Numerical Methods

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Aims

In this lecture, we will . . .

- Introduce the Fixed-Point Method
- Introduce the Newton's Method

Fixed-Point Method

The basic idea of this method which is also called successive approximation method or function iteration, is to rearrange the original equation

$$f(x) = 0, \tag{1}$$

into an equivalent expression of the form

$$x = g(x). \tag{2}$$

Any solution of (2) is called a fixed-point for the iteration function $g(x)$ and hence a root of (1).

Definition 1

(Fixed-Point of a Function)

A *fixed-point* of a function $g(x)$ is a real number α such that $\alpha = g(\alpha)$.

For example, $x = 2$ is a fixed-point of the function $g(x) = \dfrac{x^2 - 4x + 8}{2}$ because $g(2) = 2$. •

The fixed-point method essentially solves two functions simultaneously; $y = x$ and $y = g(x)$. The point of intersection of these two functions is the solution to $x = g(x)$, and thus to $f(x) = 0$, see Figure 1.

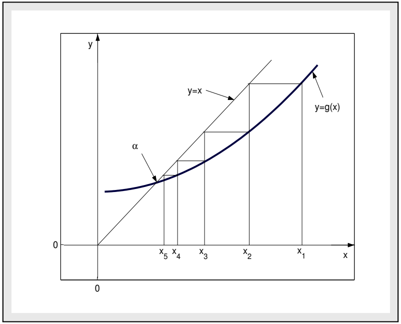

Figure: Graphical Solution of Fixed-Point Method.

Definition 2

(Fixed-Point Method)
The iteration defined in the following

$$x_{n+1} = g(x_n); \qquad n = 0, 1, 2, \ldots, \tag{3}$$

is called the *fixed-point method* or the *fixed-point iteration*. •

The value of the initial approximation x_0 is chosen arbitrarily and the hope is that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a number α which will automatically satisfies (1). Moreover, since (1) is a rearrangement of (2), α is guaranteed to be a zero of $f(x)$. In general, there are many different ways of rearranging of (2) in (1) form. However, only some of these are likely to give rise to successful iterations but sometime we don't have successful iterations. To describe such behaviour, we discuss the following example.

Example 0.1

Consider the nonlinear equation $x^3 = 2x + 1$ which has a root in the interval $[1.5, 2.0]$ using fixed-point method with $x_0 = 1.5$, take three different rearrangements for the equation.

Solution. Let us consider the three possible rearrangement of the given equation as follows:

$$
\begin{array}{llllll}
(i) & x_{n+1} & = & g_1(x_n) & = & \dfrac{(x_n^3 - 1)}{2}; \qquad n = 0, 1, 2, \ldots, \\[2ex]
(ii) & x_{n+1} & = & g_2(x_n) & = & \dfrac{1}{(x_n^2 - 2)}; \qquad n = 0, 1, 2, \ldots, \\[2ex]
(iii) & x_{n+1} & = & g_3(x_n) & = & \sqrt{\dfrac{(2x_n + 1)}{x_n}}; \qquad n = 0, 1, 2, \ldots,
\end{array}
$$

then the numerical results for the corresponding iterations, starting with the initial approximation $x_0 = 1.5$ with accuracy 5×10^{-2}, are given in Table 1.

Table: Solution of $x^3 = 2x + 1$ by fixed-point method

n	x_n	$x_{n+1} = g_1(x_n)$ $= (x_n^3 - 1)/2$	$x_{n+1} = g_2(x_n)$ $= 1/(x_n^2 - 2)$	$x_{n+1} = g_3(x_n)$ $= \sqrt{(2x_n + 1)/x_n}$
00	x_0	1.500000	1.500000	1.500000
01	x_1	1.187500	4.000000	1.632993
02	x_2	0.337280	0.071429	1.616284
03	x_3	-0.480816	-0.501279	1.618001
04	x_4	-0.555579	-0.571847	1.618037
05	x_5	-0.585745	-0.597731	1.618034

We note that the first two considered sequences diverge and the last one converges. This example asks the need for a mathematical analysis of the method. The following theorem gives sufficient conditions for the convergence of the fixed-point iteration. •

Theorem 3

(Fixed-Point Theorem)

If g is continuously differentiable on the interval $[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then

(a) *g has at-least one fixed-point in the given interval $[a, b]$.*

Moreover, if the derivative $g'(x)$ of the function $g(x)$ exists on an interval $[a, b]$ which contains the starting value x_0, with

$$k \equiv \max_{a \le x \le b} |g'(x)| < 1; \qquad \textit{for all} \qquad x \in [a, b]. \tag{4}$$

Then

(b) *The sequence (3) will converge to the attractive (unique) fixed-point α in $[a, b]$.*

(c) *The iteration (3) will converge to α for any initial approximation.*

(d) *We have the error estimate*

$$|\alpha - x_n| \le \frac{k^n}{1-k}|x_1 - x_0|, \qquad \textit{for all} \quad n \ge 1. \tag{5}$$

(e) *The limit holds:*

$$\lim_{n \to \infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha). \tag{6}$$

Now we come back to our previous Example 0.1 and discuss that why the first two rearrangements we considered, do not converge but on the other hand, last sequence has a fixed-point and converges.
Since, we observe that $f(1.5)f(2) < 0$, then the solution we seek is in the interval $[1.5, 2]$.

(i) For $g_1(x) = \dfrac{x^3 - 1}{3}$, we have $g_1'(x) = x^2$, which is greater than unity throughout the interval $[1.5, 2]$. So by Fixed-Point Theorem 3 this iteration will fail to converge.

(ii) For $g_2(x) = \dfrac{1}{x^2 - 2}$, we have $g_2'(x) = \dfrac{-2x}{(x^2 - 2)^2}$, and $|g_2'(1.5)| > 1$, so from Fixed-Point Theorem 3 this iteration will fail to converge.

(iii) For $g_3(x) = \sqrt{\dfrac{2x + 1}{x}}$, we have $g_3'(x) = x^{-3/2}/2\sqrt{2x + 1} < 1$, for all x in the given interval $[1.5, 2]$. Also, g_3 is decreasing function of x, and $g_3(1.5) = 1.63299$ and $g_3(2) = 1.58114$ both lie in the interval $[1.5, 2]$. Thus $g_3(x) \in [1.5, 2]$, for all $x \in [1.5, 2]$, so from Fixed-Point Theorem 3 the iteration will converge, see Figure 2. •

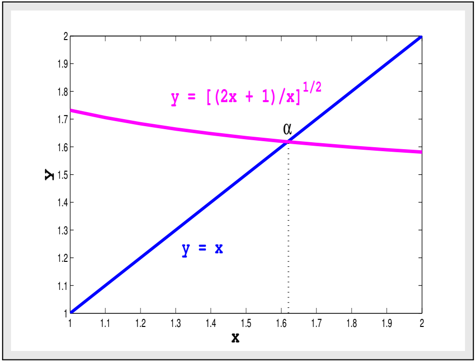

Figure: Graphical Solution of $x = \sqrt{(2x+1)/x}$.

Note1

From (5) Note that the rate of convergence of the fixed-point method depends on the factor $\dfrac{k^n}{(1-k)}$; the smaller the value of k, then faster the convergence. The convergence may be very slow if the value of k is very close to 1. •

Note2

Assume that $g(x)$ and $g'(x)$ are continuous functions of x for some open interval I, with the fixed-point α contained in this interval. Moreover assume that

$$|g'(\alpha)| < 1, \qquad \text{for} \quad \alpha \in I,$$

then, there exists an interval $[a, b]$, around the solution α for which all the conditions of Theorem 3 are satisfied. But if

$$|g'(\alpha)| > 1, \qquad \text{for} \quad \alpha \in I,$$

then the sequence (3) will not converge to α. In this case α is called a *repulsive fixed-point*. If

$$|g'(\alpha)| = 0, \qquad \text{for} \quad \alpha \in I,$$

then the sequence (3) converges very fast to the root α while if

$$|g'(\alpha)| = 1, \qquad \text{for} \quad \alpha \in I,$$

then the convergence the sequence (3) is not guaranteed and if the convergence happened, it would be very slow. Thus to get the faster convergence, the value of $|g'(\alpha)|$ should be equal to zero or very close to zero. •

Example 0.2

Find an interval $[a, b]$ on which fixed-point problem $x = \dfrac{2 - e^x + x^2}{3}$ will converges. Estimate the number of iterations n within accuracy 10^{-5}.

Solution. Since $x = \dfrac{2 - e^x + x^2}{3}$ can be written as

$$f(x) = e^x - x^2 + 3x - 2 = 0,$$

and we observe that $f(0)f(1) = (-1)(e^1) < 0$, then the solution we seek is in the interval $[0, 1]$.

For $g(x) = \dfrac{2 - e^x + x^2}{3}$, we have $g'(x) = \dfrac{2x - e^x}{3} < 1$, for all x in the given interval $[0, 1]$. Also, g is decreasing function of x and $g(0) = 0.3333$ and $g(1) = \dfrac{3 - e}{3} = 0.0939$ both lie in the interval $[0, 1]$. Thus $g(x) \in [0, 1]$, for all $x \in [0, 1]$, so from Fixed-Point Theorem 3 the g(x) has a unique fixed-point in $[0, 1]$. Taking $x_0 = 0.5$, we have

$$x_1 = g(x_0) = \frac{2 - e^{x_0} + x_0^2}{3} = 0.2004.$$

Also, we have

$$k_1 = |g'(0)| = 0.3333 \quad \text{and} \quad k_2 = |g'(1)| = 0.2394,$$

which give $k = \max\{k_1, k_2\} = 0.3333$. Thus the error estimate (5) within the accuracy 10^{-5} is

$$|\alpha - x_n| \leq 10^{-5}, \qquad \text{gives} \qquad \frac{(0.3333)^n}{1 - 0.3333}(0.2996) \leq 10^{-5},$$

and by solving this inequality, we obtain $n \geq 9.7507$. So we need ten approximations to get the desired accuracy for the given problem. •

Example 0.3

Show that the function $g(x) = 3^{-x}$ on the interval $[0, 1]$ has at least one fixed-point but it is not unique.

Solution. Given $x = g(x) = 3^{-x}$, and it can be written as

$$x - 3^{-x} = f(x) = 0.$$

So $f(0)(1) = (-1)(2/3) < 0$, so $f(x)$ has a root in the interval $[0, 1]$, see Figure 3. Note that g is decreasing function of x and $g(0) = 1$ and $g(1) = 0.3333$ both lie in the interval $[0, 1]$. Thus $g(x) \in [0, 1]$, for all $x \in [0, 1]$, so from Fixed-Point Theorem 3 the function $g(x)$ has at least one fixed-point in $[0, 1]$. Since the derivative of the function $g(x)$ is

$$g'(x) = -3^{-x} \ln 3,$$

which is less than zero on $[0, 1]$, therefore, the function g is decreasing on $[0, 1]$. But $g'(0) = -\ln 3 = -1.0986$, so

$$|g'(x)| > 1 \quad \text{on} \qquad (0, 1).$$

Thus from Fixed-Point Theorem 3 the function $g(x)$ has no unique fixed-point in $[0, 1]$. ●

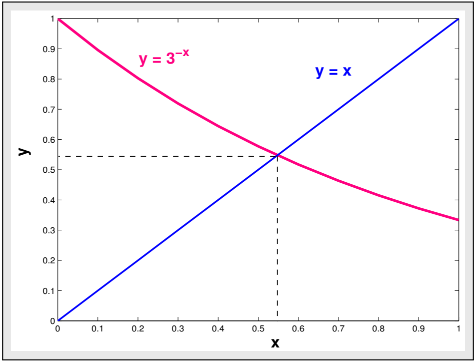

Figure: Graphical Solution of $x = 3^{-x}$.

Example 0.4

Show that the function $g(x) = \sqrt{2x-1}$ on the interval $[0,1]$ that satisfies none of the hypothesis of Theorem 3 but still has a unique fixed-point on $[0,1]$.
Solution. Since $x = g(x) = \sqrt{2x-1}$, it gives

$$x^2 - 2x + 1 = (x-1)^2 = f(x) = 0.$$

Then $x = \alpha = 1 \in [0,1]$ is the root of the nonlinear equation $f(x) = 0$ and the fixed-point of the function $g(x)$ as $g(1) = 1$. But notice that the function $g(x)$ is not continuous on the interval $[0,1]$ and the derivative of the function $g(x)$

$$g'(x) = \frac{1}{\sqrt{2x-1}},$$

does not exist on the interval $(0,1)$. So all the conditions of Fixed-Point Theorem 3 fail. •

Example 0.5

Show that the fixed point form of the equation $x = N^{1/3}$ can be written as $x = Nx^{-2}$ and the associated iterative scheme

$$x_{n+1} = Nx_n^{-2}, \qquad n \geq 0,$$

will not successful (diverge) in finding the approximation of cubic root of the positive number N.

Solution. Given $x = N^{1/3}$ and it can be written as

$$x^3 - N = 0 \quad \text{or} \quad x = \frac{N}{x^2} = Nx^{-2}.$$

It gives the iterative scheme

$$x_{n+1} = Nx_n^{-2} = g(x_n), \qquad n \geq 0.$$

From this, we have

$$g(x) = Nx^{-2} \qquad \text{and} \qquad g'(x) = -2Nx^{-3}.$$

Since $\alpha = x = N^{1/3}$, therefore

$$g'(\alpha) = -2N\alpha^{-3} \quad \text{and} \quad g'(N^{1/3}) = -2N(N^{1/3})^{-3} = -2NN^{-1} = -2.$$

Thus

$$|g'(N^{1/3})| = |-2| = 2 > 1,$$

which shows the divergence.

Example 0.6

One of the possible rearrangement of the nonlinear equation $e^x = x + 2$, which has root in $[1, 2]$ is

$$x_{n+1} = g(x_n) = \ln(x_n + 2); \qquad n = 0, 1, \ldots.$$

(a) Show that $g(x)$ has a unique fixed-point in $[1, 2]$.

(b) Use fixed-point iteration formula (3) to compute approximation x_3, using $x_0 = 1.5$.

(c) Compute an error estimate $|\alpha - x_3|$ for your approximation.

(d) Determine the number of iterations needed to achieve an approximation with accuracy 10^{-2} to the solution of $g(x) = \ln(x + 2)$ lying in the interval $[1, 2]$ by using the fixed-point iteration method.

Solution. Since, we observe that $f(1)f(2) < 0$, then the solution we seek is in the interval $[1, 2]$.

(a) For $g(x) = \ln(x+2)$, we have $g'(x) = 1/(x+2) < 1$, for all x in the given interval $[1,2]$. Also, g is increasing function of x, and $g(1) = \ln(3) = 1.0986123$ and $g(2) = \ln(4) = 1.3862944$ both lie in the interval $[1,2]$. Thus $g(x) \in [1,2]$, for all $x \in [1,2]$, so from fixed-point theorem the g(x) has a unique fixed-point, see Figure 4.

(b) using the given initial approximation $x_0 = 1.5$, we have the other approximations as

$$x_1 = g(x_0) = 1.252763, \quad x_2 = g(x_1) = 1.179505, \quad x_3 = g(x_2) = 1.156725.$$

(c) Since $a = 1$ and $b = 2$, then the value of k can be found as follows

$$k_1 = |g'(1)| = |1/3| = 0.333 \quad \text{and} \quad k_2 = |g'(2)| = |1/4| = 0.25,$$

which give $k = \max\{k_1, k_2\} = 0.333$. Thus using the error formula (5), we have

$$|\alpha - x_3| \leq \frac{(0.333)^3}{1 - 0.333}|1.252763 - 1.5| = 0.013687.$$

(d) From the error bound formula (5), we have

$$\frac{k^n}{1-k}|x_1 - x_0| \leq 10^{-2}.$$

By using above parts *(b)* and *(c)*, we have

$$\frac{(0.333)^n}{1 - 0.333}|1.252763 - 1.5| \leq 10^{-2}.$$

Solving this inequality, we obtain

$$n \ln(0.333) \leq \ln(0.02698), \quad \text{gives}, \quad n \geq 3.28539.$$

So we need four approximations to get the desired accuracy for the given problem.

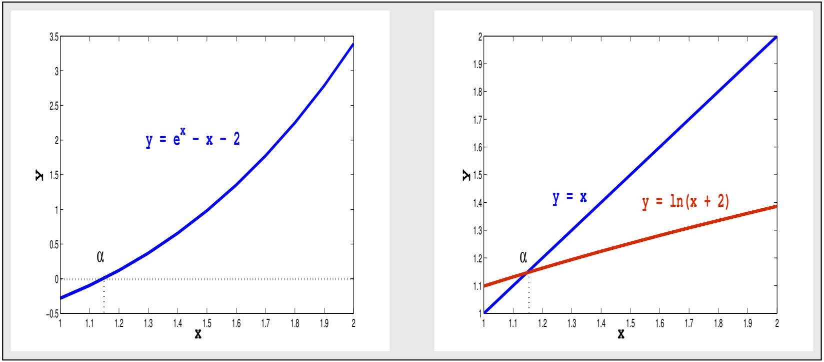

Figure: Graphical Solution of $e^x = x + 2$ Graphical solution of $x = \ln(x+2)$.

MATLAB command for the above given rearrangement $x = g(x)$ of $f(x) = x^3 - 2x - 1$ by using the initial approximation $x_0 = 1.5$, can be written as follows:

$function\ y = fn(x)$
$y = log(x+2);$
$>> x0 = 1.5; tol = 0.01;\ sol = fixpt('fn', x0, tol);$

Program 2.2

MATLAB m-file for the Fixed-Point Method

function sol=fixpt(fn,x0,tol)
old= x0+1; while abs(x0-old) $> tol$; old=x0;
$x0 = feval(fn, old)$; end; sol=$x0$;

Procedure
(Fixed-Point Method))

1. Choose an initial approximation x_0 such that $x_0 \in [a, b]$.
2. Choose a convergence parameter $\epsilon > 0$.
3. Compute new approximation x_{new} by using the iterative formula (3).
4. Check, if $|x_{new} - x_0| < \epsilon$ then x_{new} is the desire approximate root; otherwise set $x_0 = x_{new}$ and go to step *3*.

Newton's Method

This is one of the most popular and powerful iterative method for finding roots of the nonlinear equation. It is also known as the method of tangents because after estimated the actual root, the zero of the tangent to the function at that point is determined. The Newton's method consists geometrically of expanding the tangent line at a current point x_i until it crosses zero, then setting the next guess x_{i+1} to the abscissa of that zero crossing, see Figure 5. This method is also called the *Newton-Raphson method*.

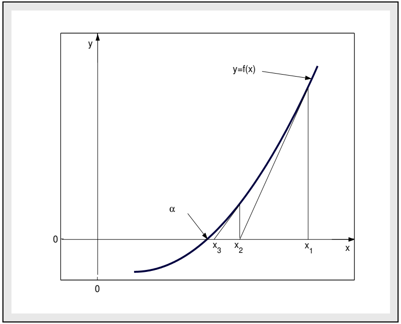

Figure: Graphical Solution of Newton's Method.

There are many description of the Newton's method. We shall derive the method from the familiar Taylor's series expansion of a function in the neighborhood of a point. Let $f \in C^2[a,b]$ and let x_n be the *nth* approximation to the root α such that $f'(x_n) \neq 0$ and $|\alpha - x_n|$ is small. Consider the first Taylor polynomial for $f(x)$ expanded about x_n, so we have

$$f(x) = f(x_n) + (x - x_n)f'(x_n) + \frac{(x - x_n)^2}{2} f''(\eta(x)), \tag{7}$$

where $\eta(x)$ lies between x and x_n. Since $f(\alpha) = 0$, then (7), with $x = \alpha$, gives

$$f(\alpha) = 0 = f(x_n) + (\alpha - x_n)f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(\eta(\alpha)).$$

Since $|\alpha - x_n|$ is small, then we neglect the term involving $(\alpha - x_n)^2$ and so

$$0 \approx f(x_n) + (\alpha - x_n)f'(x_n).$$

Solving for α, we get

$$\alpha \approx x_n - \frac{f(x_n)}{f'(x_n)}, \tag{8}$$

which should be better approximation to α than is x_n. We call this approximation as x_{n+1}, then we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \qquad \text{for all} \quad n \geq 0. \tag{9}$$

The iterative method (9) is called the Newton's method.

Example 0.7

Use Newton's method to find the approximation x_3 to the root of

$$\cos x - x = 0,$$

where $x_0 = \pi/4$. **Solution.**
Let $f(x) = \cos x - x = 0$, and use Using the Newton's iterative formula (9), we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

to find the iterations, where $x_0 = \pi/4$. Thus we get:

Table: Solution of $\cos x - x = 0$ by Newton's method

n	x_n	$f(x_n)$
1	0.7853981635	-0.078291381
2	0.7395361337	-0.000754873
3	0.7390851781	-0.000000074

Therefore, $x_3 = 0.7390851781$.

Example 0.8

Use the Newton's method to find the root of $x^3 = 2x + 1$ that is located in the interval $[1.5, 2.0]$ accurate to 10^{-2}, take an initial approximation $x_0 = 1.5$.

Solution. Given $f(x) = x^3 - 2x - 1$ and so $f'(x) = 3x^2 - 2$. Now evaluating $f(x)$ and $f'(x)$ at the give approximation $x_0 = 1.5$, gives

$$x_0 = 1.5, \qquad f(1.5) = -0.625, \qquad f'(1.5) = 4.750.$$

Using the Newton's iterative formula (9), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{(-0.625)}{4.75} = 1.631579.$$

Now evaluating $f(x)$ and $f'(x)$ at the new approximation x_1, gives

$$x_1 = 1.631579, \qquad f(1.631579) = 0.0801869, \qquad f'(1.631579) = 5.9861501.$$

Using the iterative formula (9) again to get other new approximation. The successive iterates were shown in the Table 3.

Table: Solution of $x^3 = 2x + 1$ by Newton's method

n	x_n	$f(x_n)$	$f'(x_n)$	Error $= x - x_n$
00	1.500000	-0.625000	4.750000	0.1180339
01	1.631579	0.0801869	5.9861501	-0.0135451
02	1.618184	0.000878	5.855558	-0.0001501
03	1.618034	0.00000007	5.854102	-0.0000001

Just after the third iterations the required root is approximated to be $x_3 = 1.618034$ and the functional value is reduced to 7.0×10^{-8}. Since the exact solution is 1.6180339, so the actual error is 1×10^{-7}. We see that the convergence is quite faster than the methods considered previously. $\bullet$

To get the above results using MATLAB command, firstly the function $x^3 - 2x - 1$ and its derivative $3x^2 - 2$ were saved in m-files called fn.m and dfn.m, respectively written as follows:

```
function y = fn(x)                function dy = dfn(x)
y = x.^3 - 2*x - 1;               dy = 3*x.^2 - 2;
```

after which we do the following:

```
>> x0 = 1.5; tol = 0.01;  sol = newton('fn','dfn',x0,tol);
```

Example 0.9

If the difference of two numbers x and y is 6 and the square root of their product is 4, then use Newton's method to approximate, to within 10^{-4}, the largest value of the number x and the corresponding number y using initial approximation $x_0 = 7.5$.

Solution. Given

$$x - y = 6 \quad \text{and} \quad \sqrt{xy} = 4.$$

Solving the above equations for x, we have

$$x(x-6) = 16 \quad \text{or} \quad x^2 - 6x - 16 = f(x) = 0.$$

Applying Newton's iterative formula (9) to find the approximation of this equation, we have

$$x_{n+1} = x_n - \frac{x_n^2 - 6x_n - 16}{2x_n - 6}.$$

Finding the approximation to within 10^{-4} using the initial approximation $x_0 = 7.5$, we get

$$x_1 = x_0 - \frac{x_0^2 - 6x_0 - 16}{2x_0 - 6} = 8.0278,$$

and continue in the same manner, we get the approximations within accuracy 10^{-4} as follows

$$x_2 = 8.0001, \quad x_3 = 8.0000, \quad x_4 = 8.0000.$$

Thus the largest value of number x is 8 and its corresponding y value is 2. •

Example 0.10

The graphs of $y = 2\sin x$ and $y = \ln(x) + k$ touch each other in the neighborhood of point $x = 8$. Find the value of the constant k and the coordinates of point of contact, use $x_0 = 8$.

Solution. Since we know that the graphs will touch each other if the values of derivatives at their point of contact is same. So for

$$y = 2\sin x, \quad \text{gives}, \quad y' = 2\cos x,$$

and

$$y = \ln(x) + k, \quad \text{gives}, \quad y' = \frac{1}{x}.$$

Thus

$$2\cos x = \frac{1}{x}, \quad \text{gives}, \quad x\cos x - 0.5 = 0,$$

and from this we have the function and its derivative as follows

$$f(x) = x\cos x - 0.5 \quad \text{and} \quad f'(x) = \cos x - x\sin x.$$

Using Newton's iterative formula (9), we get

$$x_{n+1} = x_n - \frac{x_n \cos x_n - 0.5}{\cos x_n - x_n \sin x_n},$$

and for finding the approximations, starting $x_0 = 8$, we obtain, $x_1 = 7.7936$ and $x_2 = 7.7897$. Taking $x = 7.79$, we have $y = 2\sin 7.79 = 1.996$. Therefore, the point of contact is $(7.79, 1.996)$. To find the value of k, we solve the equation, $1.996 = \ln(7.79) + k$, and it gives, $k = -0.0568$, the required value of k.

Example 0.11

Develop the iterative formula

$$x_{n+1} = \frac{x_n^2 - b}{2x_n - a}, \qquad n \geq 0,$$

for the approximate roots of the quadratic equation $x^2 - ax + b = 0$ using the Newton's method. Then use the formula to find the third approximation of the positive root of the equation $x^2 - 3x = 4$, starting with $x_0 = 3.5$.

Solution. Given

$$f(x) = x^2 - ax + b,$$

therefore, we have (see Figure 6),

$$f(x_n) = x_n^2 - ax_n + b \quad \text{and} \quad f'(x_n) = 2x_n - a.$$

Using these functions values in the Newton's iterative formula (9), we have

$$x_{n+1} = x_n - \frac{x_n^2 - ax_n + b}{2x_n - a} = \frac{x_n^2 - b}{2x_n - a}, \qquad n \geq 0.$$

Finding the first three approximations of the positive root of $x^2 - 3x = 4$ using the initial approximation $x_0 = 3.5$ and $a = 3, b = -4$, we use the above formula by taking $n = 0, 1, 2$ as follows

$$x_1 = \frac{x_0^2 - b}{2x_0 - a} = 4.0625, \quad x_2 = \frac{x_1^2 - b}{2x_1 - a} = 4.0008, \quad x_3 = \frac{x_2^2 - b}{2x_2 - a} = 4.0000,$$

are the possible three approximations. Note that the positive root of $x^2 - 3x - 4 = 0$ is 4, so we have

$$|4 - x_3| = |4 - 4| = 0.0000,$$

the possible absolute error. •

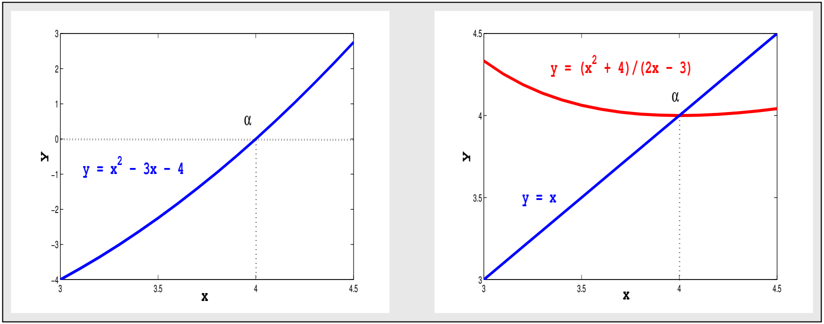

Figure: Graphical Solution of $x^2 - 3x = 4$ and $x = (x^2 + 4)/(2x - 3)$.

Example 0.12

Develop an iterative procedure for evaluating the reciprocal of a positive number N by using Newton's method. Use the developed formula to find third approximation to the reciprocal of 3, taking an initial approximation $x_0 = 0.4$. Compute absolute error.

Solution. Consider $x = 1/N$. This problem can be easily solved by noting that we seek to find a root to the nonlinear equation

$$1/x - N = 0,$$

where $N > 0$ is the number whose reciprocal is to be found. Therefore, if $f(x) = 0$, then $x = 1/N$ is the exact root. Let

$$f(x) = 1/x - N \quad \text{and} \quad f'(x) = -1/x^2.$$

Hence, assuming an initial estimate to the root, say, $x = x_0$ and by using iterative formula (9), we get

$$x_1 = x_0 - \frac{(1/x_0 - N)}{(-1/x_0^2)} = x_0 + (1/x_0 - N)x_0^2 = x_0 + x_0 - Nx_0^2 = x_0(2 - Nx_0).$$

In general, we have

$$x_{n+1} = x_n(2 - Nx_n), \qquad n = 0, 1, \ldots, \tag{10}$$

We have to find the approximation of the reciprocal of number $N = 3$. Given the initial gauss of say $x_0 = 0.4$, then by using the iterative formula (10), we get

$$x_1 = 0.3200, \qquad x_2 = 0.3328, \qquad x_3 = 0.3333.$$

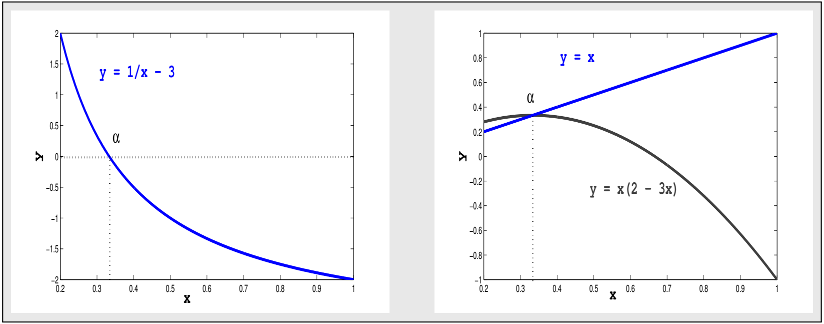

Figure: Graphical Solution of $1/x = 3$ and $x = x(2 - 3x)$.

After just three iterations the estimated value compares rather favorably with the exact value of $1/3 \approx 0.3333$, (see Figure 7). Thus the absolute error is

$$|E| = \left| \frac{1}{3} - x_3 \right| = |0.3333 - 0.3333| = 0.0000.$$

We can calculate the other reciprocal of the number in the same way by using the general iterative formula (10).

Procedure

(Newton's Method)

1. Find the initial approximation x_0 for the root by sketching the graph of the function.
2. Evaluate function $f(x)$ and the derivative $f'(x)$ at initial approximation. Check: if $f(x_0) = 0$ then x_0 is the desire approximation to a root. But if $f'(x_0) = 0$, then go back to step *1* to choose new approximation.
3. Establish Tolerance ($\epsilon > 0$) value for the function.
4. Compute new approximation for the root by using the iterative formula (9).
5. Check Tolerance. If $|f(x_n)| \leq \epsilon$, for $n \geq 0$, then end; otherwise, go back to step *4*, and repeat the process.

Summary

In this lecture, we ...

- Introduced the Fixed-Point Method
- Introduced the Newton's Method