THE INVERSE TRIGONOMETRIC FUNCTIONS

Definitions :

1. The inverse sine function is denoted by $\sin^{-1}$ and it is defined as $y = \sin^{-1} x \Leftrightarrow x = \sin y$, where $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

   The domain of the inverse sine function is $[-1, 1]$

   The range of the inverse sine function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

2. The inverse cosine function is denoted by $\cos^{-1}$ and it is defined as $y = \cos^{-1} x \Leftrightarrow x = \cos y$, where $x \in [-1, 1]$ and $y \in [0, \pi]$.

   The domain of the inverse cosine function is $[-1, 1]$

   The range of the inverse cosine function is $[0, \pi]$.

3. The inverse tangent function is denoted by $\tan^{-1}$ and it is defined as $y = \tan^{-1} x \Leftrightarrow x = \tan y$, where $x \in \mathbb{R}$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

   The domain of the inverse tangent function is $\mathbb{R}$

   The range of the inverse tangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

4. The inverse cotangent function is denoted by $\cot^{-1}$ and it is defined as $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$, where $x \in \mathbb{R}$.

   The domain of the inverse cotangent function is $\mathbb{R}$

   The range of the inverse cotangent function is $(0, \pi)$.

5. The inverse secant function is denoted by $\sec^{-1}$ and it is defined as $y = \sec^{-1} x \Leftrightarrow x = \sec y$, where $y \in \left[0, \frac{\pi}{2}\right)$ if $x \geq 1$,and $y \in \left[\pi, \frac{3\pi}{2}\right)$ if $x \leq -1$.

   The domain of the inverse secant function is $(-\infty, -1] \cup [1, \infty)$

   The range of the inverse secant function is $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.

6. The inverse cosecant function is denoted by $\csc^{-1}$ and it is defined as $\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$ where $|x| \geq 1$

   The domain of the inverse cosecant function is $(-\infty, -1] \cup [1, \infty)$

   The range of the inverse cosecant function is $\left(-\pi, -\frac{\pi}{2}\right] \cup \left(0, \frac{\pi}{2}\right]$.

Graph of $\sin^{-1} x$

Graph of $\cos^{-1} x$

Graph of $\tan^{-1} x$

Derivatives of the inverse trigonometric functions :

1. $\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$, where $|x| < 1$.
2. $\dfrac{d}{dx}\cos^{-1} x = \dfrac{-1}{\sqrt{1-x^2}}$, where $|x| < 1$.
3. $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$.
4. $\dfrac{d}{dx}\cot^{-1} x = \dfrac{-1}{1+x^2}$.
5. $\dfrac{d}{dx}\sec^{-1} x = \dfrac{1}{x\sqrt{x^2-1}}$, where $|x| > 1$.

6. $\dfrac{d}{dx}\csc^{-1} x = \dfrac{-1}{x\sqrt{x^2-1}}$, where $|x| > 1$.

Integration :

1. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c \quad , \quad (|x| < a)$

   $\displaystyle\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + c \quad , \quad (|f(x)| < a))$

2. $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

   $\displaystyle\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right) + c$

3. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c \quad , \quad (|x| > a)$

   $\displaystyle\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{f(x)}{a}\right) + c \quad , \quad (|f(x)| > a))$

Examples :

1. $\displaystyle\int \frac{x^2}{5+x^6}\,dx = \frac{1}{3}\int \frac{3x^2}{\left(\sqrt{5}\right)^2+(x^3)^2}\,dx = \frac{1}{3}\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x^3}{\sqrt{5}}\right) + c\ .$

   Here $a = \sqrt{5}$, $f(x) = x^3$ and $f'(x) = 3x^2$.

2. $\displaystyle\int \frac{3x}{\sqrt{9-x^4}}\,dx = \frac{3}{2}\int \frac{2x}{\sqrt{(3)^2-(x^2)^2}}\,dx = \frac{3}{2}\sin^{-1}\left(\frac{x^2}{3}\right) + c\ .$

   Here $a = 3$, $f(x) = x^2$ and $f'(x) = 2x$.

3. $\displaystyle\int \frac{3x}{\sqrt{9-x^2}}\,dx = \frac{3}{-2}\int \left(9-x^2\right)^{-\frac{1}{2}}(-2x)\,dx = -\frac{3}{2}\frac{\left(9-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c.$

4. $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx = \int \frac{\left(\frac{1}{x}\right)}{\sqrt{(1)^2-(\ln x)^2}}\,dx = \sin^{-1}(\ln x) + c\ .$

   Here $a = 1$, $f(x) = \ln x$ and $f'(x) = \dfrac{1}{x}$.

5. $\displaystyle\int \frac{1}{1+3x^2}\,dx = \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}}{(1)^2+\left(\sqrt{3}x\right)^2}\,dx = \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}x\right) + c\ .$

   Here $a = 1$, $f(x) = \sqrt{3}x$ and $f'(x) = \sqrt{3}$.

6. $\displaystyle\int \frac{e^{2x}}{e^{4x}+16}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{(4)^2+\left(e^{2x}\right)^2}\,dx = \frac{1}{2}\frac{1}{4}\tan^{-1}\left(\frac{e^{2x}}{4}\right) + c\ .$

Here $a = 4$, $f(x) = e^{2x}$ and $f'(x) = 2e^{2x}$.

7. $\displaystyle\int \frac{1}{\sqrt{e^{2x} - 36}}\,dx = \int \frac{e^x}{e^x\sqrt{(e^x)^2 - (6)^2}}\,dx = \frac{1}{6}\sec^{-1}\left(\frac{e^x}{6}\right) + c$.

Here $a = 6$, $f(x) = e^2$ and $f'(x) = e^x$.

8. $\displaystyle\int \frac{\sin x}{\sqrt{25 - \cos^2 x}}\,dx = -\int \frac{-\sin x}{\sqrt{(5)^2 - (\cos x)^2}}\,dx = -\sin^{-1}\left(\frac{\cos x}{5}\right) + c$.

Here $a = 5$, $f(x) = \cos x$ and $f'(x) = -\sin x$.

9. $\displaystyle\int \frac{2^x}{\sqrt{4 - 4^x}}\,dx = \frac{1}{\ln 2}\int \frac{2^x \ln 2}{\sqrt{(2)^2 - (2^x)^2}}\,dx = \frac{1}{\ln 2}\sin^{-1}\left(\frac{2^x}{2}\right) + c$.

Here $a = 2$, $f(x) = 2^x$ and $f'(x) = 2^x \ln 2$.

10. $\displaystyle\int \frac{1}{x^2 + 6x + 25}\,dx = \int \frac{1}{(x^2 + 6x + 9) + 16}\,dx = \int \frac{1}{(x + 3)^2 + (4)^2}\,dx$

$\displaystyle= \frac{1}{4}\tan^{-1}\left(\frac{x + 3}{4}\right) + c$.

Here $a = 4$, $f(x) = x + 3$ and $f'(x) = 1$.

11. $\displaystyle\int \frac{x + 2}{\sqrt{4 - x^2}}\,dx = \int \left(\frac{x}{\sqrt{4 - x^2}} + \frac{2}{\sqrt{4 - x^2}}\right)dx$

$\displaystyle= \frac{1}{-2}\int \left(4 - x^2\right)^{-\frac{1}{2}}(-2x)\,dx + 2\int \frac{1}{\sqrt{(2)^2 - (x)^2}}\,dx$

$\displaystyle= -\frac{1}{2}\frac{\left(4 - x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + 2\sin^{-1}\left(\frac{x}{2}\right) + c$.

12. $\displaystyle\int \frac{x + \tan^{-1} x}{1 + x^2}\,dx = \int \left(\frac{x}{1 + x^2} + \frac{\tan^{-1} x}{1 + x^2}\right)dx$

$\displaystyle= \frac{1}{2}\int \frac{2x}{1 + x^2}\,dx + \int \left(\tan^{-1} x\right)\frac{1}{1 + x^2}\,dx$

$\displaystyle= \frac{1}{2}\ln(1 + x^2) + \frac{\left(\tan^{-1} x\right)^2}{2} + c$.

Exercises : Solve the following integrals :

1. $\displaystyle\int \frac{x + \sin^{-1} x}{\sqrt{1 - x^2}}\,dx$.

2. $\displaystyle\int \frac{x + 1}{x^2 + 1}\,dx$