Arc-length, Unit Speed and Reparametrisation Math 473 Introduction to Differential Geometry Lecture 3

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Defnation (1):
Let $\alpha : [a, b] \mapsto \mathbb{R}^3$ be a parametrised regular curve. We define the **Length of** α as

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$$S(t) = \int_{t_0}^{t} |\alpha'(u)|du \qquad (t \in I) \ i.e. \quad S : I \mapsto \mathbb{R}.$$

Example(1): Find the Length of the curve $\alpha : \mathbb{R} \mapsto \mathbb{R}^3, \alpha(t) = (\cos t, \sin t, 6)$ from $a = 0$ to $b = 5$. Then, find the arc-length where $t_0 = 0$?

Example(2): Show that for the curve $\alpha(t) = \frac{1}{2}(t, \frac{1}{t}, \sqrt{2}\ln t)$, where $t \in (0, \infty)$, arc-length measured from $t = 1$ to $t = t_0$ is

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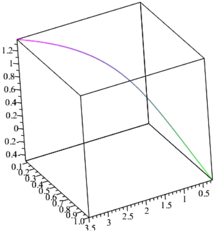

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Example(3) Show that the curve $\alpha(t) = (\cos t, \sin t, 2)$ is a unit speed curve.

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Unit speed curves are often easier to work with. But does every curve have a unit speed parametrisation?

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The curve α is a unit speed curve if and only if α is parametrised by its arc-length.

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Reparametrisation

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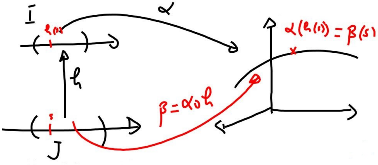
The parametrised curve $\beta = \alpha \circ h : J \mapsto \mathbb{R}^3$ is called a **reparametrisation** of the curve α.



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Proof

In the next lecture.

Thanks for listening.