at a point $x \in [a, b]$).

III. Higher the degree of interpolating polynomial better the approximate solution.

IV. We shall use interpolating polynomial at equally and unequally spaced data points.

V. Lagrange and Newton's polynomials may be used to find approximation of a function.

VI. Newton's polynomial needs a table of divided differences of a function.

VII. Piecewise linear interpolation can be used to obtain approximate solution of a function.

Theorem 1

(Weierstrass Approximation Theorem)

If $f(x)$ *is a* **continuous function** *in the closed interval* $[a, b]$ *then for every* $\epsilon > 0$ *there exists a polynomial* $p_n(x)$*, where the value of* n *depends on the value of* ϵ*, such that for all* x *in* $[a, b]$*,*

$$|f(x) - p_n(x)| < \epsilon. \tag{2}$$

Consequently, any continuous function can be approximated to any accuracy by a polynomial of high enough degree. •

Polynomial Interpolation

Suppose we have given a set of $(n+1)$ data points relating a dependent variables $f(x)$ to an independent variable x as follows

x	x_0	x_1	$\cdots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$\cdots$	$f(x_n)$

Generally the data points $x_0, x_1, \ldots, x_n$ are arbitrary and assume the interval between two **adjacent points** is not the same (unequally spaced) and assume that the data points are organized in such a way that
$x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n$. But some times this is not happen.

Lagrange Interpolating Polynomials

It is one of the popular and well known interpolation method to approximate the functions at an arbitrary point x. The **Lagrange interpolation** method provides a direct approach for determining interpolated values regardless of the data points spacing, that is, it can be fitted to unequally spaced or equally spaced data. To discuss about the Lagrange interpolation method, we start with a simplest form of interpolation, that is, *linear interpolation*. The interpolated value is obtained from the equation of straight line that passes through two tabulated values, one each side of required value. This straight line is a first-degree polynomial. The problem of determining a polynomial of degree one that passes through the distinct points (x_0, y_0) and (x_1, y_1) is the same as approximating the function $f(x)$ for which $f(x_0) = y_0$ and $f(x_1) = y_1$ by means of first degree polynomial interpolation.

Let us consider the construction of a **linear polynomial** $p_1(x)$ passing through two data points $(x_0, f(x_0))$ and $(x_1, f(x_1))$, see Figure 1. Consider a *linear polynomial* of the form

$$f(x) = p_1(x) = a_0 + a_1 x. \tag{3}$$

Since a polynomial of degree one has two coefficients, so one might expect to be able to choose two conditions, which satisfy

$$p_1(x_k) = f(x_k); \qquad k = 0, 1.$$

When $p_1(x)$ passes through point $(x_0, f(x_0))$, we have

$$p_1(x_0) = a_0 + a_1 x_0 = y_0 = f(x_0),$$

and if it passes through point $(x_1, f(x_1))$, we have

$$p_1(x_1) = a_0 + a_1 x_1 = y_1 = f(x_1).$$

Solving last two equations, gives a unique solution

$$a_0 = \frac{x_0 y_1 - x_1 y_0}{x_0 - x_1} \quad \text{and} \quad a_1 = \frac{y_1 - y_0}{x_1 - x_0}. \tag{4}$$

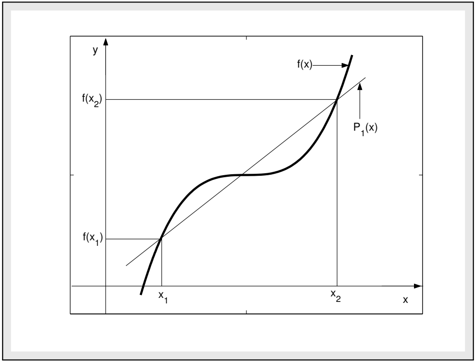

Figure: Linear Lagrange interpolation.

Putting these values in (3), we have

$$f(x) = p_1(x) = \left(\frac{x - x_1}{x_0 - x_1} \right) y_0 + \left(\frac{x - x_0}{x_1 - x_0} \right) y_1,$$

Which can also be written as

$$f(x) = p_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1), \tag{5}$$

where

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}. \tag{6}$$

Note that when $x = x_0$, then $L_0(x_0) = 1$ and $L_1(x_0) = 0$. Similarly, when $x = x_1$, then $L_0(x_1) = 0$ and $L_1(x_1) = 1$. The polynomial (5) is known as *linear Lagrange interpolating polynomial* and (6) is called the *Lagrange coefficient polynomials*.

Theorem 2

Lagrange interpolating polynomials

If $x_0, x_1, \cdots, x_n$ *are* $(n+1)$ *distinct numbers and* f *is a function whose values are given at these numbers, then there exists a unique polynomial* P *of degree at most* n *with the property that*

$$f(x_k) = p_n(x_k)$$

for each $k = 0, 1, \cdots, n$.
This polynomial, called the n**th Lagrange interpolating polynomial**, *is given by*

$$f(x) = p_n(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + \cdots + L_n(x)f(x_n) = \sum_{i=0}^{n} \prod_{k=0}^{n} \left(\frac{x - x_k}{x_i - x_k} \right) f(x_i), \tag{7}$$

where $i \neq k$.

Thus, the **Quadratic Lagrange Interpolating Polynomial** can be given by

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2), \tag{8}$$

where the Lagrange coefficients are define as follows:

$$\begin{aligned} L_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \\ L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}, \\ L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}. \end{aligned} \tag{9}$$

Also, the **Cubic Lagrange Interpolating Polynomial** is given by

$$f(x) = p_3(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3), \tag{10}$$

where the Lagrange coefficients are define as follows:

$$\begin{aligned} L_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}, \\ L_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}, \\ L_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}, \\ L_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}. \end{aligned} \tag{11}$$

Example 0.1

Let $f(x) = x + \frac{1}{x}$, with points $x_0 = 1, x_1 = 1.5, x_2 = 2.5$ and $x_3 = 3$. Find the quadratic Lagrange polynomial for the approximation of $f(2.7)$. Also, find the relative error.

Solution. Consider the quadratic Lagrange interpolating polynomial as follows:

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2).$$

Since the given interpolating point is $x = 2.7$, therefore, the best three points for the quadratic polynomial should be as follows:

$$x_0 = 1.5, \quad x_1 = 2.5, \quad x_2 = 3,$$

and the function values at these points are

$$f(x_0) = 2.1667, \quad f(x_1) = 2.9, \quad f(x_2) = 3.3333.$$

So using these values, we have

$$f(x) = p_2(x) = 2.1667L_0(x) + 2.9L_1(x) + 3.3333L_2(x),$$

where

$$\begin{aligned}
L_0(x) &= \frac{(x-2.5)(x-3)}{(1.5-2.5)(1.5-3)} = \frac{1}{1.5}(x^2 - 5.5x + 7.5),\\
L_1(x) &= \frac{(x-1.5)(x-3)}{(2.5-1.5)(2.5-3)} = \frac{1}{-0.5}(x^2 - 4.5x + 4.5),\\
L_2(x) &= \frac{(x-1.5)(x-2.5)}{(3-1.5)(3-2.5)} = \frac{1}{0.75}(x^2 - 4x + 3.75).
\end{aligned}$$

Using these Lagrange coefficients in the polynomial and after simplifying, gives

$$f(x) = p_2(x) = 0.0889x^2 + 0.3776x + 1.4003,$$

which is the required quadratic polynomial. At $x = 2.7$, we have

$$f(2.7) \approx p_2(2.7) = 3.0679.$$

The relative error is

$$\frac{|f(2.7) - p_2(2.7)|}{|f(2.7)|} = \frac{|3.0704 - 3.0679|}{|3.0704|} = 0.0008.$$

•

Example 0.2

(a) Construct the table for $(\alpha, M(\alpha))$ by evaluating the integral

$$M(\alpha) = \int_0^1 (\alpha - e^x)\ dx,$$

at $\alpha = 1, 3, 5, 6$.
(b) Use the constructed table to find the best approximation of $M(4)$ by using quadratic Lagrange polynomial. Compute the absolute error.

Solution. *(a)* Since

$$M(\alpha) = \int_0^1 (\alpha - e^x)\ dx = (\alpha x - e^x)\Big|_0^1 = \alpha - e + 1,$$

so by using the given values of α, we get

$$M(1) = -0.7183, \quad M(3) = 1.2817, \quad M(5) = 3.2817, \quad M(6) = 4.2817.$$

Thus we have the following table

α	1.00	3.00	5.00	6.00
$M(\alpha)$	-0.7183	1.2817	3.2817	4.2817

(b) Since a quadratic polynomial can be determined so that it passes through the three points, let us consider the best form of the constructed table for the quadratic Lagrange interpolating polynomial to approximate $M(4)$ as

α	3.00	5.00	6.00
$M(\alpha)$	1.2817	3.2817	4.2817

So using the quadratic Lagrange interpolating polynomial

$$M(\alpha) = p_2(\alpha) = L_0(\alpha)f(\alpha_0) + L_1(\alpha)f(\alpha_1) + L_2(\alpha)f(\alpha_2), \tag{12}$$

to get the approximation of $M(4)$, we have

$$M(4) \approx p_2(4) = 1.2817L_0(4) + 3.2817L_1(4) + 4.2817L_2(4). \tag{13}$$

The **Lagrange coefficients** can be calculate as follows:

$$\begin{aligned} L_0(4) &= \frac{(4-5)(4-6)}{(3-5)(3-6)} = \frac{1}{3}, \\ L_1(4) &= \frac{(4-3)(4-6)}{(5-3)(5-6)} = 1, \\ L_2(4) &= \frac{(4-3)(4-5)}{(6-3)(6-5)} = -\frac{1}{3}. \end{aligned}$$

Putting these values of the Lagrange coefficients in (13), we obtain

$$M(4) \approx p_2(4) = \frac{1}{3}(1.2817) + 1(3.2817) - \frac{1}{3}(4.2817) = 2.2817,$$

which is the required approximation of $M(4)$ by the quadratic interpolating polynomial.
From the given integral we can obtained the exact value as follows

$$M(4) = \int_0^1 (4 - e^x)\, dx = (4x - e^x)\Big|_0^1 = 5 - e = 2.2817,$$

so

$$|M(4) - p_2(4)| = |2.2817 - 2.2817| = 0.0000,$$

is the required absolute error in our approximation. •

Example 0.3

The equation $x - 9^{-x} = 0$ has a solution in $[0, 1]$. Compute the Lagrange polynomial on $x_0 = 0, x_1 = 0.5$ and $x_2 = 1$. By setting the interpolating polynomial equal to zero and solving the equation, find an approximate solution to the equation in the given interval $[0, 1]$.

Solution. Let us consider the form of the constructed table for the given function $f(x) = x - 9^{-x}$ at the given points as

x	0	0.5	1
$f(x)$	-1	$1/6$	$8/9$

So using the quadratic Lagrange interpolating polynomial

$$f(x) = p_2(x) = L_0(x)f(x_0)+L_1(x)f(x_1)+L_2(x)f(x_2) = -L_0(x)+\frac{1}{6}L_1(x)+\frac{8}{9}L_2(x), \tag{14}$$

where the values of the Lagrange coefficients can be calculate as follows:

$$\begin{aligned}
L_0(x) &= \frac{(x-0.5)(x-1)}{(0-0.5)(0-1)} = 2x^2 - 3x + 1, \\
L_1(x) &= \frac{(x-0)(x-1)}{(0.5-0)(0.5-1)} = -4x^2 + 4x, \\
L_2(x) &= \frac{(x-0)(x-0.5)}{(1-0)(1-0.5)} = 2x^2 - x.
\end{aligned}$$

Putting these values of the Lagrange coefficients in (14), we have

$$f(x) = p_2(x) = \frac{1}{18}(-16x^2 + 50x - 18),$$

which is the required quadratic interpolating polynomial. Now setting this polynomial equal to zero, we get

$$0 = p_2(x) = \frac{1}{18}(-16x^2 + 50x - 18),$$

which gives

$$-16x^2 + 50x - 18 = 0, \quad \text{or} \quad 8x^2 - 25x + 9 = 0.$$

Now solving this quadratic equation, one can get

$$x_1 = 2.70985 \qquad \text{and} \qquad x_2 = 0.41515.$$

Thus the approximate solution to the given equation in $[0, 1]$ is $x_2 = 0.41515$. •

Error Formula of Lagrange Polynomial

As with any numerical technique, it is important to obtain bounds for the errors involved. Now we discuss the error term when the Lagrange polynomial is used to approximate continuous function $f(x)$. It is not possible, in general, to say how accurately the interpolating polynomial p_n approximates given function f. All can be said with certainty is that $f(x) - p_n(x) = 0$ at $x = x_0, x_1, \ldots, x_n$. However, it is sometimes possible to obtain a bound on the error $f(x) - p_n(x)$ at an intermediate point x using the following theorem.

Theorem 3

(Error Formula of nth Degree Lagrange Polynomial)
If $f(x)$ has $(n+1)$ derivatives on interval I and if it is approximated by a polynomial $p_n(x)$ passing through $(n+1)$ data points on I, then the error E_n is given by

$$E_n = f(x) - p_n(x) = \frac{f^{(n+1)}(\eta(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n), \quad \eta(x) \in I, \tag{15}$$

where $p_n(x)$ is Lagrange interpolating polynomial (7) and a unknown point $\eta(x) \in (x_0, x_n)$. •

Also, Error Formula of Lagrange Polynomial can be given by

$$|f(x) - p_n(x)| \leq \frac{W(x)}{(n+1)!} M,$$

where $M = \max_{x_0 \leq x \leq x_n} |f^{(n+1)}(x)|$ and $W(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$

Error Formulas of Linear, Quadratic and Cubic Lagrange Polynomials
If $f(x)$ has second, third and fourth derivatives on interval I and if it is approximated by the polynomials $p_1(x), p_2(x), p_3(x)$ passing respectively, through $2, 3, 4$ data points on I, then the errors E_1, E_2, E_3 are given by

$$E_1 = f(x) - p_1(x) = \frac{f''(\eta(x))}{2!}(x - x_0)(x - x_1), \quad \eta(x) \in I, \tag{16}$$

where $p_1(x)$ is the linear Lagrange polynomial (5) and a unknown point $\eta(x) \in (x_0, x_1)$.

$$E_2 = f(x) - p_2(x) = \frac{f'''(\eta(x))}{3!}(x - x_0)(x - x_1)(x - x_2), \quad \eta(x) \in I, \tag{17}$$

where $p_2(x)$ is the quadratic Lagrange polynomial (8) and a unknown point $\eta(x) \in (x_0, x_2)$.

$$E_3 = f(x) - p_3(x) = \frac{f^{(4)}(\eta(x))}{4!}(x - x_0)(x - x_1)(x - x_2)(x - x_3), \quad \eta(x) \in I, \tag{18}$$

where $p_3(x)$ is the cubic Lagrange polynomial (10) and a unknown point $\eta(x) \in (x_0, x_3)$.

Example 0.4

Show that a bound for the error in the linear interpolation is

$$|f(x) - p_1(x)| \leq \frac{h^2}{8}M, \quad \text{where} \quad M = \max_{x_0 \leq x \leq x_1} |f''(x)| \quad \text{and} \quad h = x_1 - x_0. \quad (19)$$

Solution. Consider two points x_0 and x_1, then the linear polynomial $p_1(x)$ interpolating $f(x)$ at these points is

$$f(x) = p_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1).$$

By using the given data point, the error formula (15) becomes

$$f(x) - p_1(x) = \frac{(x - x_0)(x - x_1)}{2!} f''(\eta(x)),$$

where $\eta(x)$ is a unknown point between x_0 and x_1. Hence

$$|f(x) - p_1(x)| = \left| \frac{(x - x_0)(x - x_1)}{2!} \right| |f''(\eta(x))|.$$

The value of $f''(\eta(x))$ can not be computed exactly because $\eta(x)$ is not known. But we can bound the error by computing the largest possible value for $|f''(\eta(x))|$. So bound $|f''(x)|$ on $[x_0, x_1]$ can be obtain

$$M = \max_{x_0 \leq x \leq x_1} |f''(x)|,$$

and so for $|f''(\eta(x))| \leq M$, we have

$$|f(x) - p_1(x)| \leq \frac{M}{2} |(x - x_0)(x - x_1)|.$$

Since the maximum of function $g(x) = (x - x_0)(x - x_1)$ in $[x_0, x_1]$ occurs at the critical point $x = \dfrac{(x_0 + x_1)}{2}$ $(g'(x) = 0)$ and so that maximum is
$|(x - x_0)(x - x_1)| = \dfrac{(x_1 - x_0)^2}{4}$.
This follows easily by noting that the function $(x - x_0)(x - x_1)$ is a quadratic and has two roots x_0 and x_1, hence its maximum value occurs midway between these roots. Thus, for any $x \in [x_0, x_1]$, we have

$$|f(x) - p_1(x)| \leq \frac{(x_1 - x_0)^2}{8} M, \quad \text{or} \quad |f(x) - p_1(x)| \leq \frac{h^2}{8} M,$$

where $h = x_1 - x_0$. •

Example 0.5

Find the linear Lagrange polynomial that passes through the points $(0, f(0))$ and $(\pi, f(\pi))$ and then use it to approximate the function $f(x) = 2\cos x$ at $\frac{\pi}{2}$. Find absolute error and an bound for the error in the linear interpolation of $f(x)$.

Solution. Given two points $x_0 = 0$ and $x_1 = \pi$, then the linear Lagrange polynomial $p_1(x)$

$$f(x) = p_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1),$$

interpolating $f(x)$ at these points is

$$f(x) = p_1(x) = \frac{(x - \pi)}{(0 - \pi)} f(0) + \frac{(x - 0)}{(\pi - 0)} f(\pi).$$

By using the function values at the given data point, we get

$$f(x) = p_1(x) = \frac{(x - \pi)}{(0 - \pi)}(2) + \frac{(x - 0)}{(\pi - 0)}(-2) = 2 - \frac{4x}{\pi} \quad \text{and} \quad f(\pi/2) \approx p_1(\pi/2) = 0.$$

Thus absolute error, $|2\cos(\pi/2) - p_1(\pi/2)| = 0$. Since $M = \max_{0 \le x \le \pi} |f''(x)| = \max_{0 \le x \le \pi} |-2\cos x| = 2$ and $h = \pi$, so by using the linear Lagrange error formula (19), we get

$$|f(x) - p_1(x)| \le \frac{(\pi - 0)^2}{4} = \frac{\pi^2}{4},$$

which is the required bound of error in the linear interpolation of $f(x)$.

Example 0.6

Consider $f(x) = \sin x$ and its values are known at five points $\{0, 0.2, 0.4, 0.6, 0.8\}$. If the approximation of $\sin 0.28$ by four degree Lagrange interpolating polynomial is 0.2763591, then compute the error bound and the absolute error for the approximation.

Solution. To compute an error bound for the approximation of the given function in the interval $[0, 0.8]$, we use the following error formula for Lagrange polynomial degree four

$$|f(x) - p_4(x)| = \frac{|f^{(5)}(\eta(x))|}{5!}|(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)|,$$

or

$$|f(x) - p_4(x)| \leq \frac{M}{5!}|(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)|.$$

Since

$$|f^{(5)}(\eta(x))| \leq M = \max_{0 \leq x \leq 0.8} |f^{(5)}(x)| = \max_{0 \leq x \leq 0.8} |\cos x| = 1,$$

so

$$|f(0.28) - p_4(0.28)| \leq \frac{1}{120}|0.28(0.28-0.2)(0.28-0.4)(0.28-0.6)(0.28-0.6)(0.28-0.8)|$$

$$\leq 3.7 \times 10^{-6},$$

which is desired error bound. Also, we have to compute absolute error as

$$|f(0.28) - p_4(0.28)| = |\sin 0.28 - p_4(0.28)| = |0.2763556 - 0.2763591| = 3.5 \times 10^{-6},$$

which is desired result.

Example 0.7

Let $p_2(x)$ be the Lagrange polynomial which interpolating $f(x) = x^3 + x + 1$ at the points $x_i = \alpha + (i+1)h, \ i = 0, 1, 2$, where α is constant and $h > 0$. Find h such that the error at $x = \alpha$ is bounded above by 10^{-3}.

Solution. To compute an error bound for the approximation of the given function, we use the following error formula for the quadratic Lagrange polynomial as

$$|E| = |f(x) - p_2(x)| = \frac{|f^{(3)}(\eta(x))|}{3!}|(x - x_0)(x - x_1)(x - x_2)|,$$

where

$$x = \alpha, \quad x_0 = (\alpha + h), \quad x_1 = (\alpha + 2h), \quad x_2 = (\alpha + 3h).$$

Since

$$|f^{(3)}(\eta(x))| \leq M = \max_{x_0 \leq x \leq x_2} |f^{(3)}(x)| = \max_{x_0 \leq x \leq x_2} |6| = 6,$$

so

$$|E| \leq \frac{6}{6}|(\alpha - (\alpha + h))(\alpha - (\alpha + 2h))(\alpha - (\alpha + 3h))| = 6h^3.$$

Given

$$|E| < 10^{-3}, \qquad \text{so} \qquad 6h^3 < 10^{-3},$$

from this we have $h < 0.055$ and so we can take $h = 0.05$. •

Example 0.8

Consider the following table having the data for $f(x) = e^{3x} + \cos 2x$:

x	0.1	0.2	0.4	0.5
$f(x)$	2.3300	2.7432	4.0168	5.0220

Find the approximation of $f(0.45)$ using the best quadratic Lagrange interpolation formula and also estimate an **error bound** and absolute error for the approximation.

Solution. Using the data points $0.2, 0.4, 0.5$, the best Lagrange formula to find the interpolating polynomial to approximate the function is the quadratic polynomial

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2),$$

which implies that

$$\begin{aligned} f(x) = p_2(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) \\ &+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2), \end{aligned}$$

or

$$\begin{aligned} f(x) &= p_2(x) = 45.72[x^2 - 0.9x + 0.2] - 200.84[x^2 - 0.7x + 0.1] \\ &+ 167.4[x^2 - 0.6x + 0.08]. \end{aligned}$$

Thus

$$f(x) = p_2(x) = 12.28x^2 - 1.0x^2 + 2.452. \tag{20}$$

Take $x = 0.45$ in the above polynomial (20), we have

$$f(0.45) \approx p_2(0.45) = 4.4887.$$

The exact value of $f(0.45) = 4.4790$, so, the absolute error is 0.0097. Now to compute an error bound of the approximation, we use the following formula

$$|f(x) - p_2(x)| = \frac{|f^{(3)}(\eta(x))|}{3!} |(x - x_0)(x - x_1)(x - x_2)|. \tag{21}$$

Taking the third derivative of the given function, we get

$$f'(x) = 3e^{3x} - 2\sin 2x, \qquad f''(x) = 9e^{3x} - 4\cos 2x,$$

$$f'''(x) = 27e^{3x} + 8\sin 2x.$$

Thus

$$|f^{(3)}(\eta(x))| = |27e^{3\eta(x)} + 8\sin 2(\eta(x))|, \qquad \text{for} \qquad \eta(x) \in (0.2, 0.5),$$

and it gives

$$|f^{(3)}(0.2)| = 52.3126 \qquad \text{and} \qquad |f^{(3)}(0.5)| = 127.7374.$$

The value of $f^{(3)}(\eta(x))$ can not be computed exactly because $\eta(x)$ is not known. But we can bound the error by computing the largest possible value for $|f^{(3)}(\eta(x))|$. So bound $|f^{(3)}(x)|$ on $[0.2, 0.5]$ can be obtain

$$M = \max_{0.2 \leq x \leq 0.5} |f^{(3)}(x)| = 127.7374,$$

and so for $|f^{(4)}(\eta(x))| \leq M$, we have (21) as follows

$$|f(x) - p_2(x)| \leq (127.7374)(0.000625)/6 = 0.0133,$$

which is the required error bound for the approximation. •

Theorem 4

(Error Bounds for Lagrange Interpolation at *Equally Spaced Points***)**
Assume that $f(x)$ *is defined on the interval* $[a, b]$, *which contains equally spaced points* $x_k = x_0 + hk$. *Additionally, assume that* $f(x)$ *and the derivatives of* $f(x)$ *up to the order* $(n+1)$, *are continuous and bounded on the special intervals* $[x_0, x_1]$, $[x_0, x_2]$ *and* $[x_0, x_3]$, *respectively; that is*

$$|f^{(n+1)}(x)| \leq M \quad \textit{for} \quad x_0 \leq x \leq x_n,$$

for $n = 1, 2, 3$. *Then error bounds for linear, quadratic and cubic polynomials are:*

$$\begin{aligned} |E_1(x)| &\leq \frac{h^2}{8}M && \textit{for} && x_0 \leq x \leq x_1, \\ |E_2(x)| &\leq \frac{h^3}{9\sqrt{3}}M && \textit{for} && x_0 \leq x \leq x_2, \\ |E_3(x)| &\leq \frac{h^4}{24}M && \textit{for} && x_0 \leq x \leq x_3. \end{aligned}$$

Continue in the similar manner for the interval $[x_0, x_n]$, *for* $n = 1, 2, \ldots, n$, *we have*

$$|E_n(x)| \leq \frac{M}{4(n+1)}\left(\frac{b-a}{n}\right)^{n+1} = \frac{M}{4(n+1)}h^{n+1}, \quad \textit{for} \quad x_0 \leq x \leq x_n, \qquad (22)$$

the general error bound formula. It has been proved that $\displaystyle\prod_{i=0}^{n}|x - x_i| \leq \frac{h^{n+1}n!}{4}$. •

Example 0.9

Find an error bound if $f(x) = \sin x$ is approximated by an interpolation polynomial with ten equally spaced data points in $[0, 1.6875]$.

Solution. Given $n = 9$ and $a = 0,\ b = 1.6875$, then

$$M = \max_{0 \leq x \leq 1.6875} |f^{(10)}(x)| = \max_{0 \leq x \leq 1.6875} |-\sin x| \leq 1, \quad \forall\ x \in [0, 1.6875].$$

Hence, the interpolation error (using Theorem 22) can be bounded by

$$|E_9(x)| = |\sin x - p_9(x)| \leq \frac{1}{40} \left(\frac{1.6875}{9} \right)^{10} \approx 1.34 \times 10^{-9},$$

for all $x \in [0, 1.6875]$. •

Example 0.10

Find the cubic Lagrange interpolating polynomial to find the approximation of $f(x)$ if $f(1) = 0.5, f(4/3) = 1, f(5/3) = 2$ and $f(2) = 3$. If $|f^{(4)}| \leq \frac{1}{5}$ for $1 \leq x \leq 2$, then show that the error estimate is $|f(x) - p_3(x)| \leq \frac{1}{27}$.

Solution. The given number of points are, $x_0 = 1, x_1 = 4/3, x_2 = 5/3, x_3 = 2$, therefore the cubic Lagrange interpolating polynomial for the approximation of the given function is:

$$f(x) = p_3(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3),$$

and using the given function values, gives

$$f(x) = p_3(x) = L_0(x)(0.5) + L_1(x)(1) + L_2(x)(2) + L_3(x)(3).$$

The **Lagrange coefficients** can be calculate as follows:

$$\begin{aligned}
L_0(1.5) &= \frac{(x-4/3)(x-5/3)(x-2)}{(1-4/3)(1-5/3)(1-2)} = \frac{(x^3 - 5x^2 + 74/9x - 40/9)}{-2/9}, \\
L_1(1.5) &= \frac{(x-1)(x-5/3)(x-2)}{(4/3-1)(4/3-5/3)(4/3-2)} = \frac{(x^3 - 14/3x^2 + 7x - 10/3)}{2/27}, \\
L_2(1.5) &= \frac{(x-1)(x-4/3)(x-2)}{(5/3-1)(5/3-4/3)(5/3-2)} = \frac{(x^3 - 13/3x^2 + 6x - 8/3)}{-2/27}, \\
L_3(1.5) &= \frac{(x-1)(x-4/3)(x-5/3)}{(2-1)(2-4/3)(2-5/3)} = \frac{(x^3 - 4x^2 + 47/9x - 20/9)}{2/9}.
\end{aligned}$$

Putting these values of the Lagrange coefficients in the above equation, we get

$$f(x) = p_3(x) = \frac{1}{4}(-9x^3 + 45x^2 - 62x + 28),$$

which is the required cubic interpolating polynomial for the approximation of the function.
Since we know the error of cubic Lagrange polynomial is

$$f(x) - p_3(x) = \frac{f^{(4)}(\eta(x))}{4!}(x - x_0)(x - x_1)(x - x_2)(x - x_3),$$

so

$$|f(x) - p_3(x)| = \frac{|f^{(4)}(\eta(x))|}{4!}|(x - x_0)||(x - x_1)||(x - x_2)||(x - x_3)|,$$

and $|f^{(4)}| \leq \frac{1}{5}$ for $1 \leq x \leq 2$, we obtain

$$|f(x) - p_3(x)| \leq \frac{1}{120}|(x - x_0)||(x - x_1)||(x - x_2)||(x - x_3)|.$$

Now for $1 \leq x \leq 2$, we deduce that

$$|x - 1| \leq 1, \quad |x - 4/3| \leq 2/3, \quad |x - 5/3| \leq 2/3, \quad |x - 2| \leq 1.$$

Hence, the possible error in the cubic Lagrange polynomial is

$$|f(x) - p_3(x)| \leq \frac{1}{120}(1)(2/3)(2/3)(1) = 0.0037.$$

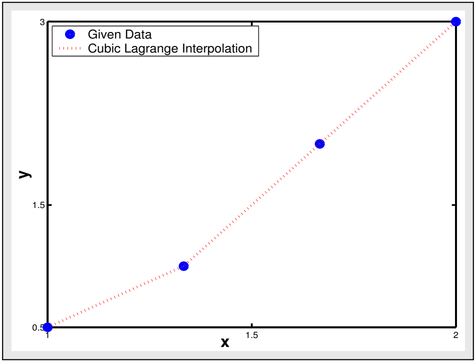

Figure: Graphical solution of the Example 0.10.

Summary

In this lecture, we ...

- Introduced the concept of interpolation
- Introduced Lagrange polynomials