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JORDAN WEAK AMENABILITY AND ORTHOGONAL FORMS ON JB*-ALGEBRAS

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Abstract. We prove the existence of a linear isometric correspondence between the Banach space of all symmetric orthogonal forms on a JB*-algebra $\mathcal{J}$ and the Banach space of all purely Jordan generalized Jordan derivations from $\mathcal{J}$ into $\mathcal{J}^*$. We also establish the existence of a similar linear isometric correspondence between the Banach spaces of all anti-symmetric orthogonal forms on $\mathcal{J}$, and of all Lie Jordan derivations from $\mathcal{J}$ into $\mathcal{J}^*$.

1. Introduction

Let φ and ψ be functionals in the dual of a C*-algebra A. The assignment

$$(a,b) \mapsto V_{\varphi,\psi}(a,b) := \varphi\Big(\frac{ab+ba}{2}\Big) + \psi\Big(\frac{ab-ba}{2}\Big)$$

defines a continuous bilinear form on A which also satisfies the following property: given $a,b \in A$ with $a \perp b$ (i.e. $ab^* = b^*a = 0$) we have $V_{\varphi,\psi}(a,b^*) = 0$. A continuous bilinear form $V : A\times A \to \mathbb{C}$ is said to be *orthogonal* when $V(a,b) = 0$ for every $a,b \in A_{sa}$ with $a \perp b$ (see [15, Definition 1.1]). A renowned and useful theorem, due to S. Goldstein [15], gives the precise expression of every continuous bilinear orthogonal form on a C*-algebra.

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Theorem 1.1. [15] *Let $V : A \times A \to \mathbb{C}$ be a continuous orthogonal form on a C*-algebra. Then there exist functionals $\varphi, \psi \in A^*$ satisfying that*

$$V(a,b) = V_{\varphi,\psi}(a,b) = \varphi(a \circ b) + \psi([a,b]),$$

for all $a, b \in A$, where $a \circ b := \frac{1}{2}(ab + ba)$, and $[a,b] := \frac{1}{2}(ab - ba)$. □

Henceforth, the term "form" will mean a "continuous bilinear form". It should be noted here that by the above Goldstein's theorem, for every orthogonal form V on a C*-algebra we also have $V(a, b^*) = 0$, for every $a, b \in A$ with $a \perp b$.

The applications of Goldstein's theorem appear in many different contexts ([5, 17]). Quite recently, an extension of Goldstein's theorem for commutative real C*-algebras has been published in [14].

Making use of the weak amenability of every C*-algebra, U. Haagerup and N.J. Laustsen gave a simplified proof of Goldstein's theorem in [17]. In the third section of the just quoted paper, and more concretely, in the proof of [17, Proposition 3.5], the above mentioned authors pointed out that for every anti-symmetric form V on a C*-algebra A which is orthogonal on A_{sa}, the mapping $D_{_V} : A \to A^*$, $D_{_V}(a)(b) = V(a,b)$ $(a, b \in A)$ is a derivation. Reciprocally, the weak amenability of A also implies that every derivation δ from A into A^* is inner and hence of the form $\delta(a) = \text{adj}_{\phi}(a) = \phi a - a\phi$ for a functional $\phi \in A^*$. In particular, the form $V_\delta(a,b) = \delta(a)(b)$ is anti-symmetric and orthogonal.

The above results are the starting point and motivation of the present note. In the setting of C*-algebras we shall complete the above picture showing that symmetric orthogonal forms on a C*-algebra A are in bijective correspondence with the *purely Jordan generalized derivations* from A into A^* (see Section 2 for definitions). However, the main goal of this note is to explore the orthogonal forms on a JB*-algebra and the similarities and differences between the associative setting of C*-algebras and the wider class of JB*-algebras.

In Section 2 we revisit the basic theory and results on Jordan modules and derivations from the associative derivations on C*-algebras to Jordan derivations on C*-algebras and JB*-algebras. The novelties presented in this section include a new study about generalized Jordan derivations from a JB*-algebra $\mathcal{J}$ into a Jordan Banach $\mathcal{J}$-module in the line explored in [24], [1, §4], and [7, §3]. We recall that, given a Jordan Banach $\mathcal{J}$-module X over a JB*-algebra, a *generalized Jordan derivation* from $\mathcal{J}$ into X is a linear mapping $G : \mathcal{J} \to X$ for which there exists $\xi \in X^{**}$ satisfying

$$G(a \circ b) = G(a) \circ b + a \circ G(b) - U_{a,b}(\xi),$$

for every a, b in $\mathcal{J}$, where

$$U_{a,b}(x) := (a \circ x) \circ b + (b \circ x) \circ a - (a \circ b) \circ x \quad (x \in X^{**}).$$

We show how the results on automatic continuity of Jordan derivations from a JB*-algebra $\mathcal{J}$ into itself or into its dual, established by S. Hejazian, A. Niknam [19] and B. Russo and the second author of this paper in [26], can be applied to prove that every generalized Jordan derivation from $\mathcal{J}$ into $\mathcal{J}$ or into $\mathcal{J}^*$ is continuous (see Proposition 2.1).

Section 3 contains the main results of the paper. In Proposition 3.8 we prove that for every generalized Jordan derivation $G : \mathcal{J} \to \mathcal{J}^*$, where $\mathcal{J}$ is a JB*-algebra, the form $V_G : \mathcal{J} \times \mathcal{J} \to \mathbb{C}$, $V_G(a,b) = G(a)(b)$ is orthogonal on the whole $\mathcal{J}$. We introduce the two new subclasses of *purely Jordan generalized Jordan derivations* and *Lie Jordan derivations*. A generalized derivation $G : \mathcal{J} \to \mathcal{J}^*$ is said to be a purely Jordan generalized derivation if $G(a)(b) = G(b)(a)$, for every $a, b \in \mathcal{J}$; while a Lie Jordan derivation is a Jordan derivation $D : \mathcal{J} \to \mathcal{J}^*$ satisfying $D(a)(b) = -D(b)(a)$, for all $a, b \in \mathcal{J}$.

Denote by $\mathcal{OF}_s(\mathcal{J})$ the Banach space of all symmetric orthogonal forms on $\mathcal{J}$, and by $\mathcal{PJGD}er(\mathcal{J}, \mathcal{J}^*)$ the Banach space of all purely Jordan generalized Jordan derivations from $\mathcal{J}$ into $\mathcal{J}^*$. The mappings

$$\mathcal{OF}_s(\mathcal{J}) \to \mathcal{PJGD}\text{er}(\mathcal{J}, \mathcal{J}^*), \quad \mathcal{PJGD}\text{er}(\mathcal{J}, \mathcal{J}^*) \to \mathcal{OF}_s(\mathcal{J}),$$
$$V \mapsto G_{_V}, \qquad\qquad G \mapsto V_{_G},$$

define two isometric linear bijections and are inverses of each other (cf. Theorem 3.6). Let now $\mathcal{OF}_{as}(\mathcal{J})$ and $\mathcal{L}ie\mathcal{JD}er(\mathcal{J}, \mathcal{J}^*)$ denote the Banach spaces of all anti-symmetric orthogonal forms on $\mathcal{J}$, and of all Lie Jordan derivations from $\mathcal{J}$ into $\mathcal{J}^*$, respectively. The mappings

$$\mathcal{OF}_{as}(\mathcal{J}) \to \mathcal{L}\text{ie}\mathcal{JD}\text{er}(\mathcal{J}, \mathcal{J}^*), \quad \mathcal{L}\text{ie}\mathcal{JD}\text{er}(\mathcal{J}, \mathcal{J}^*) \to \mathcal{OF}_{as}(\mathcal{J}),$$
$$V \mapsto D_{_V}, \qquad\qquad D \mapsto V_{_D},$$

define two isometric linear bijections and are inverses of each other (see Theorem 3.13).

We culminate the paper with a short discussion which shows that, contrary to what happens for anti-symmetric orthogonal forms on a C*-algebra, the anti-symmetric orthogonal forms on a JB*-algebra are not determined by the inner Jordan derivations from $\mathcal{J}$ into $\mathcal{J}^*$ (see Remark 3.15). It seems unnecessary to stress the high impact and deep repercussion of the theory of derivations on C*-algebras and JB*-algebras; the results in this note add a new interest and applications of Jordan derivations and generalized Jordan derivations on JB*-algebras.

Throughout this paper, we habitually consider a Banach space X as a norm closed subspace of X^{**}. Given a closed subspace Y of X, we shall identify the weak*-closure, in X^{**}, of Y with Y^{**}.

2. Derivations and generalized derivations in correspondence with orthogonal forms

A *derivation* from a Banach algebra A into a Banach A-module X is a linear map $D : A \to X$ satisfying $D(ab) = D(a)b + aD(b)$, $(a \in A)$. A *Jordan derivation* from A into X is a linear map D satisfying $D(a^2) = aD(a) + D(a)a$, $(a \in A)$, or equivalently, $D(a \circ b) = a \circ D(b) + D(a) \circ b$ $(a, b \in A)$, where $a \circ b = \frac{ab+ba}{2}$, whenever $a, b \in A$, or one of a, b is in A and the other is in X. Let x be an element of X, the mapping $\text{adj}_x : A \to X$, $a \mapsto \text{adj}_x(a) := xa - ax$, is an example of a derivation from A into X. A derivation $D : A \to X$ is said to be *inner* when it can be written in the form $D = \text{adj}_x$ for some $x \in X$.

A well known result of S. Sakai (cf. [29, Theorem 4.1.6]) states that every derivation on a von Neumann algebra is inner.

J.R. Ringrose proved in [28] that every derivation from a C*-algebra A into a Banach A-bimodule is continuous.

A Banach algebra A is *amenable* if every bounded derivation from A into a dual Banach A-bimodule is inner. Contributions of A. Connes and U. Haagerup show that a C*-algebra is amenable if and only if it is nuclear ([11, 16]). The class of weakly amenable Banach algebras is less restrictive. A Banach algebra A is *weakly amenable* if every bounded derivation from A into A^* is inner. U. Haagerup proved that every C*-algebra B is weakly amenable, that is, for every derivation $D : B \to B^*$, there exists $\varphi \in B^*$ satisfying $D(.) = \text{adj}_\varphi$ ([16, Corollary 4.2]).

In [24] J. Li and Zh. Pan introduced a concept which generalizes the notion of derivation and is more related to the Jordan structure underlying a C*-algebra. We recall that a linear mapping G from a unital C*-algebra A to a (unital) Banach A-bimodule X is called a *generalized derivation* in [24] whenever the identity

$$G(ab) = G(a)b + aG(b) - aG(1)b$$

holds for every a, b in A. The non-unital case was studied in [1, §4], where a generalized derivation from a Banach algebra A to a Banach A-bimodule X is defined as a linear operator $D : A \to X$ for which there exists $\xi \in X^{**}$ satisfying

$$D(ab) = D(a)b + aD(b) - a\xi b \ (a, b \in A).$$

Given an element x in X, it is easy to see that the operator $G_x : A \to X$, $x \mapsto G_x(a) := ax + xa$, is a generalized derivation from A into X. Clearly, every derivation from A into X is a generalized derivation. There are examples of generalized derivations from a C*-algebra A into a Banach A-bimodule X which are not derivations, for example $G_a : A \to A$ is a generalized derivation which is not a derivation when $a^* \neq -a$ (cf. [6, comments after Lemma 3]).

2.1. **Jordan algebras and modules.** We turn now our attention to Jordan structures and derivations. We recall that a real (resp., complex) *Jordan algebra* is a commutative algebra over the real (resp., complex) field which is not, in general associative, but satisfies the *Jordan identity*:

$$(a \circ b) \circ a^2 = a \circ (b \circ a^2). \tag{2.1}$$

A normed Jordan algebra is a Jordan algebra $\mathcal{J}$ equipped with a norm, $\|.\|$, satisfying $\|a \circ b\| \leq \|a\| \, \|b\|$, $a, b \in \mathcal{J}$. A *Jordan Banach algebra* is a normed Jordan algebra whose norm is complete. A JB*-algebra is a complex Jordan Banach algebra $\mathcal{J}$ equipped with an isometric algebra involution * satisfying $\| \{a, a^*, a\} \| = \|a\|^3$, $a \in \mathcal{J}$ (we recall that $\{a, a^*, a\} = 2(a \circ a^*) \circ a - a^2 \circ a^*$). A real Jordan Banach algebra $\mathcal{J}$ satisfying

$$\|a\|^2 = \|a^2\| \text{ and, } \|a^2\| \leq \|a^2 + b^2\|,$$

for every $a, b \in \mathcal{J}$ is called a *JB-algebra*. JB-algebras are precisely the self adjoint parts of JB*-algebras [9, page 174]. A JBW*-algebra is a JB*-algebra which is a dual Banach space (see [18, §4] for a detailed presentation with basic properties).

Every real or complex associative Banach algebra is a real or complex Jordan Banach algebra with respect to the natural Jordan product $a \circ b = \frac{1}{2}(ab + ba)$.

Let $\mathcal{J}$ be a Jordan algebra. A *Jordan $\mathcal{J}$-module* is a vector space X, equipped with a couple of bilinear products $(a, x) \mapsto a \circ x$ and $(x, a) \mapsto x \circ a$ from $\mathcal{J} \times X$ to X, satisfying:

$$a \circ x = x \circ a, \quad a^2 \circ (x \circ a) = (a^2 \circ x) \circ a, \text{ and,} \tag{2.2}$$

$$2((x \circ a) \circ b) \circ a + x \circ (a^2 \circ b) = 2(x \circ a) \circ (a \circ b) + (x \circ b) \circ a^2, \tag{2.3}$$

for every $a, b \in \mathcal{J}$ and $x \in X$. When X is a Banach space and a Jordan $\mathcal{J}$-module for which there exists $M > 0$ satisfying $\|a \circ x\| \leq M \ \|a\| \ \|x\|$, we say that X is a Jordan-Banach $\mathcal{J}$-module. For example, every associative Banach A-bimodule over a Banach algebra A is a Jordan-Banach A-module for the product $a \circ x = \frac{1}{2}(ax + xa)$ $(a \in A, x \in X)$. The dual, $\mathcal{J}^*$, of a Jordan Banach algebra $\mathcal{J}$ is a Jordan-Banach J-module with respect to the product

$$(a \circ \varphi)(b) = \varphi(a \circ b), \tag{2.4}$$

where $a, b \in \mathcal{J}$, $\varphi \in \mathcal{J}^*$.

Given a Banach A-bimodule X over a C*-algebra A (respectively, a Jordan Banach $\mathcal{J}$-module over a JB*-algebra $\mathcal{J}$), it is very useful to consider X^{**} as a Banach A-bimodule or as a Banach A^{**}-bimodule (respectively, as a Jordan Banach $\mathcal{J}$-module or as a Jordan Banach $\mathcal{J}^{**}$-module). The case of Banach bimodules over C*-algebras is very well dealt with in the literature (see [12] or [7, §3]), we recall here the basic facts: Let X, Y and Z be Banach spaces and let $m : X \times Y \to Z$ be a bounded bilinear mapping. Defining $m^*(z', x)(y) := z'(m(x, y))$ $(x \in X, y \in Y, z' \in Z^*)$, we obtain a bounded bilinear mapping $m^* : Z^* \times X \to Y^*$. Iterating the process, we define a mapping $m^{***} : X^{**} \times Y^{**} \to Z^{**}$. The mapping $x'' \mapsto m^{***}(x'', y'')$ is weak* to weak* continuous whenever we fix $y'' \in Y^{**}$, and the mapping $y'' \mapsto m^{***}(x, y'')$ is weak* to weak* continuous for every $x \in X$. One can consider the transposed mapping $m^t : Y \times X \to Z$, $m^t(y, x) = m(x, y)$ and the extended mapping $m^{t***t} : X^{**} \times Y^{**} \to Z^{**}$. In this case, the mapping $x'' \mapsto m^{t***t}(x'', y)$ is weak* to weak* continuous whenever we fix $y \in Y$, and the mapping $y'' \mapsto m^{t***t}(x'', y'')$ is weak* to weak* continuous for every $x'' \in X^{**}$.

In general, the mappings m^{t***t} and m^{***} do not coincide (cf. [2]). When $m^{t***t} = m^{***}$, we say that m is Arens regular. When m is Arens regular, its (unique) third Arens transpose m^{***} is separately weak* continuous (see [2, Theorem 3.3]). It is well known that the product of every C*-algebra A is Arens regular and the unique Arens extension of the product of A to $A^{**} \times A^{**}$ coincides with the product of its enveloping von Neumann algebra (cf. [12, Corollary 3.2.37]). Combining [2, Theorem 3.3] with [18, Theorem 4.4.3], we can deduce that the product of every JB*-algebra $\mathcal{J}$ is Arens regular and the unique Arens extension of the product of $\mathcal{J}$ to $\mathcal{J}^{**} \times \mathcal{J}^{**}$ coincides with the product of $\mathcal{J}^{**}$ given by [18, Theorem 4.4.3]. The literature contains some other results assuring that certain bilinear operators are Arens regular. For example, if every operator from X into Y^* is weakly compact and the same property holds for every operator from Y into X^*, then it follows from [4, Theorem 1] that every bounded bilinear mapping $m : X \times Y \to Z$ is Arens regular. It is known that every bounded operator from a JB*-algebra into the dual of another JB*-algebra is weakly compact (cf. [10, Corollary 3]), thus given a JB*-algebra $\mathcal{J}$, every bilinear mapping $m : \mathcal{J} \times \mathcal{J} \to Z$ is Arens regular.

Let X be a Banach A-bimodule over a C*-algebra A. Let us denote by

$$\pi_1 : A \times X \to X, \text{ and } \pi_2 : X \times A \to X,$$

the bilinear maps given by the corresponding module operations, that is, $\pi_1(a, x) = ax$, and $\pi_2(x, a) = xa$, respectively. The third Arens bitransposes $\pi_1^{***} : A^{**} \times X^{**} \to X^{**}$, and $\pi_2^{***} : X^{**} \times A^{**} \to X^{**}$ satisfy that $\pi_1^{***}(a, x)$ defines a weak* to weak* linear operator whenever we fix $x \in X^{**}$, or whenever we fix $a \in A$, respectively, while $\pi_2^{***}(x, a)$ defines a weak* to weak* linear operator whenever we fix $x \in X$, and $a \in A^{**}$, respectively. From now on, given $a \in A^{**}$, $z \in X^{**}$, $b \in \mathcal{J}$ and $y \in Y^{**}$, we shall frequently write $az = \pi_1^{***}(a, z)$, $za = \pi_2^{***}(z, a)$, and $b \circ y = \pi^{***}(b, y)$, respectively. Let (a_λ), and (x_μ) be nets in A and X, such that $a_\lambda \to a \in A^{**}$, and $x_\mu \to x \in X^{**}$, in the respective weak* topologies. It follows from the above properties that

$$\pi_1^{***}(a, x) = \lim_\lambda \lim_\mu a_\lambda x_\mu, \text{ and } \pi_2^{***}(x, a) = \lim_\mu \lim_\lambda x_\mu a_\lambda, \tag{2.5}$$

in the weak* topology of X^{**}. It follows from above properties that X^{**} is a Banach A^{**}-bimodule for the above operations (cf. [12, Theorem 2.6.15(iii)]).

In the Jordan setting, we do not know of any reference asserting that the bidual Y^{**} of a Jordan Banach $\mathcal{J}$-module Y over a JB*-algebra $\mathcal{J}$ is a Jordan Banach $\mathcal{J}^{**}$-module, this is for the moment an open problem. However, in the particular case of $Y = \mathcal{J}^*$, it is quite easy and natural to check that $\mathcal{J}^{***}$ is a Jordan Banach $\mathcal{J}^{**}$-module with respect to the product defined in (2.4). That is, given $\varphi \in \mathcal{J}^{***}$ and $a \in \mathcal{J}^{**}$, let us define $\varphi \circ a = a \circ \varphi \in \mathcal{J}^{***}$ as the functional determined by $(\varphi \circ a)(y) := \varphi(a \circ y)$ $(y \in \mathcal{J}^{**})$.

2.2. **Jordan derivations.** Let X be a Jordan-Banach module over a Jordan Banach algebra $\mathcal{J}$. A *Jordan derivation* from $\mathcal{J}$ into X is a linear map $D : \mathcal{J} \to X$ satisfying:

$$D(a \circ b) = D(a) \circ b + a \circ D(b).$$

Following standard notation, given $x \in X$ and $a \in \mathcal{J}$, the symbols $L(a)$ and $L(x)$ will denote the mappings $L(a) : X \to X$, $x \mapsto L(a)(x) = a \circ x$ and $L(x) : \mathcal{J} \to X$, $a \mapsto L(x)(a) = a \circ x$. By a little abuse of notation, we also denote by $L(a)$ the operator on $\mathcal{J}$ defined by $L(a)(b) = a \circ b$. Examples of Jordan derivations can be given as follows: if we fix $a \in \mathcal{J}$ and $x \in X$, the mapping

$$[L(x), L(a)] = L(x)L(a) - L(a)L(x) : \mathcal{J} \to X, \ b \mapsto [L(x), L(a)](b),$$

is a Jordan derivation. A derivation $D : \mathcal{J} \to X$ that can be written in the form $D = \sum_{i=1}^m (L(x_i)L(a_i) - L(a_i)L(x_i))$, $(x_i \in X, a_i \in \mathcal{J})$ is called *inner*.

In 1996, B.E. Johnson proved that every bounded Jordan derivation from a C*-algebra A to a Banach A-bimodule is a derivation (cf. [22]). B. Russo and the second author of this paper showed that every Jordan derivation from a C*-algebra A to a Banach A-bimodule or to a Jordan Banach A-module is continuous (cf. [26, Corollary 17]). Actually every Jordan derivation from a JB*-algebra $\mathcal{J}$ into $\mathcal{J}$ or into $\mathcal{J}^*$ is continuous (cf. [19, Corollary 2.3] and also [26, Corollary 10]).

Contrary to Sakai's theorem, which affirms that every derivation on a von Neumann algebra is inner [29, Theorem 4.1.6], there exist examples of JBW*-algebras admitting non-inner derivations (cf. [30, Theorem 3.5 and Example 3.7]). Following [20], a JB*-algebra $\mathcal{J}$ is said to be *Jordan weakly amenable*, if every (bounded) derivation from $\mathcal{J}$ into $\mathcal{J}^*$ is inner. Another difference between C*-algebras and JB*-algebras is that Jordan algebras do not exhibit a good behaviour concerning Jordan weak amenability; for example $L(H)$ and $K(H)$ are not Jordan weakly amenable when H is an infinite dimensional complex Hilbert space (cf. [20, Lemmas 4.1 and 4.3]). Jordan weak amenability is deeply connected with the more general notion of ternary weak amenability (see [20]). More interesting results on ternary weak amenability were recently developed by R. Pluta and B. Russo in [27].

Let us assume that $\mathcal{J}$ and X are unital. Following [6], a linear mapping $G:\mathcal{J}\to X$ will be called a *generalised Jordan derivation* whenever

$$G(a\circ b)=G(a)\circ b+a\circ G(b)-U_{a,b}G(1),$$

for every a,b in $\mathcal{J}$, where $U_{a,b}(x):=(a\circ x)\circ b+(b\circ x)\circ a-(a\circ b)\circ x$ ($x\in\mathcal{J}$ or $x\in X$). Following standard notation, given an element a in a JB*-algebra $\mathcal{J}$, the mapping $U_{a,a}$ is usually denoted by U_a. Every generalized Jordan derivation $G:\mathcal{J}\to X$ with $G(1)=0$ is a Jordan derivation. Every Jordan derivation from $\mathcal{J}$ into X is a generalized derivation. For each $x\in X$, the mapping $L(x):\mathcal{J}\to X$ is a generalized derivation, and, as in the associative setting, there are examples of generalized derivations which are not derivations (cf. [6, comments after Lemma 3]). In the not necessarily unital case, a linear mapping $G:\mathcal{J}\to X$ will be called a *generalized Jordan derivation* if there exists $\xi\in X^{**}$ satisfying

$$G(a\circ b)=G(a)\circ b+a\circ G(b)-U_{a,b}(\xi), \tag{2.6}$$

for every a,b in $\mathcal{J}$ (this definition was introduced in [1, §4] and in [7, §3]).

Let $\mathcal{J}$ be a JB*-algebra and let Y denote $\mathcal{J}$ or $\mathcal{J}^*$, regarded as a Jordan Banach $\mathcal{J}$-module. Suppose $G:\mathcal{J}\to Y$ is a generalized derivation, and let $\xi\in Y^{**}$ denote the element for which (2.6) holds. As we have commented before, $L(\xi):\mathcal{J}\to Y^{**}$ is a generalized Jordan derivation. If we regard G as a linear mapping from $\mathcal{J}$ into Y^{**}, it is not hard to check that $\widetilde{G}=G-L(\xi):\mathcal{J}\to Y^{**}$ is a Jordan derivation. Corollary 2.3 in [19] implies that $\widetilde{G}$ is continuous. If, in the setting of C*-algebras, we replace [19, Corollary 2.3] with [26, Corollary 17], then the above arguments remain valid and yield:

Proposition 2.1. *Every generalized Jordan derivation from a JB*-algebra $\mathcal{J}$ into itself or into $\mathcal{J}^*$ is continuous. Furthermore, every generalized derivation from a C*-algebra A into a Banach A-bimodule is continuous.* □

A consequence of the result established by T. Ho, B. Russo and the second author of this note in [20, Proposition 2.1] is that for every Jordan derivation D from a JB*-algebra $\mathcal{J}$ into its dual, its bitranspose $D^{**}:\mathcal{J}^{**}\to\mathcal{J}^{***}$ is a Jordan derivation and $D^{**}(\mathcal{J}^{**})\subseteq\mathcal{J}^{*}$. A similar technique gives:

Proposition 2.2. *Let $\mathcal{J}$ be a JB-algebra or a JB*-algebra, and suppose that $G:\mathcal{J}\to\mathcal{J}^{*}$ is a generalized Jordan derivation (respectively, a Jordan derivation). Then $G^{**}:\mathcal{J}^{**}\to\mathcal{J}^{***}$ is a weak*-continuous generalized Jordan derivation (respectively, Jordan derivation) satisfying $G^{**}(\mathcal{J}^{**})\subseteq\mathcal{J}^{*}$.*

Proof. Suppose first that $\mathcal{J}$ is a JB-algebra. It is known that $\widehat{\mathcal{J}}=\mathcal{J}+i\mathcal{J}$ can be equipped with a structure of JB*-algebra such that $\widehat{\mathcal{J}}_{sa}=\mathcal{J}$ (cf. [9, page 174]). It is easy to check that, given a generalized Jordan derivation $G:\mathcal{J}\to\mathcal{J}^{*}$, the mapping $\widehat{G}:\widehat{\mathcal{J}}\to\widehat{\mathcal{J}}^{*}$, $\widehat{G}(a+ib)=G(a)+iG(b)$ $(a,b\in\mathcal{J})$ defines a generalized Jordan derivation on $\widehat{\mathcal{J}}$, where, as usually, for $\varphi\in\mathcal{J}^{*}$, we regard $\varphi:\widehat{\mathcal{J}}\to\mathbb{C}$ as defined by $\varphi(a+ib)=\varphi(a)+i\varphi(b)$. We may therefore assume that $\mathcal{J}$ is a JB*-algebra.

By Proposition 2.1, every generalized Jordan derivation $G:\mathcal{J}\to\mathcal{J}^{*}$ is automatically continuous. Furthermore, since every bounded operator from a JB*-algebra into the dual of another JB*-algebra is weakly compact (cf. [10, Corollary 3]), we deduce that G is weakly compact. It is well known that this is equivalent to $G^{**}(\mathcal{J}^{**})\subset\mathcal{J}^{*}$.

Since $G:\mathcal{J}\to\mathcal{J}^{*}$ is a generalized Jordan derivation, there exists $\xi\in\mathcal{J}^{***}$ satisfying

$$G(x\circ y)=G(x)\circ y+x\circ G(y)-U_{x,y}(\xi),$$

for every x,y in $\mathcal{J}$. Let a and b be elements in $\mathcal{J}^{**}$. By Goldstine's Theorem, we can find two (bounded) nets (a_λ) and (b_μ) in $\mathcal{J}$ such that $(a_\lambda)\to a$ and $(b_\mu)\to b$ in the weak*-topology of $\mathcal{J}^{**}$. If we fix an element c in $\mathcal{J}^{**}$, and we take a net (ϕ_λ) in $\mathcal{J}^{***}$, converging to some $\phi\in\mathcal{J}^{***}$ in the $\sigma(\mathcal{J}^{***},\mathcal{J}^{**})$-topology, the net $(\phi_\lambda\circ c)$ converges in the $\sigma(\mathcal{J}^{***},\mathcal{J}^{**})$-topology to $\phi\circ c$. The weak*-continuity of the mapping G^{**} implies that

$$G^{**}(a\circ c)=\text{w}^*\text{-}\lim_{\lambda}G(a_\lambda\circ c)=\text{w}^*\text{-}\lim_{\lambda}G(a_\lambda)\circ c+a_\lambda\circ G(c)-U_{a_\lambda,c}(\xi)$$

$$=G^{**}(a)\circ c+a\circ G(c)-U_{a,c}(\xi),$$

for every $c\in\mathcal{J}$. This shows that $G^{**}(a\circ c)=G^{**}(a)\circ c+a\circ G(c)-U_{a,c}(\xi)$, for every $c\in\mathcal{J}$, $a\in\mathcal{J}^{**}$. Therefore

$$G^{**}(a\circ b)=\text{w}^*\text{-}\lim_{\mu}G^{**}(a\circ b_\mu)=\text{w}^*\text{-}\lim_{\mu}G^{**}(a)\circ b_\mu+a\circ G(b_\mu)-U_{a,b_\mu}(\xi)$$

$$=G^{**}(a)\circ b+a\circ G^{**}(b)-U_{a,b}(\xi),$$

giving the desired conclusion. □

Remark 2.3. Let $G:\mathcal{J}\to\mathcal{J}^{*}$ be a generalized Jordan derivation, where $\mathcal{J}$ is a JB*-algebra. Let $\xi\in\mathcal{J}^{***}$ satisfy

$$G(a\circ b)=G(a)\circ b+a\circ G(b)-U_{a,b}(\xi),$$

for every a, b in $\mathcal{J}$. The previous Proposition 2.2 assures that $G^{**} : \mathcal{J}^{**} \to \mathcal{J}^{***}$ is a weak*-continuous generalized Jordan derivation, $G^{**}(\mathcal{J}^{**}) \subseteq \mathcal{J}^*$, and

$$G^{**}(a \circ b) = G^{**}(a) \circ b + a \circ G^{**}(b) - U_{a,b}(\xi),$$

for every a, b in $\mathcal{J}^{**}$. In particular, $G^{**}(1) = \xi \in \mathcal{J}^*$, and G is a Jordan derivation if and only if $G^{**}(1) = 0$.

3. Orthogonal forms

In the non-associative setting of JB*-algebras, a Jordan version of Goldstein's theorem remains unexplored. In this section we shall study the structure of the orthogonal forms on a JB*-algebra $\mathcal{J}$. In this non-associative setting, the lacking of a Jordan version of Goldstein's theorem makes, a priori, unclear whether a form on $\mathcal{J}$ which is orthogonal on $\mathcal{J}_{sa}$ is orthogonal on the whole of $\mathcal{J}$. We shall prove that symmetric orthogonal forms on a JB*-algebra $\mathcal{J}$ are in one to one correspondence with the *purely Jordan generalized Jordan derivations* from $\mathcal{J}$ into $\mathcal{J}^*$ (see Theorem 3.6), while anti-symmetric orthogonal forms on $\mathcal{J}$ are in one to one correspondence with the *Lie Jordan derivations* from $\mathcal{J}$ into $\mathcal{J}^*$ (see Theorem 3.13). These results, together with the existence of JB*-algebras $\mathcal{J}$ which are not Jordan weakly amenable (i.e., they admit Jordan derivations from $\mathcal{J}$ into $\mathcal{J}^*$ which are not inner), show that a Jordan version of Goldstein's theorem for anti-symmetric orthogonal forms on a JB*-algebra is a hopeless task (see Remark 3.15).

We introduce next the exact definitions. In a JB*-algebra $\mathcal{J}$ we consider the following triple product

$$\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*.$$

When equipped with this triple product and its norm, every JB*-algebra becomes an element in the class of JB*-triples introduced by W. Kaup in [23]. The precise definition of JB*-triples reads as follows: A *JB*-triple* is a complex Banach space E equipped with a continuous triple product $\{\cdot, \cdot, \cdot\} : E \times E \times E \to E$ which is linear and symmetric in the outer variables, conjugate linear in the middle one and satisfies the following conditions:

(JB*-1) (Jordan identity) for a, b, x, y, z in E,

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\};$$

(JB*-2) $L(a, a) : E \to E$ is an hermitian (linear) operator with non-negative spectrum, where $L(a, b)(x) = \{a, b, x\}$ with $a, b, x \in E$;

(JB*-3) $\|\{x, x, x\}\| = \|x\|^3$ for all $x \in E$.

We refer to the monographs [18], [9], and [8] for the basic background on JB*-algebras and JB*-triples.

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual [3]). It is known that the triple product of a JBW*-triple is separately weak*-continuous [3]. A result due to S. Dineen establishes that the second dual of a JB*-triple E is a JBW*-triple with a product extending that of E (compare [9, Corollary 3.3.5]).

An element e in a JB*-triple E is said to be a *tripotent* if $\{e,e,e\} = e$. Each tripotent e in E gives raise to the so-called *Peirce decomposition* of E associated to e, that is,

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0, 1, 2$, $E_i(e)$ is the $\frac{i}{2}$ eigenspace of $L(e,e)$. The Peirce decomposition satisfies certain rules known as *Peirce arithmetic*:

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e),$$

if $i - j + k \in \{0, 1, 2\}$ and is zero otherwise. In addition,

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0.$$

The corresponding *Peirce projections* are denoted by $P_i(e) : E \to E_i(e)$, $(i = 0, 1, 2)$. The Peirce space $E_2(e)$ is a JB*-algebra with product $x \bullet_e y := \{x, e, y\}$ and involution $x^{\sharp_e} := \{e, x, e\}$.

For each element x in a JB*-triple E, we shall denote $x^{[1]} := x$, $x^{[3]} := \{x,x,x\}$, and $x^{[2n+1]} := \left\{x, x, x^{[2n-1]}\right\}$, $(n \in \mathbb{N})$. The symbol E_x will stand for the JB*-subtriple generated by the element x. It is known that E_x is JB*-triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space Ω contained in $(0, \|x\|]$, such that $\Omega \cup \{0\}$ is compact, where $C_0(\Omega)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. It is also known that we can find a triple isomorphism Ψ from E_x onto $C_0(\Omega)$, such that $\Psi(x)(t) = t$ $(t \in \Omega)$ (cf. Corollary 1.15 in [23]).

Therefore, for each $x \in E$, there exists a unique element $y \in E_x$ satisfying that $\{y,y,y\} = x$. The element y, denoted by $x^{[\frac{1}{3}]}$, is termed the *cubic root* of x. We can inductively define, $x^{[\frac{1}{3^n}]} = \left(x^{[\frac{1}{3^{n-1}}]}\right)^{[\frac{1}{3}]}$, $n \in \mathbb{N}$. The sequence $(x^{[\frac{1}{3^n}]})$ converges in the weak*-topology of E^{**} to a tripotent denoted by $r(x)$ and called the *range tripotent* of x. The element $r(x)$ is the smallest tripotent $e \in E^{**}$ such that x is positive in the JBW*-algebra $E_2^{**}(e)$ (compare [13], Lemma 3.3).

Elements a, b in a JB*-algebra $\mathcal{J}$, or more generally, in a JB*-triple E, are said to be *orthogonal* (denoted by $a \perp b$) when $L(a,b) = 0$, that is, the triple product $\{a,b,c\}$ vanishes for every $c \in \mathcal{J}$ or in E ([5]). An application of [5, Lemma 1] assures that $a \perp b$ if and only if one of the following statements holds:

$$\begin{array}{lll} \{a,a,b\} = 0; & a \perp r(b); & r(a) \perp r(b); \\ E_2^{**}(r(a)) \perp E_2^{**}(r(b)); & r(a) \in E_0^{**}(r(b)); & a \in E_0^{**}(r(b)); \\ b \in E_0^{**}(r(a)); & E_a \perp E_b & \{b,b,a\} = 0. \end{array} \quad (3.1)$$

The above equivalences imply, in particular, that the relation of being orthogonal is a "local concept", more precisely, $a \perp b$ in $\mathcal{J}$ (respectively in E) if and only if $a \perp b$ in a JB*-subalgebra (respectively, JB*-subtriple) $\mathcal{K}$ containing a and b.

Suppose $a \perp b$ in $\mathcal{J}$, applying the above arguments we can always assume that $\mathcal{J}$ is unital. In this case, $a \circ b^* = \{a, b, 1\} = 0$ and $(a \circ a^*) \circ b - (a \circ b) \circ a^* =$

$(a \circ a^*) \circ b + (b \circ a^*) \circ a - (a \circ b) \circ a^* = 0$, therefore $a \circ b^* = 0$ and $(a \circ a^*) \circ b = (a \circ b) \circ a^*$. Actually the last two identities also imply that $a \perp b$. It follows that

$$a \perp b \Leftrightarrow a \circ b^* = 0 \text{ and } (a \circ a^*) \circ b = (a \circ b) \circ a^*. \tag{3.2}$$

So, if $a \perp b$ and c is another element in $\mathcal{J}$, we deduce, via Jordan identity, that

$$\{U_a(c), U_a(c), b\} = \{\{a, c^*, a\}, \{a, c^*, a\}, b\} = -\{c^*, a, \{\{a, c^*, a\}, a, b\}\}$$
$$+\{\{c^*, a, \{a, c^*, a\}\}, a, b\} + \{\{a, c^*, a\}, a, \{c^*, a, b\}\} = 0,$$

which shows that $U_a(c) \perp b$.

We shall also make use of the following fact

$$a \perp b \text{ in } \mathcal{J} \Rightarrow (c \circ b^*) \circ a = (a \circ c) \circ b^*, \tag{3.3}$$

for every $c \in \mathcal{J}$, this means that a and b^* operator commute in $\mathcal{J}$ (cf. [5, page 225]). For the proof, we observe that, since $a \perp b$, $a \circ b^* = 0$, and the involution preserves triple products, we have $0 = \{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$, which proves the desired equality. A direct application of (3.3) and (3.2) shows that

$$a \perp b \text{ in } \mathcal{J} \Rightarrow (a^2) \circ b^* = (a \circ b^*) \circ a = 0. \tag{3.4}$$

When a C*-algebra A is regarded with its structure of JB*-algebra, elements a, b in A are orthogonal in the associative sense if and only if they are orthogonal in the Jordan sense.

Definition 3.1. A form $V : \mathcal{J} \times \mathcal{J} \to \mathbb{C}$ is said to be orthogonal when $V(a, b^*) = 0$ for every $a, b \in \mathcal{J}$ with $a \perp b$. If $V(a, b) = 0$ only for elements $a, b \in \mathcal{J}_{sa}$ with $a \perp b$, we shall say that V is orthogonal on $\mathcal{J}_{sa}$.

3.1. **Purely Jordan generalized Jordan derivations and symmetric orthogonal forms.** We begin this subsection by dealing with symmetric orthogonal forms on a C*-algebra, a setting in which these forms have been already studied. Let $V : A \times A \to X$ be a symmetric, orthogonal form on a C*-algebra. By Goldstein's theorem (cf. Theorem [15]), there exists a unique functional $\phi_V \in A^*$ satisfying that $V(a, b) = \phi_V(a \circ b)$ for all $a, b \in A$. The statement also follows from the studies of orthogonally additive n-homogeneous polynomials on C*-algebras developed in [25].

Given an element a in the self adjoint part $\mathcal{J}_{sa}$ of a JBW*-algebra $\mathcal{J}$, there exists a smallest projection $r(a)$ in $\mathcal{J}$ with the property that $r(a) \circ a = a$. We call $r(a)$ the range projection of a, and it is further known that $r(a)$ belongs JBW*-subalgebra of $\mathcal{J}$ generated by a. It is easy to check that $r(a)$ coincides with the range tripotent of a in $\mathcal{J}$ when the latter is seen as a JBW*-triple, so, our notation is consistent with the previous definitions.

We explore now the symmetric orthogonal forms on a JB*-algebra.

Proposition 3.2. *Let $V : \mathcal{J} \times \mathcal{J} \to \mathbb{C}$ be a symmetric form on a JB*-algebra which is orthogonal on $\mathcal{J}_{sa}$. Then there exists a unique $\phi \in \mathcal{J}^*$ satisfying*

$$V(a, b) = \phi(a \circ b),$$

for every $a, b \in \mathcal{J}$.

Proof. We have already commented that the (unique) third Arens transpose $V^{***}: \mathcal{J}^{**} \times \mathcal{J}^{**} \to \mathbb{C}$ is separately weak*-continuous (cf. Subsection 2.1). Let a be a self-adjoint element in $\mathcal{J}$. It is known that the JB*-subalgebra $\mathcal{J}_a$ generated by a is JB*-isometrically isomorphic to a commutative C*-algebra (cf. [18, §3]). Since the restricted mapping $V|_{\mathcal{J}_a \times \mathcal{J}_a} : \mathcal{J}_a \times \mathcal{J}_a \to \mathbb{C}$ is a symmetric orthogonal form, there exists a functional $\phi_a \in (\mathcal{J}_a)^*$ satisfying that

$$V(c, d) = \phi_a(c \circ d),$$

for every $c, d \in \mathcal{J}_a$ (cf. Theorem 1.1). It follows from the weak*-density of $\mathcal{J}_a$ in $(\mathcal{J}_a)^{**}$ together with the separate weak*-continuity of V^{***}, and the weak*-continuity of ϕ_a, that

$$V^{***}(c, d) = \phi_a(c \circ d),$$

for every $c, d \in (\mathcal{J}_a)^{**}$. Taking $c = a$ and $d = r(a)$ the range projection of a we get

$$V(a, a) = \phi_a(a \circ a) = \phi_a(a^2 \circ r(a)) = V^{***}(a^2, r(a)) = V^{***}(r(a), a^2), \tag{3.5}$$

for every $a \in \mathcal{J}_{sa}$.

We claim that

$$V^{***}(a, r(a)) = V^{***}(r(a), a) = V^{***}(a, 1) = V^{***}(1, a), \tag{3.6}$$

for every positive $a \in \mathcal{J}_{sa}$. We may assume that $\|a\| = 1$. We actually know that there is a set $L \subset [0, 1]$ with $L \cup \{0\}$ compact such that $\mathcal{J}_a$ is isomorphic to the C*-algebra $C_0(L)$ of all continuous complex-valued functions on L vanishing at 0, and under this isometric identification the element a is identified with the function $t \mapsto t$. Given $\varepsilon > 0$, let $p_\varepsilon = \chi_{[\varepsilon,1]}$ denote the projection in $(\mathcal{J}_a)^{**}$, which coincides with the characteristic function of the set $[\varepsilon, 1] \cap L$. Clearly, $p_\varepsilon \leq r(a)$ in $\mathcal{J}^{**}$. Suppose we have a function $g \in \mathcal{J}_a \equiv C_0(L)$ satisfying $p_\varepsilon \circ g = g \geq 0$, that is, the cozero set of g is inside the interval $[\varepsilon, 1]$.

Take a sequence $(h_n) \subset C_0(L)$ defined by

$$h_n(t) := \begin{cases} 1, & \text{if } t \in L \cap [\varepsilon - \frac{1}{2n}, 1]; \\ \text{affine}, & \text{if } t \in L \cap [\varepsilon - \frac{1}{n}, \varepsilon - \frac{1}{2n}]; \\ 0, & \text{if } t \in L \cap [0, \varepsilon - \frac{1}{n}] \end{cases}$$

for n large enough ($n \geq m_0$). The sequence (h_n) converges to p_ε in the weak*-topology of $(\mathcal{J}_a)^{**}$ and $1 - h_n \perp p_\varepsilon, g$. So, $\mathcal{J} \ni U_{1-h_n}(c) \perp g$ for every $c \in \mathcal{J}$ and $n \geq m_0$. Since $1 \in \mathcal{J}^{**}$, we can find, via Goldstine's theorem, a net $(c_\gamma) \subset \mathcal{J}$ converging to 1 in the weak* topology of $\mathcal{J}^{**}$. By hypothesis, $0 = V(U_{1-h_n}(c_\gamma), g)$, for every $\lambda, n \geq m_0$. Taking weak* limits in γ and in n, it follows from the separate weak* continuity of V^{***}, that

$$V^{***}(1 - p_\varepsilon, g) = 0 \tag{3.7}$$

for every p_ε and g as above. If we take

$$g_\varepsilon(t) := \begin{cases} t, & \text{if } t \in L \cap [2\varepsilon, 1]; \\ \text{affine}, & \text{if } t \in L \cap [\varepsilon, 2\varepsilon]; \\ 0, & \text{if } t \in L \cap [0, \varepsilon], \end{cases}$$

then $0 \leq g_\varepsilon \leq p_\eta$, for every $\eta \leq \varepsilon$, $\lim_{\varepsilon\to 0} \|g_\varepsilon - a\| = 0$ and weak*-$\lim_{\eta\to 0} p_\eta = r(a)$. Combining these facts with (3.7) and the separate weak*-continuity of V^{***}, we get $V^{***}(1 - r(a), a) = 0$, which proves (3.6).

The identities in (3.5) and (3.6) show that $V(a, a) = V^{***}(1, a^2)$, for every $a \in \mathcal{J}_{sa}$. Let us define $\phi = V^{***}(1, .) \in A^*$. A polarization formula, and V being symmetric imply that $V(a, b) = V^{***}(1, a \circ b) = \phi(a \circ b)$, for every $a, b \in \mathcal{J}_{sa}$, and by bilinearity $V(a, b) = \phi(a \circ b)$, for every $a, b \in \mathcal{J}$. □

The previous proposition is a generalization of Goldstein's theorem for symmetric orthogonal forms. It can be also regarded as a characterization of orthogonally additive 2-homogeneous polynomials on a JB*-algebra $\mathcal{J}$. More concretely, according to the notation in [25], a 2-homogeneous polynomial $P : \mathcal{J} \to \mathbb{C}$ is orthogonally additive on $\mathcal{J}_{sa}$ (i.e., $P(a+b) = P(a)+P(b)$ for every $a \perp b$ in $\mathcal{J}_{sa}$) if, and only if, there exists a unique $\phi \in \mathcal{J}^*$ satisfying $P(a) = \phi(a^2)$, for every $a \in \mathcal{J}$. This characterization constitutes an extension of [25, Theorem 2.8] to the setting of JB*-algebras.

Remark 3.3. Let $V : \mathcal{J} \times \mathcal{J} \to \mathbb{C}$ be a symmetric form on a JB*-algebra. The above Proposition 3.2 implies that V is orthogonal if and only if it is orthogonal on $\mathcal{J}_{sa}$.

Let $V : \mathcal{J} \times \mathcal{J} \to \mathbb{C}$ be a symmetric orthogonal form on a JB*-algebra, and let ϕ_V be the unique functional in $\mathcal{J}^*$ given by Proposition 3.2. If we define $G_{_V} : \mathcal{J} \to \mathcal{J}^*$, the operator given by $G_{_V}(a) = V(a, .)$, we can conclude that $G_{_V}(a) = \phi_{_V} \circ a = G_{\phi_{_V}}(a)$, and hence $G_{_V} : \mathcal{J} \to \mathcal{J}^*$ is a generalized Jordan derivation and $V(a, b) = G_{_V}(a)(b)$ $(a, b \in \mathcal{J})$. Moreover, for every $a, b \in \mathcal{J}$, $G_{_V}(a)(b) = V(a, b) = V(b, a) = G_{_V}(b)(a)$. This fact motivates the following definition:

Definition 3.4. Let $\mathcal{J}$ be a JB*-algebra. A *purely Jordan generalized Jordan derivation* from $\mathcal{J}$ into $\mathcal{J}^*$ is a generalized Jordan derivation $G : \mathcal{J} \to \mathcal{J}^*$ satisfying $G(a)(b) = G(b)(a)$, for every $a, b \in \mathcal{J}$.

We have already seen that every symmetric orthogonal form V on a JB*-algebra $\mathcal{J}$ determines a purely Jordan generalized Jordan derivation $G_V : \mathcal{J} \to \mathcal{J}^*$. To explore the reciprocal implication we shall prove that every generalized derivation from $\mathcal{J}$ into $\mathcal{J}^*$ defines an orthogonal form on $\mathcal{J}_{sa}$.

Proposition 3.5. *Let $G : \mathcal{J} \to \mathcal{J}^*$ be a generalized Jordan derivation, where $\mathcal{J}$ is a JB*-algebra. Then the form $V_G : \mathcal{J} \times \mathcal{J} \to \mathbb{C}$, $V_G(a, b) = G(a)(b)$ is orthogonal on $\mathcal{J}_{sa}$.*

Proof. Let $G : \mathcal{J} \to \mathcal{J}^*$ be a generalized Jordan derivation. By Proposition 2.1, G is continuous, and by Proposition 2.2, $G^{**} : \mathcal{J}^{**} \to \mathcal{J}^*$ is a generalized Jordan derivation too. Let ξ denote $G^{**}(1)$.

Let p be a projection in $\mathcal{J}^{**}$ and let b be any element in $\mathcal{J}^{**}$ such that $p \perp b$. Since

$$G^{**}(p) = G^{**}(p \circ p) = 2p \circ G^{**}(p) + U_p(\xi),$$

we deduce that

$$G^{**}(p)(b^*) = 2G^{**}(p)(p \circ b^*) + \xi(U_p(b^*)) = 0. \tag{3.8}$$

Let a be a symmetric element in $\mathcal{J}^{**}$, and let b be any element in $\mathcal{J}^{**}$ satisfying $a \perp b$. By (3.1), the JBW*-algebra $\mathcal{J}_a^{**}$ generated by a is orthogonal to b, that is, $c \perp b$ for every $c \in \mathcal{J}_a^{**}$. It is well known that a can be approximated in norm by finite linear combinations of mutually orthogonal projections in $\mathcal{J}_a^{**}$ (cf. [18, Proposition 4.2.3]). It follows from (3.8), the continuity of G^{**}, and the previous comments that

$$V_{G^{**}}(a, b^*) = G^{**}(a)(b^*) = 0,$$

for every $a \in \mathcal{J}_{sa}^{**}$ and every $b \in \mathcal{J}^{**}$ with $a \perp b$. □

Our next result follows now as a consequence of Proposition 3.2, Remark 3.3, and Proposition 3.5.

Theorem 3.6. *Let $\mathcal{J}$ be a JB*-algebra. Let $\mathcal{OF}_s(\mathcal{J})$ denote the Banach space of all symmetric orthogonal forms on $\mathcal{J}$, and let $\mathcal{PJGDer}(\mathcal{J}, \mathcal{J}^*)$ the Banach space of all purely Jordan generalized Jordan derivations from $\mathcal{J}$ into $\mathcal{J}^*$. For each $V \in \mathcal{OF}_s(\mathcal{J})$ define $G_V : \mathcal{J} \to \mathcal{J}^*$ in $\mathcal{PJGDer}(\mathcal{J}, \mathcal{J}^*)$ given by $G_V(a)(b) = V(a, b)$, and for each $G \in \mathcal{PJGDer}(\mathcal{J}, \mathcal{J}^*)$ we set $V_G : \mathcal{J} \times \mathcal{J} \to C$, $V_G(a, b) := G(a)(b)$ $(a, b \in \mathcal{J})$. Then the mappings*

$$\mathcal{OF}_s(\mathcal{J}) \to \mathcal{PJGDer}(\mathcal{J}, \mathcal{J}^*), \quad \mathcal{PJGDer}(\mathcal{J}, \mathcal{J}^*) \to \mathcal{OF}_s(\mathcal{J}),$$

$$V \mapsto G_V, \qquad\qquad G \mapsto V_G,$$

define two isometric linear bijections and are inverses of each other. □

Actually, Proposition 3.2 gives a bit more:

Corollary 3.7. *Let $\mathcal{J}$ be a JB*-algebra. Then, for every purely Jordan generalized Jordan derivation $G : \mathcal{J} \to \mathcal{J}^*$ there exists a unique $\phi \in \mathcal{J}^*$, such that $G = G_\phi$, that is, $G(a) = \phi \circ a$ $(a \in \mathcal{J})$.*

3.2. **Derivations and anti-symmetric orthogonal forms.** We focus now our study on the anti-symmetric orthogonal forms on a JB*-algebra. We motivate our study with the case of a C*-algebra A. By Goldstein's theorem every anti-symmetric orthogonal form V on A writes in the form $V(a, b) = \psi([a, b]) = \psi(ab - ba)$ $(a, b \in A)$, where $\psi \in A^*$ (cf. Theorem 1.1). Unfortunately, ψ is not uniquely determined by V (see [15, Proposition 2.6 and comments prior to it]). Anyway, the operator $D_V : A \to A^*$, $D_V(a)(b) = V(a, b) = [\psi, a](b)$ defines a derivation from A into A^* and $D_V(a)(b) = -D_V(b)(a)$ $(a, b \in A)$. On the other hand, if $D : A \to A^*$ is a derivation, it follows from the weak amenability of A (cf. [16, Corollary 4.2]), that there exists $\psi \in A^*$ satisfying $D(a) = [a, \psi]$. Therefore, the form $V : A \times A \to \mathbb{C}$, $V_D(a, b) = D(a)(b)$ is orthogonal and anti-symmetric. However, when A is replaced with a JB*-algebra, the Lie product doesn't make any sense. To avoid the gap, we shall consider Jordan derivations.

It seems natural to ask whether the class of anti-symmetric orthogonal forms on a JB*-algebra $\mathcal{J}$ is empty or not. Here is an example: let $c_1, \ldots, c_m \in \mathcal{J}$ and

$\phi_1, \ldots, \phi_m \in \mathcal{J}^*$, and define $V : \mathcal{J} \times \mathcal{J} \to \mathbb{C}$,

$$V(a,b) := \left(\sum_{i=1}^{m} \left[L(\phi_i), L(c_i) \right](a) \right)(b) \tag{3.9}$$

$$= \left(\sum_{i=1}^{m} \left(\phi_i \circ (c_i \circ a) - c_i \circ (\phi_i \circ a) \right) \right)(b) = \sum_{i=1}^{m} \phi_i \left(b \circ (c_i \circ a) - (c_i \circ b) \circ a \right),$$

for every $a, b \in \mathcal{J}$. Clearly, V is an anti-symmetric form on $\mathcal{J}$. It follows from (3.3) that $V(a, b^*) = 0$ for every $a \perp b$ in $\mathcal{J}$, that is, V is an orthogonal form on $\mathcal{J}$. Further, the inner Jordan derivation $D : \mathcal{J} \to \mathcal{J}^*$, $D = \sum_{i=1}^{m} \left(L(\phi_i) L(a_i) - L(a_i) L(\phi_i) \right)$ satisfies $V(a,b) = D(a)(b)$ for every $a, b \in \mathcal{J}$.

We shall see now that, like in the case of C*-algebras and in the previous example, Jordan derivations from a JB*-algebra $\mathcal{J}$ into its dual exhaust all the possibilities to produce an anti-symmetric orthogonal form on $\mathcal{J}$. We begin with an strengthened version of Proposition 3.5.

Proposition 3.8. *Let $G : \mathcal{J} \to \mathcal{J}^*$ be a generalized Jordan derivation, where $\mathcal{J}$ is a JB*-algebra. Then the form $V_G : \mathcal{J} \times \mathcal{J} \to \mathbb{C}$, $V_G(a,b) = G(a)(b)$ is orthogonal (on the whole $\mathcal{J}$).*

Proof. We already know that every generalized Jordan derivation $G : \mathcal{J} \to \mathcal{J}^*$ is continuous (cf. Proposition 2.1). By Proposition 2.2, $G^{**} : \mathcal{J}^{**} \to \mathcal{J}^*$ is a generalized Jordan derivation too. Let $\xi = G^{**}(1)$.

Let e be a tripotent in $\mathcal{J}^{**}$ and let b be any element in $\mathcal{J}^{**}$ such that $e \perp b$. Since $\{e, e, e\} = 2(e \circ e^*) \circ e - e^2 \circ e^* = e$ we deduce that

$$\begin{aligned} G^{**}(e) &= 2G^{**}((e \circ e^*) \circ e) - G^{**}(e^2 \circ e^*) \\ &= 2G^{**}(e \circ e^*) \circ e + 2(e \circ e^*) \circ G^{**}(e) - 2U_{e \circ e^*, e}(\xi) \\ &\quad - G^{**}(e^2) \circ e^* - e^2 \circ G^{**}(e^*) + U_{e^2, e^*}(\xi). \end{aligned}$$

Therefore,

$$\begin{aligned} G^{**}(e)(b^*) &= 2G^{**}(e \circ e^*)\Big(b^* \circ e\Big) + 2G^{**}(e)\Big((e \circ e^*) \circ b^*\Big) \qquad (3.10) \\ &\quad -2\xi\Big((e \circ e^*) \circ (e \circ b^*) + ((e \circ e^*) \circ b^*) \circ e - ((e \circ e^*) \circ e) \circ b^*\Big) \\ &\quad -G^{**}(e^2)\Big(e^* \circ b^*\Big) - G^{**}(e^*)\Big(e^2 \circ b^*\Big) + \xi\Big(e^2 \circ (e^* \circ b^*) + (e^2 \circ b^*) \circ e^* - (e^2 \circ e^*) \circ b^*\Big) \\ &= \text{(by (3.2), (3.3), and (3.4))} \ = 2G^{**}(e)\Big((e \circ e^*) \circ b^*\Big) - G^{**}(e^2)\Big(e^* \circ b^*\Big) \\ &\quad + \xi\Big(e^2 \circ (e^* \circ b^*) - (e^2 \circ e^*) \circ b^*\Big) \\ &= 2G^{**}(e)\Big((e \circ e^*) \circ b^*\Big) - 2\left(e \circ G^{**}(e)\right)\Big(e^* \circ b^*\Big) + U_e(\xi)(e^* \circ b^*) \\ &\quad + \xi\Big(e^2 \circ (e^* \circ b^*) - (e^2 \circ e^*) \circ b^*\Big) \\ &= 2G^{**}(e)\Big((e \circ e^*) \circ b^* - (b^* \circ e^*) \circ e\Big) + \xi\Big(2e \circ (e \circ (e^* \circ b^*)) - e^2 \circ (e^* \circ b^*)\Big) \\ &\quad + \xi\Big(e^2 \circ (e^* \circ b^*) - (e^2 \circ e^*) \circ b^*\Big) \end{aligned}$$

$$= (\text{by } (3.3)) \ = \xi\Big(2e\circ(e\circ(e^*\circ b^*)) - (e^2\circ e^*)\circ b^*\Big)$$
$$= ((3.3) \text{ applied twice}) \ = \xi\Big(2b^*\circ(e\circ(e^*\circ e)) - b^*\circ(e^2\circ e^*)\Big)$$
$$= \xi\Big(b^*\circ\Big(2(e\circ(e^*\circ e)) - (e^2\circ e^*)\Big)\Big) = \xi\Big(b^*\circ\{e,e,e\}\Big) = \xi\Big(b^*\circ e\Big) = 0,$$

where in the last step we applied (3.2).

Let us take $a.b$ in $\mathcal{J}^{**}$, with $a\perp b$. The characterizations given in (3.1) imply that the JBW*-triple $\mathcal{J}_a^{**}$ generated by a is orthogonal to b, that is, $c\perp b$ for every $c\in\mathcal{J}_a^{**}$. Lemma 3.11 in [21] guarantees that the element a can be approximated in norm by finite linear combinations of mutually orthogonal projections in $\mathcal{J}_a^{**}$. Finally, the fact proved in (3.10), the continuity of G^{**}, and the previous comments imply that $V_{G^{**}}(a,b^*) = G^{**}(a)(b^*) = 0$. □

We shall prove next that every anti-symmetric orthogonal form is given by a Jordan derivation.

Proposition 3.9. *Let $V:\mathcal{J}\times\mathcal{J}\to\mathbb{C}$ be an anti-symmetric form on a JB*-algebra which is orthogonal on $\mathcal{J}_{sa}$. Then the mapping $D_V:\mathcal{J}\to\mathcal{J}^*$, $D_V(a)(b) = V(a,b)$ $(a,b\in\mathcal{J})$ is a Jordan derivation.*

Our strategy will follow some of the arguments given by U. Haagerup and N.J. Laustsen in [17, §3], the Jordan setting will require some simple adaptations and particularizations. The proof will be divided into several lemmas. The next lemma was established in [17, Lemma 3.3] for associative Banach algebras, however the proof, which is left to the reader, is also valid for JB*-algebras.

Lemma 3.10. *Let $V:\mathcal{J}\times\mathcal{J}\to\mathbb{C}$ be a form on a JB*-algebra. Suppose that $f,g:\mathbb{R}\to\mathcal{J}$ are infinitely differentiable functions at a point $t_0\in\mathbb{R}$. Then the map $t\mapsto V(f(t),g(t))$, $\mathbb{R}\to\mathbb{C}$, is infinitely differentiable at t_0 and its n'th derivative is given by*

$$\sum_{k=0}^{n}\binom{n}{k} V(f^{(k)}(t_0), g^{(n-k)}(t_0)).$$

□

The next lemma is also due to Haagerup and Laustsen, who established it for associative Banach algebras in [17, Lemma 3.4]. The proof given in the just quoted paper remains valid in the Jordan setting, the details are included here for completeness reasons.

Lemma 3.11. *Let $\mathcal{J}$ be a Jordan Banach algebra, let $\mathcal{U}$ be an additive subgroup of $\mathcal{J}$ whose linear span coincides with $\mathcal{J}$. Let $V:\mathcal{J}\times\mathcal{J}\to\mathbb{C}$ be an anti-symmetric form satisfying $V(a^2,a)=0$ for every $a\in\mathcal{U}$. Then the bounded linear operator $D_V:\mathcal{J}\to\mathcal{J}^*$ given by $D_V(a)(b)=V(a,b)$ for all $a,b\in\mathcal{J}$ is a Jordan derivation.*

Proof. Let us take $a,b\in\mathcal{U}$. It follows from our hypothesis that

$$D_V(a^2)(b) - 2\Big(a\circ D_V(a)\Big)(b) = D_V(a^2)(b) - 2D_V(a)(a\circ b)$$
$$= V(a^2,b) + 2V(a\circ b,a) = V(a^2,b) - 2V(a,a\circ b)$$

$$= \frac{V((a+b)^2, a+b) - V((a-b)^2, a-b) - 2V(b^2, b)}{2} = 0.$$

This implies that $D_V(a^2)(b) = 2\Big(a \circ D_V(a)\Big)(b)$, for every $a, b \in \mathcal{U}$. It follows from the bilinearity and continuity of V, and the norm density of the linear span of $\mathcal{U}$ that $D_V(a^2) = 2a \circ D_V(a)$, for every $a \in \mathcal{J}$, witnessing that $D_V : \mathcal{J} \to \mathcal{J}^*$ is a Jordan derivation. □

We deal now with the proof of Proposition 3.9.

Proof of Proposition 3.9. For each $a \in \mathcal{J}_{sa}$, let B denote the JB*-subalgebra of $\mathcal{J}$ generated by a. It is known that B is isometrically isomorphic to a commutative C*-algebra (see [18, Theorem 3.2.2 and 3.2.3]). Clearly, $V|_{B\times B} : B \times B \to \mathbb{C}$ is an anti-symmetric form which is orthogonal on B_{sa} (and hence orthogonal on B). Since B is a commutative unital C*-algebra, an application of Goldstein's theorem (cf. Theorem 1.1) shows that $V(x, y) = 0$. for every $x, y \in B$. In particular, $V(a^2, a) = 0$ for every $a \in \mathcal{J}_{sa}$. Lemma 3.11 guarantees that $D_V : \mathcal{J} \to \mathcal{J}^*$ is a Jordan derivation. Clearly, $D_V(a)(b) = -D_V(b)(a)$, for every $a, b \in \mathcal{J}$. □

Definition 3.12. Let $\mathcal{J}$ be a JB*-algebra. A Jordan derivation D from $\mathcal{J}$ into $\mathcal{J}^*$ is said to be a *Lie Jordan derivation* if $D(a)(b) = -D(b)(a)$, for every $a, b \in \mathcal{J}$.

Propositions 3.8 and 3.9 give:

Theorem 3.13. *Let $\mathcal{J}$ be a JB*-algebra. Let $\mathcal{OF}_{as}(\mathcal{J})$ denote the Banach space of all anti-symmetric orthogonal forms on $\mathcal{J}$, and let $\mathcal{L}ie\mathcal{J}\mathcal{D}er(\mathcal{J}, \mathcal{J}^*)$ the Banach space of all Lie Jordan derivations from $\mathcal{J}$ into $\mathcal{J}^*$. For each $V \in \mathcal{OF}_{as}(\mathcal{J})$ we define $D_V : \mathcal{J} \to \mathcal{J}^*$ in $\mathcal{L}ie\mathcal{J}\mathcal{D}er(\mathcal{J}, \mathcal{J}^*)$ given by $D_V(a)(b) = V(a, b)$, and for each $D \in \mathcal{L}ie\mathcal{J}\mathcal{D}er(\mathcal{J}, \mathcal{J}^*)$ we set $V_D : \mathcal{J} \times \mathcal{J} \to C$, $V_D(a, b) := D(a)(b)$ $(a, b \in \mathcal{J})$. Then the mappings*

$$\begin{array}{cc} \mathcal{OF}_{as}(\mathcal{J}) \to \mathcal{L}ie\mathcal{J}\mathcal{D}er(\mathcal{J}, \mathcal{J}^*), & \mathcal{L}ie\mathcal{J}\mathcal{D}er(\mathcal{J}, \mathcal{J}^*) \to \mathcal{OF}_{as}(\mathcal{J}), \\ V \mapsto D_V, & D \mapsto V_D, \end{array}$$

define two isometric linear bijections and are inverses of each other. □

Our final result subsumes the main conclusions of the last subsections.

Corollary 3.14. *Let $V : \mathcal{J} \times \mathcal{J} \to \mathbb{C}$ be a form on a JB*-algebra. The following statements are equivalent:*

(a) V is orthogonal;
(b) V is orthogonal on $\mathcal{J}_{sa}$;
(c) There exist a (unique) purely Jordan generalized Jordan derivation $G : \mathcal{J} \to \mathcal{J}^*$ and a (unique) Lie Jordan derivation $D : \mathcal{J} \to \mathcal{J}^*$ such that $V(a, b) = G(a)(b) + D(a)(b)$, for every $a, b \in \mathcal{J}$;
(d) There exist a (unique) functional $\phi \in \mathcal{J}^*$ and a (unique) Lie Jordan derivation $D : \mathcal{J} \to \mathcal{J}^*$ such that $V(a, b) = G_\phi(a)(b) + D(a)(b)$, for every $a, b \in \mathcal{J}$.

Proof. $(a) \Rightarrow (b)$ is clear. To see $(b) \Rightarrow (c)$ and $(b) \Rightarrow (d)$, we recall that every form $V : \mathcal{J} \times \mathcal{J} \to \mathbb{C}$ writes uniquely in the form $V = V_s + V_{as}$, where $V_s, V_{as} : \mathcal{J} \to \mathcal{J}^*$ are a symmetric and an anti-symmetric form on $\mathcal{J}$, respectively. Furthermore, since $V_s(a,b) = \frac{1}{2}(V(a,b)+V(b,a))$ and $V_{as}(a,b) = \frac{1}{2}(V(a,b)-V(b,a))$ $(a,b \in \mathcal{J})$, we deduce that V is orthogonal (on $\mathcal{J}_{sa}$) if and only if both V_s and V_{as} are orthogonal (on $\mathcal{J}_{sa}$). Therefore, the desired implications follow from Theorems 3.6 and 3.13. The same theorems also prove $(c) \Rightarrow (a)$ and $(d) \Rightarrow (a)$. □

We shall finish this note with an observation which helps us to understand the limitations of Goldstein theorem in the Jordan setting.

Remark 3.15. Let A be a C*-algebra, since the anti-symmetric orthogonal forms on A and the Lie Jordan derivations from A into A^* are mutually determined, we can deduce, via Goldstein's theorem (cf. Theorem 1.1), that every Lie Jordan derivation $D : A \to A^*$ is an inner derivation, i.e., a derivation given by a functional $\psi \in A^*$, that is, $D(a) = \text{adj}_\psi(a) = \psi a - a\psi$ $(a \in A)$. We shall see that a finite number of functionals in the dual of a JB*-algebra $\mathcal{J}$ and a finite collection of elements in $\mathcal{J}$, i.e. the inner Jordan derivations, are not enough to determine the Lie Jordan derivations from $\mathcal{J}$ into $\mathcal{J}^*$ nor the anti-symmetric orthogonal forms on $\mathcal{J}$. Indeed, as we have commented before, there exist examples of JB*-algebras which are not Jordan weakly amenable, that is the case of $L(H)$ and $K(H)$ when H is an infinite dimensional complex Hilbert space (cf. [20, Lemmas 4.1 and 4.3]). Actually, let $B = K(H)$ denote the ideal of all compact operators on H, and let ψ be an element in B^* whose trace is not zero. The proof of [20, Lemmas 4.1] shows that the derivation $D = \text{adj}_\psi : B \to B^*$, $a \mapsto \psi a - a\psi$ is not inner in the Jordan sense. Therefore the anti-symmetric form $V(a,b) = D(a)(b) = (\psi a - a\psi)(b) = \psi[a,b]$ cannot be represented in the form given in (3.9). A similar example holds for $B = B(H)$ (cf. [20, Lemma 4.3]).

Remark 3.16. We have already shown the existence of JBW*-algebras which are not Jordan weakly amenable (cf. [20, Lemmas 4.1 and 4.3]). Thus, the problem of determining whether in a JB*-algebra $\mathcal{J}$, the inner Jordan derivations on $\mathcal{J}$ are norm-dense in the set of all Jordan derivations on $\mathcal{J}$, takes on a new importance. If the problem has an affirmative answer for a JB*-algebra $\mathcal{J}$, Theorem 3.13 allows us to approximate anti-symmetric orthogonal forms on $\mathcal{J}$ by a finite collection of functionals in $\mathcal{J}^*$ and a finite number of elements in $\mathcal{J}$. Related to this problem, we note that Pluta and Russo recently proved that if the set of inner triple derivations from a von Neumann algebra M into its predual is norm dense in the real vector space of all triple derivations, then M must be finite, and the reciprocal statement holds if M acts on a separable Hilbert space, or is a factor [27, Theorem 1]. It would be interesting to explore the connections between normal orthogonal forms and normal Jordan weak amenability or norm approximation by normal inner derivations on JBW*-algebras.

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