Legendre Polynomials

- Introduced in 1784 by the French mathematician A. M. Legendre(1752-1833).
- We only study Legendre *polynomials* which are special cases of Legendre *functions*. See sections 4.3, 4.7, 4.8, and 4.9 of Kreyszig.
- Legendre functions are important in problems involving spheres or spherical coordinates. Due to their orthogonality properties they are also useful in numerical analysis.
- Also known as spherical harmonics or zonal harmonics. Called *Kugelfunktionen* in German. (Kugel = Sphere).

Introduction: Consider Laplace's equation(see Kreyszig Sec. 8.8)

$$\nabla^2 V = 0 \tag{1}$$

In rectangular coordinates

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{2}$$

In spherical coordinates

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}\right] = 0 \tag{3}$$

Equations (2) and (3) express exactly the same fact in different coordinates. Which one to use is a matter of convenience. Even when spherical coordinates are more natural and equation (3) is used, equation (2) may give some additional insight due to its greater simplicity.

It is easy to verify that $V = 1/r = 1/\sqrt{x^2+y^2+z^2}$, the potential due to a point source(charge or mass), satisfies (2) or equivalently (3). This solution is spherically symmetric. Are there solutions which depend on θ and ϕ?

One approach to create new solutions from $V = 1/r$ is to take partial derivatives of this function with respect to x, y, or z. In fact, since $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ operators commute, equation (2) shows that partial derivatives of all orders of $V = 1/r$ also satisfy the Laplace equation. Let us consider the partial derivatives of $1/r$ with respect to z. This will lead us to the Legendre polynomials. (If we consider partial derivatives with respect to x and y, we will encounter the *Associated Legendre functions.*)

Some equations relating (x, y, z) to (r, θ, ϕ) are,

$$x = r\sin\theta\cos\phi \tag{4}$$

$$y = r\sin\theta\sin\phi \tag{5}$$

$$z = r\cos\theta \tag{6}$$

$$r^2 = x^2 + y^2 + z^2 \tag{7}$$

From (7) it follows that

$$\frac{\partial r}{\partial z} = \frac{z}{r} \tag{8}$$

Let $V = 1/r$. Then the first few partial derivatives of V with respect to z are

$$\frac{\partial V}{\partial z} = \frac{\partial(1/r)}{\partial z} = -\frac{1}{r^2}\frac{\partial r}{\partial z} = -\frac{z}{r^3} = -\frac{\cos\theta}{r^2}$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{3z^2 - r^2}{r^5} = \frac{3\cos^2\theta - 1}{r^3}$$

$$\frac{\partial^3 V}{\partial z^3} = -\frac{15z^3 - 9zr^2}{r^7} = -\frac{15\cos^3\theta - 9\cos\theta}{r^4}$$

We notice that $\partial^n V/\partial z^n$ is an n-th degree polynomial of $\cos\theta$ divided by r^{n+1}. One formula which combines all partial derivatives of V with respect to z is

$$V(x,y,z-h) = V(x,y,z) - h\frac{\partial V}{\partial z}(x,y,z) + h^2\frac{1}{2!}\frac{\partial^2 V}{\partial z^2}(x,y,z) + \cdots + h^n\frac{(-1)^n}{n!}\frac{\partial^n V}{\partial z^n}(x,y,z) + \cdots \tag{9}$$

This is just the potential due to a translation of the source, and also satisfies (2), (3).

Definition:

$$\frac{1}{\sqrt{x^2+y^2+(z-h)^2}} = \frac{1}{\sqrt{r^2-2rh\cos\theta+h^2}} = \sum_{n=0}^{\infty}\frac{h^n P_n(\cos\theta)}{r^{n+1}} \tag{10}$$

for $|h|<r$. Equation (10) defines the Legendre polynomial of degree n, P_n. Comparison with (9) shows that

$$\frac{P_n(\cos\theta)}{r^{n+1}} = \frac{(-1)^n}{n!}\frac{\partial^n(1/r)}{\partial z^n} \tag{11}$$

satisfies the Laplace equation. (This solution of the Laplace equation arises naturally in the study of electric source configurations known as multipoles. So (10) is actually a multipole potential expansion. Here it expresses the potential due to a displaced charge in terms of the potentials of multipoles at the origin. The case n=1 is called a dipole, and the case n=2 is called a quadrupole. In general we have 2^n-poles. In fluid mechanics a *doublet* is akin to a dipole.) Let $w=P_n(\cos\theta)$. Then substituting w/r^{n+1} in (3) in place of V and simplifying we see that $w=P_n(\cos\theta)$ satisfies

$$n(n+1)w+\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dw}{d\theta}\right)=0$$

Or, upon substituting $t=\cos\theta$,

$$\frac{d}{dt}\left((1-t^2)\frac{dw}{dt}\right)+n(n+1)w=0 \tag{12}$$

This is known as Legendre's differential equation. $w=P_n(t)$ is one of two linearly independent solutions of this equation. So far we have got the following.

1. A definition of the Legendre polynomials P_n given by (10). We note that (11) can be considered an equivalent definition.

2. A differential equation (12) which is satisfied by the Legendre polynomials.

In fact, (10) can be written as

$$\frac{1}{r}\frac{1}{\sqrt{1-2(h/r)\cos\theta+(h/r)^2}} = \sum_{n=0}^{\infty}\frac{h^n P_n(\cos\theta)}{r^{n+1}} = \frac{1}{r}\sum_{n=0}^{\infty}(h/r)^n P_n(\cos\theta)$$

Cancelling the $1/r$ factor, calling h/r as u, and $\cos\theta$ as t we get the simpler-looking definition

$$\frac{1}{\sqrt{1-2ut+u^2}} = \sum_{n=0}^{\infty}u^n P_n(t) \tag{13}$$

The left-hand side of (13) is called the generating function of the Legendre polynomials. Many important properties of the Legendre polynomials can be obtained from (13). We derive several of these properties now.

Let $u=0$ in (13). Then the left-hand side is 1 and the right-hand side is $P_0(t)$. So,

$$P_0(t)=1 \tag{14}$$

Let $t=1$ in (13). Then the left-hand side is $1/\sqrt{1-2u+u^2}=1/(1-u)=1+u+u^2+\cdots$. The right-hand side is $P_0(1)+uP_1(1)+u^2P_2(1)+\cdots$. Comparing the coefficients of u^n on both sides we get,

$$P_n(1)=1 \tag{15}$$

Substituting $t=-1$ we can derive,

$$P_n(-1)=(-1)^n \tag{16}$$

A recurrence relation: Using (11), we get

$$\frac{P_{n+1}(\cos\theta)}{r^{n+2}} = \frac{(-1)^{n+1}}{(n+1)!}\frac{\partial^{n+1}(1/r)}{\partial z^{n+1}} = \frac{-1}{n+1}\frac{\partial}{\partial z}\left(\frac{P_n(\cos\theta)}{r^{n+1}}\right)$$

Or,

$$\frac{P_{n+1}(\cos\theta)}{r^{n+2}} = \frac{-1}{n+1}\left(-\frac{(n+1)P_n(\cos\theta)}{r^{n+2}}\frac{\partial r}{\partial z}+\frac{1}{r^{n+1}}\frac{d(P_n(\cos\theta))}{d\cos\theta}\frac{\partial(\cos\theta)}{\partial z}\right) \tag{17}$$

Now

$$\frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta \tag{18}$$

What about $\partial(\cos\theta)/\partial z$? Taking the partial derivative of both sides of (6), that is $z = r\cos\theta$, we get

$$1 = \frac{\partial r}{\partial z}\cos\theta + r\frac{\partial(\cos\theta)}{\partial z} = \cos\theta.\cos\theta + r\frac{\partial(\cos\theta)}{\partial z} = \cos^2\theta + r\frac{\partial(\cos\theta)}{\partial z}$$

So,

$$\frac{\partial(\cos\theta)}{\partial z} = \frac{1-\cos^2\theta}{r} \tag{19}$$

Using (18) and (19) in (17) we get

$$\frac{P_{n+1}(\cos\theta)}{r^{n+2}} = \frac{-1}{n+1}\left(-\frac{(n+1)P_n(\cos\theta)}{r^{n+2}}\cos\theta + \frac{1}{r^{n+1}}\frac{d(P_n(\cos\theta))}{d\cos\theta}\frac{1-\cos^2\theta}{r}\right)$$

Cancelling the common $1/r^{n+2}$ factor from both sides and simplifying we get

$$P_{n+1}(\cos\theta) = \cos\theta P_n(\cos\theta) - \frac{1-\cos^2\theta}{n+1}\frac{d(P_n(\cos\theta))}{d\cos\theta}$$

Writing $t = \cos\theta$ we get,

$$P_{n+1}(t) = tP_n(t) - \frac{1-t^2}{n+1}P_n^{'}(t) \tag{20}$$

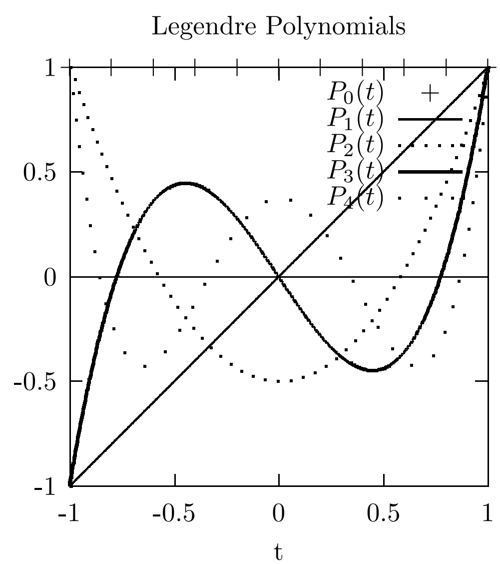
where the prime($'$) means differentiation with respect to the argument. (20) expresses P_{n+1} in terms of P_n and its derivative. We will use it and the differential equation for P_n to derive several other recurrence relations later. We use $P_0(t) = 1$ as a starting condition. Then applying (20) repeatedly we get

$$P_1(t) = t \tag{21}$$

$$P_2(t) = \frac{3t^2-1}{2} \tag{22}$$

$$P_3(t) = \frac{5t^3-3t}{2} \tag{23}$$

$$P_4(t) = \frac{35t^4-30t^2+3}{8} \tag{24}$$



Even-odd symmetry: The recurrence (20) also shows that if $P_n(t)$ is an even(odd) function of t then $P_{n+1}(t)$ is an odd(even) function of t. Since P_0 is even, it follows that Legendre polynomials of even degrees are even functions of t, and those of odd degrees are odd functions of t.

Orthogonality: Now we prove that $\int_{-1}^{1} P_m(t)P_n(t)dt = 0$, if integers $m \geq 0$ and $n \geq 0$ are unequal. Let $v = P_m(t)$ and $w = P_n(t)$. Then by Legendre's differential equation (12)

$$\left((1-t^2)v'\right)' + m(m+1)v = 0 \tag{25}$$

$$\left((1-t^2)w'\right)' + n(n+1)w = 0 \tag{26}$$

Now we multiply (25) by w and integrate from $t = -1$ to $t = 1$ to obtain

$$\int_{-1}^{1} \left((1-t^2)v'\right)' w dt + m(m+1)\int_{-1}^{1} vw dt = 0$$

Integrating the first integral by parts we get,

$$\left[(1-t^2)v'w\right]_{-1}^{1} - \int_{-1}^{1}(1-t^2)v'w'dt + m(m+1)\int_{-1}^{1} vw dt = 0$$

But since $(1-t^2)$ is zero both at $t = -1$ and $t = 1$ this becomes,

$$-\int_{-1}^{1}(1-t^2)v'w'dt + m(m+1)\int_{-1}^{1} vw dt = 0 \tag{27}$$

In exactly the same way we can multiply (26) by v and integrate from $t = -1$ to $t = 1$ to obtain

$$-\int_{-1}^{1}(1-t^2)v'w'dt + n(n+1)\int_{-1}^{1} vw dt = 0 \tag{28}$$

Subtracting (28) from (27) we get

$$(m(m+1) - n(n+1))\int_{-1}^{1} vw dt = 0$$

Or, since $v = P_m(t)$ and $w = P_n(t)$

$$(m(m+1) - n(n+1))\int_{-1}^{1} P_m(t)P_n(t)dt = 0$$

This gives the orthogonality relationship:

$$\int_{-1}^{1} P_m(t)P_n(t)dt = 0 \tag{29}$$

for $m \neq n$, and $m \geq 0$, $n \geq 0$. This is a very important property of the Legendre polynomials.

To determine $\int_{-1}^{1} P_n^2(t)dt$ we square (13) and integrate from $t = -1$ to $t = 1$. Due to orthogonality only the integrals of terms having $P_n^2(t)$ survive on the right-hand side. So we get

$$\int_{-1}^{1} \frac{1}{1-2ut+u^2}dt = \sum_{n=0}^{\infty} u^{2n}\int_{-1}^{1} P_n^2(t)dt$$

Or,

$$\frac{1}{u}\ln\frac{1+u}{1-u} = \sum_{n=0}^{\infty}\frac{2u^{2n}}{2n+1} = \sum_{n=0}^{\infty} u^{2n}\int_{-1}^{1} P_n^2(t)dt$$

Comparing coefficients of u^{2n} we get

$$\int_{-1}^{1} P_n^2(t)dt = \frac{2}{2n+1} \tag{30}$$

More recurrences: Rearranging (20) we get

$$(1-t^2)P_n^{'}(t) = (n+1)(tP_n(t) - P_{n+1}(t)) \tag{31}$$

Differentiating both sides of (20) with respect to t and using the fact(from (12)) that $((1-t^2)P_n^{'}(t))' = -n(n+1)P_n(t)$ we get after some simplification

$$P_{n+1}^{'}(t) = (n+1)P_n(t) + tP_n^{'}(t) \tag{32}$$

Multiplying both sides of (32) by $(1-t^2)$ we get

$$(1-t^2)P_{n+1}^{'}(t) = (n+1)(1-t^2)P_n(t) + t(1-t^2)P_n^{'}(t)$$

But since by (31) $(1-t^2)P_n^{'}(t) = (n+1)(tP_n(t) - P_{n+1}(t))$

$$(1-t^2)P_{n+1}^{'}(t) = (n+1)(1-t^2)P_n(t) + t(n+1)(tP_n(t) - P_{n+1}(t))$$

On simplification we get

$$(1-t^2)P_{n+1}^{'}(t) = (n+1)(P_n(t) - tP_{n+1}(t)) \tag{33}$$

Rearranging (33) we get

$$P_n(t) = tP_{n+1}(t) + \frac{(1-t^2)}{n+1}P_{n+1}^{'}(t) \tag{34}$$

If in (33) we decrement n by 1 we get

$$(1-t^2)P_n^{'}(t) = n(P_{n-1}(t) - tP_n(t)) \tag{35}$$

Comparing (31) and (35) we get

$$n(P_{n-1}(t) - tP_n(t)) = (n+1)(tP_n(t) - P_{n+1}(t))$$

On simplification this gives **Bonnet's recursion** formula.

$$(n+1)P_{n+1}(t) = (2n+1)tP_n(t) - nP_{n-1}(t) \tag{36}$$

Since this formula involves no derivatives it is much used in programs to compute the Legendre polynomials. One usually starts with $P_0(t) = 1$ and $P_1(t) = t$.

Location and interlacing of zeros: Plotting $P_n(t)$ for the first few values of n we see that:

1. Between two consecutive zeros of $P_{n+1}(t)$ there is one of $P_n(t)$.
2. Between two consecutive zeros of $P_n(t)$ there is one of $P_{n+1}(t)$. Between the smallest zero of $P_n(t)$ and -1 there is one zero of $P_{n+1}(t)$. Between the largest zero of $P_n(t)$ and $+1$ there is one zero of $P_{n+1}(t)$.
3. All n zeros of $P_n(t)$ lie in $-1 < t < 1$.

These statements, which are of great use in the numerical calculation of the zeros of $P_n(t)$, may be proved using the recurrence relationships derived earlier. A rough sketch of these proofs follows.

To prove the first statement we multiply both sides of (34) by $(n+1)/(1-t^2)^{(n+3)/2}$ to obtain

$$\frac{(n+1)P_n(t)}{(1-t^2)^{(n+3)/2}} = \frac{(n+1)tP_{n+1}(t)}{(1-t^2)^{(n+3)/2}} + \frac{P_{n+1}^{'}(t)}{(1-t^2)^{(n+1)/2}}$$

Or,

$$\frac{(n+1)P_n(t)}{(1-t^2)^{(n+3)/2}} = \left(\frac{P_{n+1}(t)}{(1-t^2)^{(n+1)/2}}\right)' \tag{37}$$

By Rolle's theorem the first statement follows.

We multiply both sides of (20) by $(n+1)(1-t^2)^{(n-1)/2}$ to obtain

$$(n+1)(1-t^2)^{(n-1)/2}P_{n+1}(t) = (n+1)t(1-t^2)^{(n-1)/2}P_n(t) - (1-t^2)^{(n+1)/2}P_n^{'}(t)$$

Or,

$$(n+1)(1-t^2)^{(n-1)/2}P_{n+1}(t) = (-(1-t^2)^{(n+1)/2}P_n^{'}(t))' \tag{38}$$

So, by Rolle's theorem between two zeros of $-(1-t^2)^{(n+1)/2}P_n^{'}(t)$ there is at least one of $(n+1)(1-t^2)^{(n-1)/2}P_{n+1}(t)$. But the first function is zero when $P_n(t)$ is zero or when $t = -1$ or $t = +1$. This proves the second statement about the zeros mentioned above. It follows that if $P_n(t)$ has n zeros in $(-1, 1)$, then $P_{n+1}(t)$ has $n+1$ zeros in $(-1, 1)$. But it is known that $P_1(t)$, which equals t, has one zero in $(-1, 1)$. Then using the method of mathematical induction we can prove that $P_n(t)$ has n zeros in $(-1, 1)$ for all integral $n \geq 0$. This proves the third statement.