

THE GENERAL EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Definition (The general exponential function) :

It has the form a^x where $a > 0$ and $a \neq 1$.

Note : $a^x = e^{x \ln a}$.

Derivative of the general exponential function :

1. $\frac{d}{dx} a^x = a^x \ln a$.
2. $\frac{d}{dx} a^{f(x)} = a^{f(x)} f'(x) \ln a$.

Integration :

1. $\int a^x dx = \frac{a^x}{\ln a} + c$.
2. $\int a^{f(x)} f'(x) dx = \frac{a^{f(x)}}{\ln a} + c$.

Definition (The general logarithmic function) :

The general logarithmic function of base a where $a > 0$ and $a \neq 1$ is denoted by $\log_a x$ and it is the inverse function of the general exponential function a^x .

Notes :

1. $\log_a x = y \Leftrightarrow a^y = x$.
2. $\log_a x = \frac{\ln x}{\ln a}$.

Notations :

1. $\log x = \log_{10} x$.
2. $\ln x = \log_e x$.

Derivative of the general logarithmic function :

1. $\frac{d}{dx} \log_a |x| = \frac{1}{x} \frac{1}{\ln a}$.
2. $\frac{d}{dx} \log_a |f(x)| = \frac{f'(x)}{f(x)} \frac{1}{\ln a}$.

Examples :

1. Find the value of x if $\log_2 x = 3$?
 $\log_2 x = 3 \Leftrightarrow x = 2^3 = 8$.
2. Find the value of a if $\log_a 125 = 3$?
 $\log_a 125 = 3 \Leftrightarrow 125 = a^3 \Leftrightarrow a = \sqrt[3]{125} = 5$.
3. Find the value of x if $2 \log |x| = \log 2 + \log |3x - 4|$? .
 $2 \log |x| = \log 2 + \log |3x - 4| \Rightarrow \log x^2 = \log |2(3x - 4)|$
 $\Rightarrow x^2 = 2(3x - 4) \Rightarrow x^2 = 6x - 8 \Rightarrow x^2 - 6x + 8 = 0$
 $(x - 4)(x - 2) = 0 \Rightarrow x = 4 \text{ or } x = 2$.
4. Find y' if $2x = 4^y$?
Differentiate both sides : $2 = 4^y y' \ln 4 \Rightarrow y' = \frac{2}{4^y \ln 4} = \frac{2}{2x \ln 4} = \frac{1}{x \ln 4}$.
Another way : $2x = 4^y \Rightarrow \ln |2x| = \ln 4^y = y \ln 4 \Rightarrow y = \frac{\ln |2x|}{\ln 4}$
Hence $y' = \frac{1}{\ln 4} \frac{2}{2x} = \frac{1}{x \ln 4}$.
5. Find $f'(x)$ if $f(x) = 7^{\sqrt[3]{x}}$?
 $f'(x) = 7^{\sqrt[3]{x}} \frac{1}{3} x^{-\frac{2}{3}} \ln 7$.
6. Find $f'(x)$ if $f(x) = \pi^{3x}$?
 $f'(x) = \pi^{3x} (3) \ln \pi = 3\pi^{3x} \ln \pi$.
7. Find y' if $y = (\sin x)^x$?
 $y = (\sin x)^x \Rightarrow \ln y = \ln (\sin x)^x = x \ln |\sin x|$
Differentiate both sides : $\frac{y'}{y} = \ln |\sin x| + x \frac{\cos x}{\sin x} = \ln |\sin x| + x \cot x$
 $y' = y [\ln |\sin x| + x \cot x] = (\sin x)^x [\ln |\sin x| + x \cot x]$
8. Find y' if $y = (1 + x^2)^{2x+1}$?
 $y = (1 + x^2)^{2x+1} \Rightarrow \ln y = \ln (1 + x^2)^{2x+1} = (2x + 1) \ln(1 + x^2)$
Differentiate both sides : $\frac{y'}{y} = 2 \ln(1 + x^2) + (2x + 1) \frac{2x}{1 + x^2}$
 $y' = y \left[2 \ln(1 + x^2) + \frac{2x(2x + 1)}{1 + x^2} \right] = (1 + x^2)^{2x+1} \left[2 \ln(1 + x^2) + \frac{2x(2x + 1)}{1 + x^2} \right]$
9. $\int x^2 6^{x^3} dx = \frac{1}{3 \ln 6} \int 6^{x^3} (3x^2) \ln 6 dx = \frac{6^{x^3}}{3 \ln 6} + c$.
10. $\int \frac{2^x}{2^x + 1} dx = \frac{1}{\ln 2} \int \frac{2^x \ln 2}{2^x + 1} dx = \frac{\ln(2^x + 1)}{\ln 2} + c$.
11. $\int \frac{3^{-\cot x}}{\sin^2 x} dx = \frac{1}{\ln 3} \int 3^{-\cot x} \csc^2 x \ln 3 dx = \frac{3^{-\cot x}}{\ln 3} + c$
12. $\int 2^{x \ln x} (1 + \ln |x|) dx = \frac{1}{\ln 2} \int 2^{x \ln x} (1 + \ln |x|) \ln 2 dx = \frac{2^{x \ln x}}{\ln 2} + c$
13. $\int 4^x 5^{4^x} dx = \frac{1}{\ln 4 \ln 5} \int 5^{4^x} 4^x \ln 4 \ln 5 dx = \frac{5^{4^x}}{\ln 4 \ln 5} + c$

$$\begin{aligned}
 14. \quad \int 3^x (1 + \sin 3^x) \, dx &= \int (3^x + 3^x \sin 3^x) \, dx = \int 3^x \, dx + \int 3^x \sin 3^x \, dx \\
 &= \frac{1}{\ln 3} \int 3^x \ln 3 \, dx + \frac{1}{\ln 3} \int \sin(3^x) 3^x \ln 3 \, dx = \frac{3^x}{\ln 3} - \frac{\cos 3^x}{\ln 3} + c
 \end{aligned}$$

Exercises :

1. Find $f'(x)$ if $f(x) = (x^2 + 1)^x$?

2. Evaluate $\int \frac{3^{\sqrt{x}}}{\sqrt{x}} \, dx$?