

Numerical Methods

King Saud University

Aims

In this lecture, we will . . .

- ▶ Considering how to numerically find roots of algebraic equations
- ▶ Introduce the Bisection method

Nonlinear Equations

We will consider any one of the following types:

1. A polynomial equation of degree n :

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0, \quad a_n \neq 0, \quad n > 1,$$

where a_n, a_{n-1}, \dots, a_1 and a_0 are constants. For example, the following equations are nonlinear.

$$x^2 + 5x + 6 = 0; \quad x^3 = 2x + 1; \quad x^{200} - 2x + 1 = 0.$$

2. The power of the unknown variable (not a positive integer number). For example, the following non-polynomial equations are nonlinear

$$x^{-1} + 2x = 1; \quad \sqrt{x} + x = 2; \quad x^{2/3} + \frac{2}{x} + 4 = 0.$$

3. The equation which involves the trigonometric functions, exponential functions and logarithmic functions. For example, all the following transcendental equations are nonlinear

$$x = \cos(x); \quad e^x + x - 10 = 0; \quad x + \ln x = 10.$$

Roots of nonlinear equations

One of the most basic problems that numerical techniques are used for is finding roots of nonlinear equations. This process involves finding a root, or solution, of an equation of the form

$$f(x) = 0. \tag{1}$$

A root of this equation is also called a zero of the function f . Here, we are going to look at some common numerical methods for finding roots of equations.

Important Points

I. Nonlinear equations may have many roots but we will seek the approximation of only one of its real root α lies in the given interval $[a, b]$, that is

$$f(\alpha) = 0, \quad \text{where } \alpha \in [a, b].$$

II. If $f(x)$ is continuous function in in a interval $[a, b]$ and $f(x)$ has opposite signs at the end points of the interval, then there must be a root of nonlinear equation $f(x) = 0$ in $[a, b]$.

III. Root of a nonlinear equation may be **simple** (not repeating) or **multiple** (repeating). Simple root means

$$f(\alpha) = 0 \quad \text{but} \quad f'(\alpha) \neq 0.$$

For example, $\alpha_1 = -3$ and $\alpha_2 = -2$ are the simple roots of the nonlinear equation $x^2 + 5x + 6 = 0$. For the multiple root, we mean

$$f(\alpha) = 0 \quad \text{but} \quad f'(\alpha) = 0.$$

For example, $\alpha_1 = -2$ and $\alpha_2 = -2$ are the multiple roots of the nonlinear equation $x^2 + 4x + 4 = 0$.

IV. We will consider iterative methods which include bisection method, fixed-point method, Newton method (also called, Newton-Raphson method) and secant method which give us the approximation of single (or simple) root of the nonlinear equation. For the multiple roots of the nonlinear equation we will use other iterative methods, called, the first modified Newton's method (also called the Schroeder's method) and the second modified Newton's method.

V. Remember that the best method for the approximation of the simple root of nonlinear equation is Newton's method (called quadratic convergent method) and for multiple root of nonlinear equation is modified Newton's method (called quadratic convergent method). Newton's method for multiple root of nonlinear root is called a linear convergent method.

The Bisection Method

The Bisection method is used to determine, to any specified accuracy that your computer will permit, a solution to $f(x) = 0$ on an interval $[a, b]$, provided:

- ▶ $f(x)$ is continuous on $[a, b]$;
- ▶ $f(a)$ and $f(b)$ are of opposite sign.

The concept of the Bisection method is simple, and is based on utilizing the Intermediate Value Theorem. Essentially, due the continuity of f on $[a, b]$, and since $f(a)f(b) < 0$, then there must be a point $a < \alpha < b$ such that $f(\alpha) = 0$. The implication is that one of the values is negative and the other is positive. These conditions can be easily satisfied by sketching the function, see Figure 1.

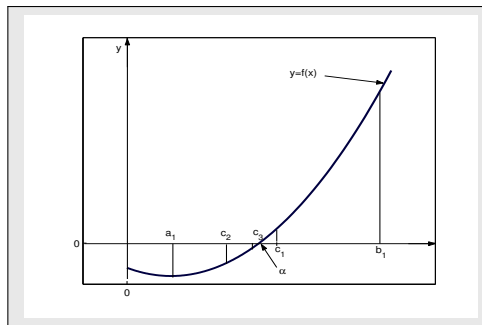


Figure: Graphical Solution of Bisection Method.

Therefore the root must lie between a and b (by Intermediate Value Theorem) and a new approximation to the root α be calculated as

$$c = \frac{a + b}{2},$$

and, in general

$$c_n = \frac{a_n + b_n}{2}, \quad n \geq 1. \quad (2)$$

The iterative formula (2) is known as the **bisection method**.

If $f(c) \approx 0$, then $c \approx \alpha$ is the desired root, and, if not, then there are two possibilities.

- ▶ Firstly, if $f(a)f(c) < 0$, then $f(x)$ has a zero between point a and point c . The process can then be repeated on the new interval $[a, c]$.
- ▶ Secondly, if $f(a)f(c) > 0$ it follows that $f(b)f(c) < 0$ since it is known that $f(b)$ and $f(c)$ have opposite signs. Hence, $f(x)$ has zero between point c and point b and the process can be repeated with $[c, b]$. We see that after one step of the process, we have found either a zero or a new bracketing interval which is precisely half the length of the original one.
- ▶ The process continue until the desired accuracy is achieved.

Example 0.1

Use the bisection method to find the approximation to the root of the equation

$$x^3 = 2x + 1,$$

that is located in the interval $[1.5, 2.0]$ accurate to within 10^{-2} .

Solution. Since the given function $f(x) = x^3 - 2x - 1$ is a polynomial function and so is continuous on $[1.5, 2.0]$, starting with $a_1 = 1.5$ and $b_1 = 2$, we compute:

$$\begin{aligned} a_1 &= 1.5 : & f(a_1) &= -0.625 \\ b_1 &= 2.0 : & f(b_1) &= 3.0, \end{aligned}$$

and since $f(1.5)f(2.0) < 0$, so that a root of $f(x) = 0$ lies in the interval $[1.5, 2.0]$. Using formula (2) (when $n = 1$), we get:

$$c_1 = \frac{a_1 + b_1}{2} = 1.75; \quad f(c_1) = 0.859375.$$

Hence the function changes sign on $[a_1, c_1] = [1.5, 1.75]$. To continue, we squeeze from right and set $a_2 = a_1$ and $b_2 = c_1$. Then the midpoint is:

$$c_2 = \frac{a_2 + b_2}{2} = 1.625; \quad f(c_2) = 0.041056.$$

Continue in this way we obtain a sequence $\{c_k\}$ of approximation shown by Table 1.

Table: Solution of $x^3 = 2x + 1$ by bisection method

n	Left Endpoint a_n	Midpoint c_n	Right Endpoint b_n	Function Value $f(c_n)$
01	1.500000	1.750000	2.000000	0.8593750
02	1.500000	1.625000	1.750000	0.0410156
03	1.500000	1.562500	1.625000	-0.3103027
04	1.562500	1.593750	1.625000	-0.1393127
05	1.593750	1.609375	1.625000	-0.0503273
06	1.609375	1.617188	1.625000	-0.0049520

We see that the functional values are approaching zero as the number of iterations is increase. We got the desired approximation to the root of the given equation is $c_6 = 1.617188 \approx \alpha$ after 6 iterations with accuracy $\epsilon = 10^{-2}$. ●

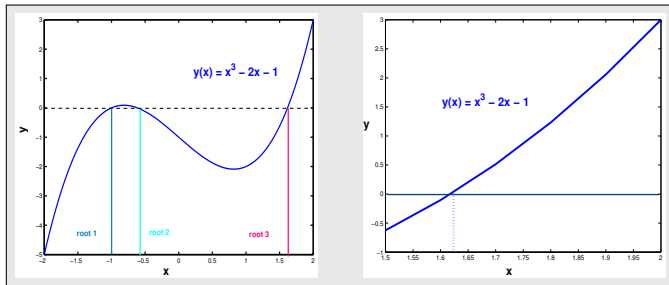


Figure: Graphical Solution of $x^3 = 2x + 1$ in the intervals $[-2, 2]$ and $[-1.5, 2]$.

Note that to use MATLAB command for the bisection method, first we define a function m-file as `fn.m` for the equation as follows:

```
function y = fn(x)
y = x.^ 3 - 2 * x - 1;
```

then use the single commands:

```
>> s = bisect('fn', 1.5, 2, 1e - 2)
```

We can easily find the roots (1.61803399, -1.00, -0.61803399) of the equation $x^3 = 2x + 1$ by defining the coefficients of the polynomial equation using MATLAB commands as:

```
>> CP = [1 0 - 2 - 1]; Sol = roots(CP);
```

Example 0.2

Find the point of intersection of the graphs $y = x^3 + 2x - 1$ and $y = \sin x$, then use bisection method within accuracy 10^{-3} .

Solution. The graphs in the Figure 3 show that there is an intersection at about point $(0.66, 0.61)$. Using the function $f(x) = x^3 + 2x - \sin x - 1$ and the starting interval $[0.5, 1.0]$, we compute:

$$\begin{aligned} a_1 &= 0.5 : & f(a_1) &= -0.3544, \\ b_1 &= 1.0 : & f(b_1) &= 1.1585. \end{aligned}$$

Since $f(x)$ is continuous on $[0.5, 1.0]$ and $f(0.5) \cdot f(1.0) < 0$, so that a root of $f(x) = 0$ lies in the interval $[0.5, 1.0]$. Using formula (2) (when $n = 1$), we get:

$$c_1 = \frac{a_1 + b_1}{2} = 0.75; \quad f(c_1) = 0.240236.$$

Hence the function changes sign on $[a_1, c_1] = [0.5, 0.75]$. To continue, we squeeze from right and set $a_2 = a_1$ and $b_2 = c_1$. Then the midpoint is:

$$c_2 = \frac{a_2 + b_2}{2} = 0.625; \quad f(c_2) = -0.090957.$$

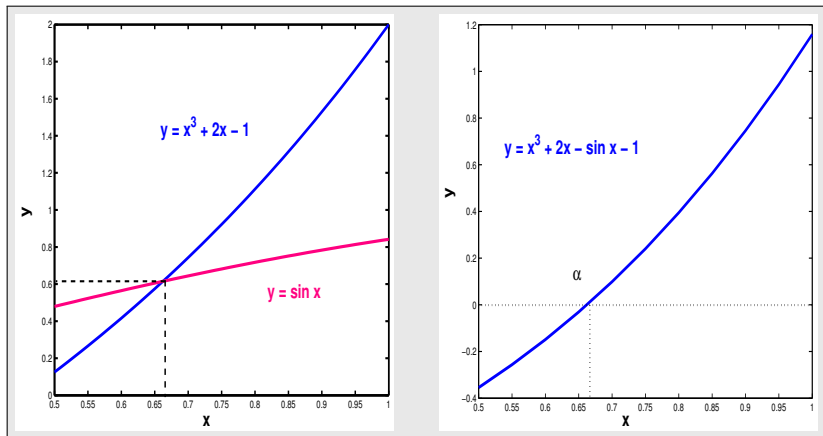


Figure: Graphical Solution of $\sin x = x^3 + 2x - 1$ and $x^3 + 2x - \sin x = 1$.

Then continue in this manner we obtain a sequence $\{c_k\}$ of approximation shown by Table 2.

Table: Solution of $x^3 + 2x - \sin x - 1$ by bisection method

n	Left Endpoint a_n	Right Endpoint b_n	Midpoint c_n	Function Value $f(c_n)$
01	0.5000	1.0000	0.750000	0.240236
02	0.5000	0.7500	0.625000	-0.090957
03	0.6250	0.7500	0.687500	0.065344
⋮	⋮	⋮	⋮	⋮
07	0.6563	0.6641	0.660156	-0.005228
08	0.6602	0.6641	0.662109	-0.000302

We see that the functional values are approaching zero as the number of iterations is increase. We got the desired approximation to the root of the given equation is $c_8 = 0.662109 \approx \alpha$ after 8 iterations with accuracy $\epsilon = 10^{-3}$. ●

Program 2.1

MATLAB m-file for the Bisection Method

```
function sol=bisect(fn,a,b,tol)
fa = feval(fn,a); fb = feval(fn,b);
if fa * fb > 0; fprintf('Endpoints have same sign') return end
while abs (b - a) > tol c = (a + b)/2; fc = feval(fn,c);
if fa * fc < 0; b = c; else a = c; end; end; sol=(a + b)/2;
```


Theorem 1

(Bisection Convergence and Error Theorem)

Let $f(x)$ be continuous function defined on the given initial interval $[a_0, b_0] = [a, b]$ and suppose that $f(a)f(b) < 0$. Then bisection method (2) generates a sequence $\{c_n\}_{n=1}^{\infty}$ approximating $\alpha \in (a, b)$ with the property

$$|\alpha - c_n| \leq \frac{b-a}{2^n}, \quad n \geq 1. \quad (3)$$

Moreover, to obtain accuracy of

$$|\alpha - c_n| \leq \epsilon,$$

(for $\epsilon = 10^{-k}$) it suffices to take

$$n \geq \frac{\ln \{10^k(b-a)\}}{\ln 2}, \quad (4)$$

where k is nonnegative integer.

Note:

The above Theorem 1 gives us information about bounds for errors in approximation and the number of bisections needed to obtain any given accuracy.

Example 0.3

Show that number of iterations of bisection will require to attain an accuracy of 10^{-4} using the starting interval $[a, b]$ is

$$n \geq \frac{\ln(b-a) + 4 \ln 5}{\ln 2} + 4.$$

Determine the number of iterations needed to achieve the an approximation with same above given accuracy to the solution of $x^3 - 2x - 1 = 0$ lying in the interval $[1.5, 2]$.

Solution. By using the inequality (4), we get

$$n \geq \frac{\ln \{10^k(b-a)\}}{\ln 2} = \frac{\ln(b-a) + \ln[(5)(2)]^4}{\ln 2},$$

or

$$n \geq \frac{\ln(b-a) + 4[\ln 5 + \ln 2]}{\ln 2} = \frac{\ln(b-a) + 4 \ln 5}{\ln 2} + 4.$$

Now by taking $a = 1.5$ and $b = 2$ in the above inequality, we get

$$n \geq \frac{\ln(2-1.5) + 4 \ln 5}{\ln 2} + 4 = 8.2877 + 4 = 12.2877.$$

So no more than thirteen iterations are required to obtain an approximation accurate to within the given accuracy 10^{-4} . ●

Example 0.4

Find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-1} to the solution of $xe^x = 1$ lying in the interval $[0.5, 1]$ using the bisection method. Find an approximation to the root with this degree of accuracy.

Solution. Here $a = 0.5$, $b = 1$ and $k = 1$, then by using inequality (4), we get

$$n \geq \frac{\ln[10^1(1 - 0.5)]}{\ln 2} \approx 2.3219.$$

So no more than three iterations are required to obtain an approximation accurate to within 10^{-1} .

The given function $f(x) = xe^x - 1$ is continuous on $[0.5, 1.0]$, so starting with $a_1 = 0.5$ and $b_1 = 1$, we compute:

$$\begin{array}{ll} a_1 & = 0.5 : & f(a_1) = -0.1756, \\ b_1 & = 1 : & f(b_1) = 1.7183, \end{array}$$

since $f(0.5)f(1) < 0$, so that a root of $f(x) = 0$ lies in the interval $[0.5, 1]$. Using formula (2) (when $n = 1$), we get:

$$c_1 = \frac{a_1 + b_1}{2} = 0.75; \quad f(c_1) = 0.5878.$$

Hence the function changes sign on $[a_1, c_1] = [0.5, 0.75]$. To continue, we squeeze from right and set $a_2 = a_1$ and $b_2 = c_1$. Then the bisection formula gives

$$c_2 = \frac{a_2 + b_2}{2} = 0.625; \quad f(c_2) = 0.1677.$$

Finally, we have in the similar manner as

$$c_3 = \frac{a_3 + b_3}{2} = 0.5625,$$

the value of the third approximation which is accurate to within 10^{-1} . •

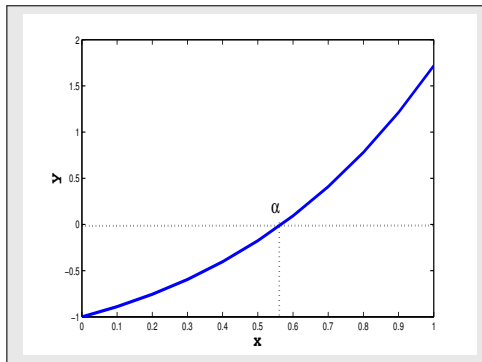


Figure: Graphical Solution of $xe^x = 1$.

Example 0.5

Use the bisection method to compute the first three approximate values for $\sqrt[4]{18}$. Also, compute an error bound and absolute error for your approximation.

Solution. Consider

$$x = \sqrt[4]{18} = (18)^{1/4}, \quad \text{or} \quad x^4 - 18 = 0.$$

Choose the interval $[2, 2.5]$ on which the function $f(x) = x^4 - 18$ is continuous and the function $f(x)$ satisfies the sign property, that is

$$f(2)f(2.5) = (-2)(21.0625) = -42.125 < 0.$$

Hence root $\alpha = \sqrt[4]{18} = 2.0598 \in [2, 2.5]$ and we compute its first approximate value by using formula (2) (when $n = 1$) as follows:

$$c_1 = \frac{2.0 + 2.5}{2} = 2.2500 \quad \text{and} \quad f(2.25) = 7.6289.$$

Since the function $f(x)$ changes sign on $[2.0, 2.25]$. To continue, we squeeze from right and use formula (2) again to get the following second approximate value of the root α as:

$$c_2 = \frac{2.0 + 2.25}{2} = 2.1250 \quad \text{and} \quad f(2.1250) = 2.3909.$$

Then continue in the similar way, the third approximate value of the root α is $c_3 = 2.0625$ with $f(2.0625) = 0.0957$.

Note that the value of the function at each new approximate value is decreasing which shows that the approximate values are coming closer to the root α . Now to compute the error bound for the approximation we use the formula (3) and get

$$|\alpha - c_3| \leq \frac{2.5 - 2.0}{2^3} = 0.0625,$$

which is the possible maximum error in our approximation and

$$|E| = |2.0598 - 2.0625| = 0.0027,$$

be the absolute error in the approximation. ●

Procedure

(Bisection Method)

1. Establish an interval $a \leq x \leq b$ such that $f(a)$ and $f(b)$ are of opposite sign, that is, $f(a) \cdot f(b) < 0$.
2. Choose an error tolerance ($\epsilon > 0$) value for the function.
3. Compute a new approximation for the root:
$$c_n = \frac{(a_n + b_n)}{2}; \quad n = 1, 2, 3, \dots$$
4. Check tolerance. If $|f(c_n)| \leq \epsilon$, use c_n , $n \geq 1$ for desired root; otherwise continue.
5. Check, if $f(a_n)f(c_n) < 0$, then set $b_n = c_n$; otherwise set $a_n = c_n$.
6. Go back to step 3, and repeat the process.

Summary

In this lecture, we ...

- ▶ Considered how to numerically find roots of algebraic equations
- ▶ Introduced the Bisection method.