Numerical Methods

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Aims

In this lecture, we will . . .

- find the approximate solutions of the differential equations.

Important Points

I. In this chapter we shall find the approximate solutions of the differential equations.
II. Differential equation may be ordinary differential equation (only one independent variable) or partial differential equation (more than one independent variable).
III. Given data points should be equally spaced only (length of each subinterval should be same). Smaller the length of the interval better the approximation.
IV. Given differential equations may be linear or nonlinear and first degree and first-order.
V. We shall discuss the first-order ordinary differential equations and sets of simultaneous first-order differential equations, since, one can easily find that *nth*-order differential equation may be solved by transforming it to a set of n-simultaneous first-order differential equations. All the specified conditions are on the same endpoints. These are *initial-value problems*. Many numerical methods are discussed for the approximate solutions of such initial value problem.
VI. We shall use single step numerical methods for the approximate solution of the ordinary differential equations.

Ordinary Differential Equations

Here, we will discuss about the ordinary differential equations and their numerical solutions.

Definition 1

(Differential Equation)
An equation which involving functions and their derivatives. For example, the following equations

$$(a) \quad \frac{dy}{dx} = 3x, \qquad (b) \quad \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0,$$

$$(c) \quad \frac{dy}{dx} = x^2 + y^2, \quad (d) \quad \left(\frac{d^3y}{dx^3}\right)^2 - 5\frac{d^2y}{dx^2} + 2y = 5,$$

are differential equations. •

Definition 2

(Dependent Variable)
It is the variable that has being differentiated. For example, in each of above differential equations *(a)*-*(d)*, y is the dependent variable. •

Definition 3

(Independent Variable)
It is the variable with respect to which the differentiation is performed. For example, in each of above differential equations *(a)*-*(d)*, x is the independent variable. •

Definition 4

(Order of Differential Equation)
The *order* of the differential equation is the order of the highest derivative involved. For example, the differential equations *(a)* and *(c)* are of first-order since the highest derivatives that appear is of first-order, whereas the differential equations *(b)* and *(d)* are respectively, the second-order and the third-order. •

Definition 5

(Degree of Differential Equation)
The *degree* of the differential equation is the power to which the highest-order derivative is raised. For example, the differential equations *(a)*-*(c)* are of degree 1 while the differential equation *(d)* is of degree 2. •

Definition 6

(Linear Differential Equation)
An differential equation is *linear* if

(1) The dependent variable y and all its derivatives are of the first degree, that is, the power of each term involving y or its derivatives is one.

(2) Each coefficient depends on only independent variable x or constant.

For example, the above differential equations *(a)* and *(b)* are the linear differential equations while the differential equations *(c)* and *(d)* are the nonlinear differential equations. •

Definition 7

(Initial Conditions)

When all of the conditions are given at starting value of independent variable x to solve a given differential equation, is called a *initial condition*. When the conditions are given at the endpoints of x-values, then the conditions are called the *boundary conditions*. •

Classification of Differential Equations

There are two major types of differential equations, called, *ordinary differential equations (ODE)* and *partial differential equations (PDE)*. If an equation contains only ordinary derivatives of one or more dependent variables, with respect to a single independent variable, it is then said to be an *ordinary differential equation*. For example, all the differential equations *(a)-(d)* are ordinary differential equations because there is only one independent variable, called x.
An equation involving the partial derivatives of one or more dependent variables of two or more independent variables is called it partial differential equation. For example, the following differential equation

$$\frac{\partial^2 y}{\partial x^2} = c\frac{\partial^2 y}{\partial t^2},$$

is the partial differential because it involves two independent variables, x and t. Although partial differential equations are very useful and important, their study demands a good foundation in the theory of ordinary differential equations. Consequently, in this chapter the discussion that follows we shall confine our attention to ordinary differential equations.

As a mathematical form, the ordinary differential equation is a very useful tool. The *solution* of a differential equation is the function which satisfies the differential equation. In solving a differential equation analytically, one usually compute a general solution containing arbitrary constant. The simplest form of the differential equation is

$$y' = f(x), \tag{1}$$

with $f(x)$ a given function. The general solution of this equation is

$$y(x) = \int f(x)dx + C, \tag{2}$$

where C is an arbitrary constant. For example, the differential equation of the form

$$y' = \cos x, \tag{3}$$

has general solution of the form

$$y(x) = \sin x + C. \tag{4}$$

The more general equation is

$$y' = f(x, y(x)). \tag{5}$$

Since the general solution of differential equation is depends on an arbitrary constant C, so this constant can be calculated by specifying the value of function $y(x)$ at a particular point x_0

$$y(x_0) = y_0.$$

The point x_0 is called initial point, and the number y_0 is called the initial value. We call the problem of solving

$$y' = \frac{dy}{dx} = f(x, y); \quad x_0 \leq x \leq x_n, \quad y(x_0) = y_0, \tag{6}$$

the initial-value problem (IVP). For example, for finding the solution of the differential equation (3) satisfying $y(\pi) = 1$, we have the value of the constant $C = 1$, so (4) becomes

$$y(x) = \sin x + 1,$$

and it is called the particular solution of the differential equation (3), or called the solution of the initial-value problem

$$y' = \cos x, \qquad y(\pi) = 1.$$

The main concern of this chapter is approximating the solution to the problem (6). The initial-value problems are problems in which the value of the dependent variable y is known at a point x_0. Such a large number of methods are available to handle problems of this type that one may have difficulty in deciding which to use. Solving initial-value problem numerically we will assume that the solution is being sought on a given finite interval $x_0 \leq x \leq x_n$ with $h = (b - a)/n$, where $x_0 = a, x_n = b$ and n be the number of subintervals. In this chapter the most widely used numerical methods are discussed in some details to find the solution of the initial-value problem. If the analytical process of finding a exact solution $y(x)$ is not feasible, it is still useful to know whether a solution exists and unique using numerical methods. To make precise preceding discussion, we give the following theorem which gives a sufficient condition for the existence and uniqueness of the initial-value problem (6).

Theorem 8

(Existence and Uniqueness Theorem)

Let $f(x, y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of x and y at all points (x, y) in some neighborhood of the initial point (x_0, y_0). Then there is a unique function $y(x)$ defined on some interval $[x_0 - \epsilon, x_0 + \epsilon]$ and satisfying

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0, \quad x \in [x_0 - \epsilon, x_0 + \epsilon], \quad \epsilon > 0 \tag{7}$$

For example, the initial-value problem

$$y'(x) = 2xy^2, \qquad y(0) = 1,$$

has a unique solution

$$y(x) = \frac{1}{1 - x^2}, \quad -1 < x < 1,$$

because the both functions

$$f(x, y) = 2xy^2, \qquad \frac{\partial f}{\partial y} = 4xy,$$

are continuous for all (x, y). Note that this example also showed that the continuity of the function $f(x, y)$ and $\frac{\partial f}{\partial y}$ for all (x, y) does not imply the existence of a $y(x)$ that is continuous for all x.

Numerical Methods for Solving IVP

By a numerical method for solving the initial-value problem (6) is meant a procedure for finding approximate values $y_0, y_1, \ldots, y_n$ of the exact solution $y(x)$ at the given points $x_0 < x_1 < \ldots < x_n$. We will let y_i denote the numerical value obtained as approximation to the exact solution $y(x_i)$, with $x_i = x_0 + ih$ for $i = 0, 1, \ldots, n$, where h (constant) is the size of the interval. Numerical methods for differential equations are of great importance to the engineer and physicist because practical problems often lead to differential equations that cannot be solved by any analytical method or to equations for which the solutions in terms of formulas are so complicated that are often prefers to calculate a table of values by applying a numerical method to such an equation.
Two different types of numerical methods are available to solve initial-value problem (6). These are called the *single-step* and the *multi-steps* methods. The methods discussed will vary in complexity, since in general, the greater the accuracy of a method, the greater is its complexity. We shall discuss many numerical methods for solving the approximate solution of the initial-value problems (6) and the error analysis of each of the methods is explained in detail. Now we shall discuss the single-step methods for solving the problem (6).

Single-Step Methods for IVP

This type of method called self-starting, refers to estimate $y'(x)$, from a initial condition $y(x_0) = y_0$ and $y'_0 = f(x_0, y_0)$ from (6) and proceed step-wise. In the first-step we compute an approximate value y_1 of the solution $y(x)$ at $x = x_1 = x_0 + h$. In the second-step we compute an approximate value y_2 of that solution at $x = x_2 = x_0 + 2h$ and so on. Although these methods generally use functional evaluation information for x_i and x_{i+1}, they do not retain that information for direct use in future approximations. All the information used by these methods is consequently obtained within the interval over which the solution is being approximated. Among of them we will discuss here, the Euler's method , the Taylor's method of higher-orders, and the Runge-Kutta method of order two only.

Euler's Method

One of the simplest and most straight forward numerical method for solving first-order ordinary differential equation of the form (6) is called method of *Euler*. This method is not an efficient numerical method and so seldom used, but it is relatively easy to analysis and many of the ideas involved in the numerical solution of differential equations are introduced most simply with it.
In principle, the Euler's method uses the forward difference formula approximation of $y'(x)$ which we discussed in the previous Chapter 5. That is

$$y' = \frac{dy}{dx} \approx \frac{y(x_{i+1}) - y(x_i)}{h}, \tag{8}$$

where h is the stepsize and it is equal to $x_{i+1} - x_i$. Given that $\frac{dy}{dx} = f(x, y)$ and the initial conditions $x = x_0,\ y(x) = y(x_0)$, we have

$$\frac{y(x_1) - y(x_0)}{h} \approx f(x_0, y(x_0)), \quad \text{or} \quad y(x_1) \approx y(x_0) + hf(x_0, y(x_0)),$$

which shows that $y(x_1)$ is approximately given by $y(x_0) + hf(x_0, y(x_0))$. We can now use this approximation for $y(x_1)$ to estimate $y(x_2)$, that is

$$y(x_2) \approx y(x_1) + hf(x_1, y(x_1)),$$

and so on. In general,

$$y(x_{i+1}) \approx y(x_i) + hf(x_i, y(x_i)), \quad i = 0, 1, \ldots, n-1.$$

Taking $y_i \approx y(x_i)$, for each $i = 1, 2, \ldots, n$, we have

$$y_{i+1} = y_i + hf(x_i, y_i), \quad i = 0, 1, \ldots, n-1. \tag{9}$$

This simple integration strategy is known as the Euler's method, or the *Euler-Cauchy method.* It is called an explicit method because the value of $y(x)$ at the next step is calculated only from the value of $y(x)$ at the previous step. Given the approximate formula, one can solve for y_{i+1} in terms of x_i, y_i and $f(x_i, y_i)$, all of which are known. Note that the above formula (9) can be derive by using the Taylor series expansion of the unknown solution $y(x)$ to the problem (6) about the point $x = x_i$, for each $i = 0, 1, \dots, n-1$

$$y(x_{i+1}) = y(x_i)+(x_{i+1}-x_i)y'(x_i)+\frac{(x_{i+1}-x_i)^2}{2!}y''(\eta_i) = y(x_i)+hy'(x_i)+\frac{h^2}{2!}y''(\eta_i), \tag{10}$$

where unknown point η_i lies in the interval (x_i, x_{i+1}). For the smaller value of stepsize h, the higher power h^2 will be very small and may be neglected. Using $f(x_i, y_i)$ to evaluate $y'(x_i)$ and $y_i \approx y(x_i)$, we have the formula (9).

Geometrically interpretation of the method is shown by Figure 1.

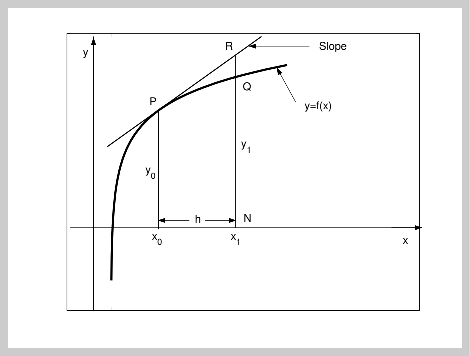

Figure: Geometrically interpretation of the Euler's method.

Example 0.1

Use the Euler's method to find the approximate value of $y(1)$ for the given initial-value problem

$$y' = xy + x, \qquad 0 \leq x \leq 1, \quad y(0) = 0, \quad \text{with} \quad h = 0.1,\ 0.2.$$

Compare your approximate solutions with the exact solution $y(x) = -1 + e^{x^2/2}$.

Solution. Since $f(x, y) = xy + x$, and $x_0 = 0$, $y_0 = 0$, then

$$y_{i+1} = y_i + hf(x_i, y_i), \quad \text{for} \quad i = 0, 1, \ldots, 9$$

Then for $h = 0.1$ and taking $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(x_0 y_0 + x_0) = 0 + (0.1)[(0)(0) + (0)] = 0.0000.$$

Similar way, we have other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 9$, as follows

$$y_2 = 0.0100, \quad y_3 = 0.0302, \quad y_4 = 0.0611, \quad y_5 = 0.1036, \quad y_6 = 0.1587,$$

$$y_7 = 0.2283, \quad y_8 = 0.3142, \quad y_9 = 0.4194, \quad y_{10} = 0.5471,$$

with possible absolute error

$$|y(1) - y_{10}| = |0.6487 - 0.5471| = 0.1016.$$

Similarly, the approximations for $h = 0.2$, give

$$y_1 = 0.0000, \quad y_2 = 0.0400, \quad y_3 = 0.1232, \quad y_4 = 0.2580, \quad y_5 = 0.4592,$$

with possible absolute error

$$|y(1) - y_5| = |0.6487 - 0.4592| = 0.1895.$$

Example 0.2

Use the Euler's method to find the approximate value of $y(1.4)$ for the given initial-value problem

$$\frac{1}{x}y' - y^2 = 0, \quad y(1) = 1, \quad \text{with} \quad n = 2.$$

Compare your approximate solutions with the exact solution $y(x) = 2/(3 - x^2)$.

Solution. Since $f(x, y) = xy^2$, and $x_0 = 1$, $y_0 = 1$, $h = 0.2$, then using the Euler's method

$$y_{i+1} = y_i + hf(x_i, y_i), \quad \text{for} \quad i = 0, 1, \ldots, n-1,$$

for $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(x_0 y_0^2) = 1 + (0.2)[(1)(1)] = 1.2.$$

Similar way, we have other approximations by taking $i = 1$, as follows

$$y_2 = y_1 + hf(x_1, y_1) = y_1 + h(x_1 y_1^2) = 1.2 + (0.2)[(1.2)(1.44)] = 1.5456,$$

the required approximation of $y(1.4)$ and

$$|y(1.4) - y_2| = |1.9231 - 1.5456| = 0.3775,$$

is the possible absolute error. •

Analysis of the Euler's Method

The preceding Example 0.1 demonstrates that the error in applying the Euler's method is reduced when h is reduced. The question of how well the Euler's method for solving the initial-value problem (6) works is closely related to the truncation error of the method. There are two types of such error, *local* and *global* truncation error.
In case of local truncation error one consider the size of the error made during one step and for global truncation error, one can consider the errors in the entire interval $x_0 \leq x \leq x_n$ over which the solution is sought.
We turn to the Taylor series to find an expression that represents the error, we have

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(\eta(x)),$$

for unknown $\eta(x) \in [x, x+h]$.
If $y(x_{i+1})$ is the true value of $y(x)$, then the Taylor series expression at x_i is

$$y(x_{i+1}) = y(x_i) + hf(x_i, y_i) + \frac{h^2}{2!}y''(\eta(x_i)), \quad \eta(x_i) \in (x_i, x_{i+1})$$

as $y' = f(x_i, y_i)$. The Euler's formula uses the recurrence relation

$$y_{i+1} = y(x_i) + hf(x_i, y_i),$$

to estimate y_{i+1} assuming that $y(x_i)$ is the true solution.

The error in y_{i+1} is given by $y_{i+1} - y(x_{i+1})$ which can be written as

$$y_{i+1} - y(x_{i+1}) = (y(x_i) + hf(x_i, y_i)) - (y(x_i) + hf(x_i, y_i) + \frac{h^2}{2!} y''(\eta(x_i))) = -\frac{h^2}{2!} y''(\eta(x_i)), \quad (11)$$

for $i = 0, 1, \ldots, n-1$. We call the term $-\frac{h^2}{2} y''(\eta(x_i))$, the *local* truncation error for the Euler's method. It is of order h^2.

Note that this error term only applies in the region (x_i, x_{i+1}), hence it is only the error in estimating y_{i+1} from $y(x_i)$. It does not take into account the compounded error from previous estimates. If we assume that the error is increasing linearly with n, then the error will be proportional to nh^2, but n is dependent on h as $h = \frac{x_n - x_0}{n}$, so the error will be proportional to

$$\frac{x_n - x_0}{h} h^2 = (x_n - x_0)h,$$

which is order h. This error is called the *global* truncation error.

The analysis above leads to important theorem in the analysis of numerical methods

Theorem 9

For the differential equations $\dfrac{dy}{dx} = f(x, y)$, *if the leading term in the local truncation error involves* h^{p+1}, *for some integer* p, *then the global error, for small* h, *is of order* h^p, *that is*

$$y_{i+1} - y(x_{i+1}) \approx ch^p,$$

where c *does not depend on stepsize* h. •

Note that the Euler's method is called the first-order method because of its local truncation error given by the formula (11), since this arises on each application of the method. Thus, in generating the solution point (x_k, y_k) the truncation error appears k times, once for each application of the method.

Higher-Order Taylor Methods

The basis for many numerical techniques finding the approximate solution of the initial-value problem can be depend to the Taylor's series, as we used this series in the previous section in finding the Euler's method which also called the Taylor's method of order one. One can, of course, develop the Taylor's method for higher-order to obtain better accuracy, and in general, one expect that higher the order of the method, greater the accuracy for a given stepsize. The Taylor's method is relatively easy to use, however, the necessity of calculating the higher derivatives makes the Taylor's method completely unsuitable. Nevertheless, it is of great theoretical interest because the most of the practical methods attempt to achieve the same accuracy as the Taylor's method of a given order without the disadvantage of having to calculate the higher derivatives. Assuming that the solution $y(x)$ of the initial-value problem (6) has $(n+1)$ continuous derivatives and expanding $y(x)$ in terms of its *nth* degree Taylor polynomial about x_i, we get

$$\begin{aligned} y(x_{i+1}) &= y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \cdots \\ &+ \frac{h^n}{n!}y^{(n)}(x_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\eta(x_i)), \end{aligned} \tag{12}$$

for some $\eta(x_i) \in (x_i, x_{i+1})$.

The derivatives in this expansion are not known explicitly since the solution is not known. However, if f is sufficiently differentiable, they can be obtained by taking the total derivative of (6) with respect to x, keeping in mind that f is an implicit function of y. Thus

$$\begin{aligned} y' &= f(x,y) = f \\ y'' &= f' = f_x + f_y f \\ y''' &= f'' = f_{xx} + 2f_{xy}f + f_{yy}f^2 + f_x f_y + f_y^2 f \\ &\vdots \end{aligned} \tag{13}$$

Continuing in this manner, we can express any derivative of y in terms of $f(x, y)$ and its partial derivatives. It is already clear, however, that unless $f(x, y)$ is a very simple function, the higher total derivatives become increasingly complex. Now substituting these results into (12), gives

$$\begin{aligned} y(x_{i+1}) &= y(x_i) + hf(x_i, y(x_i)) + \frac{h^2}{2!} f'(x_i, y(x_i)) + \cdots \\ &+ \frac{h^n}{n!} f^{(n-1)}(x_i, y(x_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\eta(x_i), y(\eta(x_i))). \end{aligned} \tag{14}$$

By taking $y_i \approx y(x_i)$, that the approximation to the exact solution at x_i, for each $i = 0, 1, \ldots, n-1$, we have

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2!} f'(x_i, y_i) + \cdots + \frac{h^n}{n!} f^{(n-1)}(x_i, y_i). \tag{15}$$

Then this formula is called the Taylor's method of order n. The last term of (14), called remainder, shows that the local error of Taylor's method of order n is

$$E = \frac{h^{n+1}}{(n+1)!} f^{(n)}(\eta_i, y(\eta(x_i))) = \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\eta(x_i)), \tag{16}$$

for some $x_i < \eta(x_i) < x_{i+1}$.

Example 0.3

Use the Taylor's method of order 2 to find the approximate value of $y(1)$ for the given initial-value problem.

$$y' = xy + x, \quad 0 \le x \le 1, \quad y(0) = 0, \quad \text{with} \quad h = 0.2$$

Compare your approximate solution with the exact solution $y(x) = -1 + e^{x^2/2}$.

Solution. Since $f(x, y) = xy + x$, and $x_0 = 0,\ y_0 = 0$, then

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i), \quad \text{for} \quad i = 0, 1, 2, 3, 4$$

where $f'(x_i, y_i) = y_i + x_i^2 y_i + x_i^2 + 1$. Then for $i = 0$, we have

$$y_1 = y_0 + h(x_0 y_0 + x_0) + \frac{h^2}{2}(y_0 + x_0^2 y_0 + x_0^2 + 1) = 0 + (0.2)(0) + (0.02)(1) = 0.0200,$$

and similar way, we have for $i = 1, 2, 3, 4$, as follows

$$y_2 = 0.0820, \quad y_3 = 0.1937, \quad y_4 = 0.3694, \quad y_5 = 0.6334,$$

with absolute possible error

$$|y(1) - y_5| = |0.6487 - 0.6334| = 0.0153.$$

Example 0.4

Use the Taylor's method of order 3 to find the approximate value of $y(1)$ for the given initial-value problem

$$4y' - y = 0, \quad 0 \leq x \leq 1, \quad y(0) = 1, \quad \text{with} \quad n = 2.$$

Compare your approximate solution with the exact solution $y(x) = e^{x/4}$.
Solution. Since the Taylor's method of order 3 is

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2!} f'(x_i, y_i) + \frac{h^3}{3!} f''(x_i, y_i),$$

for $i = 0, 1, \ldots, n-1$, and using the given values $x_0 = 0$, $y_0 = 1$ and $f(x, y) = 1/4y$, we get, $f'(x, y) = 1/16y$ and $f''(x, y) = 1/64y$. So using these values we obtain Taylor's method of order 3 of the form

$$y_{i+1} = y_i + h(1/4y_i) + \frac{h^2}{2}(1/16y_i) + \frac{h^3}{6}(1/64y_i).$$

Then for $i = 0$, we have

$$y_1 = y_0 \left[1 + \frac{h}{4} + \frac{h^2}{32} + \frac{h^3}{384}\right],$$

and by taking $y_0 = 1$, $h = 0.5$, we get

$$y(0.5) \approx y_1 = 1(1 + 0.125 + 0.0078 + 0.0003) = 1.1331,$$

and similar way, we have other approximation for taking $i = 1$, as follows

$$y(1) \approx y_2 = y_1(1 + 0.125 + 0.0078 + 0.0003) = 1.1331(1.1331) = 1.2839,$$

the required approximation of $y(1)$ and

$$|y(1) - y_2| = |1.2840 - 1.2839| = 0.0001,$$

is the possible absolute error. $\bullet$

Runge-Kutta Methods

Since we studied that the Euler's method is not very useful in practical problems because it requires a very small stepsize for reasonable accuracy. the Taylor's method of higher-order is difficult to use because it needs to obtain higher total derivatives of $y(x)$. An important group of methods which allow us to obtain greater accuracy at each step and yet require only initial value of $y(x)$ to be given with the differential equation are called the Runge-Kutta methods. The Runge-Kutta methods attempt to obtain greater accuracy, and at the same time avoid the need of higher derivatives by evaluating the function $f(x, y)$ at selected points on each subintervals. These methods can be used to generate not only starting values but, in fact, in whole solution. They are self-starting and easy to program for a digital computer. We shall begin by showing how to derive the simplest formulas in this class. These are of the form

$$y_{i+1} = y_i + (w_1 k_1 + w_2 k_2), \tag{17}$$

where

$$k_1 = hf(x_i, y_i) \qquad \text{and} \qquad k_2 = hf(x_i + ah, y_i + bk_1).$$

The parameters $w_1,\ w_2,\ a$, and b are chosen in order to make the formula (17) as accurate as possible, that is, to make the order of accuracy as large as possible. To this end, we substitute the exact value $y(x), y(x_{i+1})$ by the local solution into the formula (17) and expand about the point x_i. The parameters are then chosen to make the resulting expansion agree as much as possible with the Taylor series for $y(x_{i+1})$ about x_i. Upon substituting into (17), we first expanding $y(x_{i+1})$ in the Taylor series through terms of order h^3, we obtain

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2!} y''(x_i) + \frac{h^3}{3!} y'''(x_i) + \cdots \tag{18}$$

Since

$$\begin{aligned} y' &= f(x,y) \\ y'' &= f'(x_i,y_i) = (f_x + f_y f)_i \\ y''' &= f''(x_i,y_i) = (f_{xx} + 2f_{xy}f + f_{yy}f^2 + f_x f_y + f_y^2 f)_i + O(h^4). \end{aligned} \tag{19}$$

So

$$\begin{aligned} y(x_{i+1}) &= y(x_i) + hf(x_i,y_i) + \frac{h^2}{2!}(f_x + ff_y)_i \\ &+ \frac{h^3}{3!}(f_{xx} + 2f_{xy}f + f_{yy}f^2 + f_x f_y + f_y^2 f)_i + O(h^4), \end{aligned} \tag{20}$$

where the subscripts on f denote partial derivatives with respect to the indicated variables, and the subscript i means that all functions involved are to be evaluated at (x_i, y_i). Now using the Taylor's expansion for functions of two variables, we find that

$$\begin{aligned} k_2 &= hf(x_i + ah, y_i + bk_1) = h[f + h(af_x + bff_y) \\ &+ \frac{h^2}{2}(a^2 f_{xx} + 2abff_{xy} + b^2 f^2 f_{yy}) + O(h^4)]_i. \end{aligned} \tag{21}$$

Now we substitute this expression for k_2 into (17), gives

$$\begin{aligned} y_{i+1} &= y_i + h[w_1 f(x_i, y_i) + w_2 f(x_i + ah, y_i + bk_1)] \\ &= y_i + h[(w_1 + w_2)f]_i + h^2 w_2[(af_x + bff_y)]_i \\ &+ \frac{h^3}{2} w_2[a^2 f_{xx} + 2abff_{xy} + b^2 f^2 f_{yy}]_i + O(h^4). \end{aligned} \quad (22)$$

On comparing (20) and (22), we see that to make the corresponding powers of h and h^2 agree, we must have

$$w_1 + w_2 = 1 \quad \text{and} \quad a = \frac{1}{2w_2} = b.$$

This is a system of two nonlinear equations in the four unknowns a, b, w_1, and w_2 and its solution can be written in the form

$$b = a = \frac{1}{2w_2}, \quad w_1 = 1 - w_2. \quad (23)$$

There are many solutions to (23) depending on the choices of w_2. These choices leads to the numerical method which has order 2 and some of them do correspond to any of the standard numerical integration formulas. Taking the first choice when $w_2 = 1/2$, we have

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))]. \quad (24)$$

Runge-Kutta Method of Order Two (Modified Euler's Method)

The equation (24) can be written in a standard form as

$$y_{i+1} = y_i + \frac{h}{2}[k_1 + k_2], \tag{25}$$

where

$$k_1 = f(x_i, y_i) \quad \text{and} \quad k_2 = f(x_{i+1}, y_i + hk_1),$$

for each $i = 0, 1, \ldots n-1$. Then the relation (25) is called the *Runge-Kutta method of order 2* which is also known as the *Modified Euler's method.* This method corresponds to using the Trapezoidal rule to estimate the integral where a preliminary (full) Euler step is taken to obtain the (approximate) value at x_{i+1}. Geometrically interpretation of the method is shown by Figure 2.

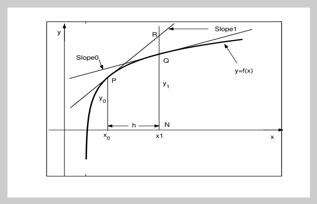

Figure: Geometrically interpretation of the Modified Euler method.

The local error of this formula is, however, of order h^3, whereas that of the Euler's method is h^2. We can therefore expect to be able to use a large stepsize with this formula.

Example 0.5

Use Runge-Kutta method of order two (Modified Euler's method) to find the approximate value of $y(1)$ for the given initial-value problem

$$y' = xy + x, \quad 0 \leq x \leq 1, \quad y(0) = 0, \quad \text{with} \quad h = 0.2.$$

Compare your approximate solution with the exact solution $y(x) = -1 + e^{x^2/2}$.

Solution. Since $f(x, y) = xy + x$, and $x_0 = 0,\ y_0 = 0$, then for $i = 0$, we have

$$\begin{aligned} k_1 &= f(x_0, y_0) = (x_0 y_0 + x_0) = 0.0000 \\ k_2 &= f(x_1, y_0 + hk_1) = (x_1(y_0 + hk_1) + x_1) = (0 + 0.2) = 0.2000, \end{aligned}$$

and using these values, we have

$$y_1 = y_0 + \frac{h}{2}[k_1 + k_2] = 0 + 0.1(0 + 0.2000) = 0.0200.$$

Continuing in this manner, we have

$$\begin{array}{llllll} k_1 = & 0.204, & k_2 = & 0.4243, & \text{then} & y_2 = 0.0828, \\ k_1 = & 0.4331, & k_2 = & 0.7017, & \text{then} & y_3 = 0.1963, \\ k_1 = & 0.7178, & k_2 = & 1.0719, & \text{then} & y_4 = 0.3753, \\ k_1 = & 1.1002, & k_2 = & 1.5953, & \text{then} & y_5 = 0.6449, \end{array}$$

with possible error

$$|y(1) - y_5| = |0.6487 - 0.6449| = 0.0039$$

Example 0.6

Use the Runge-Kutta method of order two (the Modified Euler's method) to find the approximate value of $y(1.2)$ for the given initial-value problem

$$x^2y' - y = 0, \quad y(1) = 2, \quad \text{with} \quad n = 2.$$

and compare your approximate solution with the exact solution $y(x) = e^{(x-1)/x}$.
Solution. Since $f(x,y) = x^{-2}y$ and $x_0 = 1,\ y_0 = 2,\ h = (1.2-1)/2 = 0.1$, then for $i = 0$, we have

$$\begin{aligned} k_1 &= f(x_0, y_0) = f(1,2) = (1)^{-2}(2) = 2, \\ k_2 &= f(x_1, y_0 + hk_1) = f(1.1, 2.2) = (1.1)^{-2}(2.2) = 1.8182, \end{aligned}$$

and by using these values, we have

$$y(1.1) \approx y_1 = y_0 + \frac{h}{2}[k_1 + k_2] = 2 + 0.05(2 + 1.8182) = 2.1909.$$

Similar manner, we have the other approximation for taking $i = 1$, as follows

$$k_1 = 1.8107 \qquad \text{and} \qquad k_2 = 1.6035,$$

and by using these values, we have

$$y(1.2) \approx y_2 = y_1 + \frac{h}{2}[k_1 + k_2] = 2.1909 + 0.05(1.8107 + 1.6035) = 2.3616,$$

the required approximation of $y(1.2)$ and

$$|y(1.2) - y_2| = |2.3627 - 2.3616| = 0.0011,$$

is the possible absolute error. •