

# Introduction to Operations Research

*Deterministic Models*

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# Contents

<b>1</b>	<b>Mathematical modeling by example</b>	<b>1</b>
1.1	Activity-based formulation . . . . .	3
<b>2</b>	<b>Linear Programming</b>	<b>5</b>
2.1	Formulating a linear program . . . . .	5
2.2	Summary and further tricks . . . . .	8
<b>3</b>	<b>Solving linear programs</b>	<b>10</b>
3.1	Graphical method . . . . .	10
3.2	Fourier-Motzkin Elimination (FME) . . . . .	13
<b>4</b>	<b>Simplex method</b>	<b>16</b>
4.1	Canonical form . . . . .	16
4.2	Simplex method by example . . . . .	17
4.3	Two phase Simplex method . . . . .	21
4.4	Special cases . . . . .	21
4.5	Phase I. . . . .	23
<b>5</b>	<b>Linear Algebra Review</b>	<b>27</b>
5.1	Systems of linear equations . . . . .	29
5.2	Summary . . . . .	31
<b>6</b>	<b>Sensitivity Analysis</b>	<b>33</b>
6.1	Changing the objective function . . . . .	35
6.2	Changing the right-hand side value . . . . .	37
6.3	Detailed example . . . . .	39
6.4	Adding a variable/activity . . . . .	43

6.5	Adding a constraint . . . . .	43
6.6	Modifying the left-hand side of a constraint . . . . .	44
<b>7</b>	<b>Duality</b>	<b>45</b>
7.1	Pricing interpretation . . . . .	45
7.2	Duality Theorems and Feasibility . . . . .	47
7.3	General LPs . . . . .	47
7.4	Complementary slackness . . . . .	48
<b>8</b>	<b>Other Simplex Methods</b>	<b>52</b>
8.1	Dual Simplex Method . . . . .	52
8.2	Upper-Bounded Simplex . . . . .	54
8.3	Lower bounds . . . . .	55
8.4	Dual Simplex with Upper Bounds . . . . .	56
8.5	Goal Programming . . . . .	57
<b>9</b>	<b>Transportation Problem</b>	<b>59</b>
9.1	Transportation Simplex Method . . . . .	60
<b>10</b>	<b>Network problems</b>	<b>65</b>
10.1	Shortest Path Problem . . . . .	66
10.2	Minimum Spanning Tree . . . . .	68
10.3	Maximum Flow problem . . . . .	69
10.4	Minimum-cost Flow problem . . . . .	72
10.5	Network Simplex Algorithm . . . . .	72
10.6	Network Simplex Algorithm with capacities . . . . .	76
10.7	Complete example . . . . .	77
10.8	Summary . . . . .	83
<b>11</b>	<b>Game Theory</b>	<b>85</b>
11.1	Pure and Mixed strategies . . . . .	86
11.2	Nonconstant-sum Games . . . . .	88
<b>12</b>	<b>Integer programming</b>	<b>89</b>
12.1	Problem Formulation . . . . .	89
12.2	Cutting Planes . . . . .	92
12.3	Branch and Bound . . . . .	95
<b>13</b>	<b>Dynamic Programming</b>	<b>103</b>
<b>14</b>	<b>Analysis of efficiency</b>	<b>113</b>
14.1	Algorithmic Complexity . . . . .	115

## Preface

These lecture notes were written during the Fall/Spring 2013/14 semesters to accompany lectures of the course *IEOR 4004: Introduction to Operations Research - Deterministic Models*. The notes were meant to provide a succinct summary of the material, most of which was loosely based on the book *Winston-Venkataramanan: Introduction to Mathematical Programming (4th ed.)*, Brooks/Cole 2003. Other material (such as the dictionary notation) was adapted from *Chvátal: Linear Programming*, Freeman 1983 and *Dantzig-Thapa: Linear Programming*, Springer-Verlag 1997. Various other bits were inspired by other lecture notes and sources on the Internet. These notes are not meant to replace any book; interested reader will find more details and examples in the Winston book in particular. I would like to thank students that helped me correct numerous mistakes in the earlier versions of the notes. Most likely all mistakes have not been yet caught, and so the reader should exercise caution should there be inconsistencies in the text. I am passing on the notes to Prof. Strickland who will continue making adjustments to the material as needed for the upcoming offerings of the course. Of course, any suggestions for improvements are welcomed from anyone interested.

Juraj Stacho  
July 26, 2014

## Mathematical modeling by example

### Product mix

A toy company makes two types of toys: *toy soldiers* and *trains*. Each toy is produced in two stages, first it is constructed in a carpentry shop, and then it is sent to a finishing shop, where it is varnished, waxed, and polished. To make one toy soldier costs \$10 for raw materials and \$14 for labor; it takes 1 hour in the carpentry shop, and 2 hours for finishing. To make one train costs \$9 for raw materials and \$10 for labor; it takes 1 hour in the carpentry shop, and 1 hour for finishing.

There are 80 hours available each week in the carpentry shop, and 100 hours for finishing. Each toy soldier is sold for \$27 while each train for \$21. Due to decreased demand for toy soldiers, the company plans to make and sell at most 40 toy soldiers; the number of trains is not restricted in any way.

What is the optimum (*best*) product mix (i.e., what quantities of which products to make) that *maximizes* the profit (assuming all toys produced will be sold)?

### Terminology

<b>decision variables:</b>	$x_1, x_2, \dots, x_i, \dots$
<b>variable domains:</b> values that variables can take	$x_1, x_2 \geq 0$
<b>goal/objective:</b>	maximize/minimize
<b>objective function:</b> function to minimize/maximize	$2x_1 + 5x_2$
<b>constraints:</b> equations/inequalities	$3x_1 + 2x_2 \leq 10$

### Example

Decision variables:

- $x_1$  = # of toy soldiers
- $x_2$  = # of toy trains

Objective: maximize profit

- $\$27 - \$10 - \$14 = \$3$  profit for selling one toy soldier  $\Rightarrow 3x_1$  profit (in \$) for selling  $x_1$  toy soldier
- $\$21 - \$9 - \$10 = \$2$  profit for selling one toy train  $\Rightarrow 2x_2$  profit (in \$) for selling  $x_2$  toy train

$\Rightarrow z = \underbrace{3x_1 + 2x_2}_{\text{objective function}}$  profit for selling  $x_1$  toy soldiers and  $x_2$  toy trains

Constraints:

- producing  $x_1$  toy soldiers and  $x_2$  toy trains requires
  - (a)  $1x_1 + 1x_2$  hours in the carpentry shop; there are 80 hours available
  - (b)  $2x_1 + 1x_2$  hours in the finishing shop; there are 100 hours available
- the number  $x_1$  of toy soldiers produced should be at most 40

Variable domains: the numbers  $x_1, x_2$  of toy soldiers and trains must be non-negative (**sign restriction**)

$$\begin{aligned} \text{Max } & 3x_1 + 2x_2 \\ & x_1 + x_2 \leq 80 \\ & 2x_1 + x_2 \leq 100 \\ & x_1 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

We call this a **program**. It is a **linear** program, because the objective is a linear function of the decision variables, and the constraints are linear inequalities (in the decision variables).

## Blending

A company wants to produce a certain alloy containing 30% lead, 30% zinc, and 40% tin. This is to be done by mixing certain amounts of existing alloys that can be purchased at certain prices. The company wishes to minimize the cost. There are 9 available alloys with the following composition and prices.

Alloy	1	2	3	4	5	6	7	8	9	Blend
Lead (%)	20	50	30	30	30	60	40	10	10	30
Zinc (%)	30	40	20	40	30	30	50	30	10	30
Tin (%)	50	10	50	30	40	10	10	60	80	40
Cost (\$/lb)	7.3	6.9	7.3	7.5	7.6	6.0	5.8	4.3	4.1	minimize

Designate a *decision* variables  $x_1, x_2, \dots, x_9$  where

$x_i$  is the **amount** of Alloy  $i$  in a **unit of blend**

In particular, the decision variables must satisfy  $x_1 + x_2 + \dots + x_9 = 1$ . (It is a common mistake to choose  $x_i$  the **absolute** amount of Alloy  $i$  in the blend. That may lead to a non-linear program.)

With that we can setup constraints and the objective function.

$$\begin{aligned} \text{Min } & 7.3x_1 + 6.9x_2 + 7.3x_3 + 7.5x_4 + 7.6x_5 + 6.0x_6 + 5.8x_7 + 4.3x_8 + 4.1x_9 = z \quad [\text{Cost}] \\ \text{s.t. } & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 = 1 \\ & 0.2x_1 + 0.5x_2 + 0.3x_3 + 0.3x_4 + 0.3x_5 + 0.6x_6 + 0.4x_7 + 0.1x_8 + 0.1x_9 = 0.3 \quad [\text{Lead}] \\ & 0.3x_1 + 0.4x_2 + 0.2x_3 + 0.4x_4 + 0.3x_5 + 0.3x_6 + 0.5x_7 + 0.3x_8 + 0.1x_9 = 0.3 \quad [\text{Zinc}] \\ & 0.5x_1 + 0.1x_2 + 0.5x_3 + 0.3x_4 + 0.4x_5 + 0.1x_6 + 0.1x_7 + 0.6x_8 + 0.8x_9 = 0.4 \quad [\text{Tin}] \end{aligned}$$

Do we need **all** the four equations?

## Product mix (once again)

Furniture company manufactures four models of chairs. Each chair requires certain amount of raw materials (wood/steel) to make. The company wants to decide on a production that maximizes profit (assuming all produced chair are sold). The required and available amounts of materials are as follows.

	Chair 1	Chair 2	Chair 3	Chair 4	Total available
Steel	1	1	3	9	4,400 (lbs)
Wood	4	9	7	2	6,000 (lbs)
Profit	\$12	\$20	\$18	\$40	maximize

Decision variables:

$x_i$  = the number of chairs of type  $i$  produced  
 each  $x_i$  is non-negative

Objective function:

maximize profit  $z = 12x_1 + 20x_2 + 18x_3 + 40x_4$

Costraints:

at most 4,400 lbs of steel available:  $x_1 + x_2 + 3x_3 + 9x_4 \leq 4,400$   
 at most 6,000 lbs of wood available:  $4x_1 + 9x_2 + 7x_3 + 2x_4 \leq 6,000$

Resulting program:

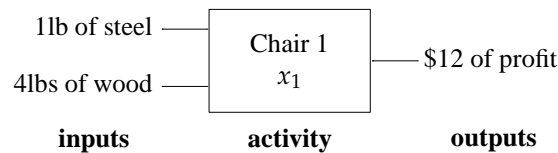
$$\begin{aligned} \text{Max } & 12x_1 + 20x_2 + 18x_3 + 40x_4 = z && \text{[Profit]} \\ \text{s.t. } & x_1 + x_2 + 3x_3 + 9x_4 \leq 4,400 && \text{[Steel]} \\ & 4x_1 + 9x_2 + 7x_3 + 2x_4 \leq 6,000 && \text{[Wood]} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

## 1.1 Activity-based formulation

Instead of constructing the formulation as before (row-by-row), we can proceed by columns.

We can view columns of the program as **activities**. An activity has

- inputs:** materials consumed per unit of activity (1lb of steel and 4lbs of wood)
- outputs:** products produced per unit of activity (\$12 of profit)
- activity level:** a level at which we operate the activity (indicated by a variable  $x_1$ )



Operating the activity “Chair 1” at level  $x_1$  means that we produce  $x_1$  chairs of type 1, each consuming 1lb of steel, 4lbs of wood, and producing \$12 of profit. Activity levels are always assumed to be **non-negative**.

The materials/labor/profit consumed or produced by an activity are called **items** (correspond to rows).

The effect of an activity on items (i.e. the amounts of items that are consumed/produced by an activity) are **input-output coefficients**.

The total amount of items available/supplied/required is called the **external flow of items**.

We choose **objective** to be one of the items which we choose to maximize or minimize.

Last step is to write **material balance equations** that express the flow of items in/out of activities and with respect to the external flow.

**Example**

Items: Steel  
 Wood  
 Profit

External flow of items:  
 Steel: 4,400lbs of available (flowing in)  
 Wood: 6,000lbs of available (flowing in)

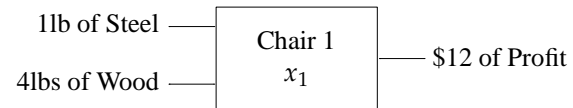
Objective:

Profit: maximize (flowing out)

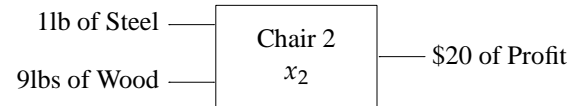
Activities:

producing a chair of type  $i$  where  $i = 1, 2, 3, 4$ , each is assigned an activity level  $x_i$

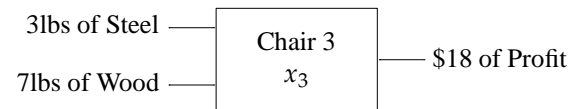
Chair 1: Producing 1 chair of type 1  
 consumes 1 lb of Steel  
 4 lbs of Wood  
 produces \$12 of Profit



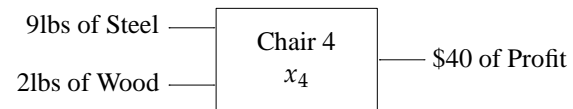
Chair 2: Producing 1 chair of type 2  
 consumes 1 lb of Steel  
 9 lbs of Wood  
 produces \$20 of Profit



Chair 3: Producing 1 chair of type 3  
 consumes 3 lbs of Steel  
 7 lbs of Wood  
 produces \$18 of Profit

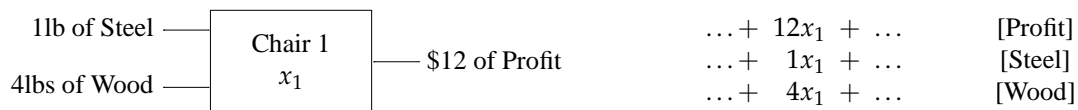


Chair 4: Producing 1 chair of type 4  
 consumes 9 lbs of Steel  
 2 lbs of Wood  
 produces \$40 of Profit



The material balance equations:

To see how to do this, consider activity Chair 1: consumes 1lb of Steel, 4lbs of Wood, and produces \$12 of Profit. Thus at level  $x_1$ , we consume  $1x_1$  lbs of Steel,  $4x_1$  lbs of Wood, and produce  $12x_1$  dollars of Profit.



On the right, you see the effect of operating the activity at level  $x_1$ . (Note in general we will adopt a different **sign convention**; we shall discuss in a later example.)

Thus considering all activities we obtain:

$$\begin{array}{rcl} 12x_1 + 20x_2 + 18x_3 + 40x_4 & \text{[Profit]} \\ x_1 + x_2 + 3x_3 + 9x_4 & \text{[Steel]} \\ 4x_1 + 9x_2 + 7x_3 + 2x_4 & \text{[Wood]} \end{array}$$

Finally, we incorporate the external flow and objective: 4,400lbs of Steel available, 6,000lbs of Wood available, maximize profit:

$$\begin{array}{rcl} \text{Max } 12x_1 + 20x_2 + 18x_3 + 40x_4 = z & \text{[Profit]} \\ \text{s.t. } x_1 + x_2 + 3x_3 + 9x_4 \leq 4,400 & \text{[Steel]} \\ 4x_1 + 9x_2 + 7x_3 + 2x_4 \leq 6,000 & \text{[Wood]} \\ x_1, x_2, x_3, x_4 \geq 0 & \end{array}$$



## Linear Programming

Linear program (LP) in a **standard form** (maximization)

$$\begin{array}{rcll}
 \max & c_1x_1 & + & c_2x_2 & + & \dots & + & c_nx_n & & \text{Objective function} \\
 \text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & \leq & b_2 \\
 & \vdots & & \vdots & & & & \vdots & & \vdots \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & x_1, x_2, \dots, x_n & \geq & 0 & \text{Sign restrictions}
 \end{array}
 \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \text{Constraints}$$

**Feasible solution** (point)  $P = (p_1, p_2, \dots, p_n)$  is an assignment of values to the  $p_1, \dots, p_n$  to variables  $x_1, \dots, x_n$  that satisfies **all** constraints and **all** sign restrictions.

**Feasible region**  $\equiv$  the set of all feasible points.

**Optimal solution**  $\equiv$  a feasible solution with maximum value of the objective function.

### 2.1 Formulating a linear program

1. Choose decision variables
2. Choose an objective and an objective function – linear function in variables
3. Choose constraints – linear inequalities
4. Choose sign restrictions

#### Example

You have \$100. You can make the following three types of investments:

**Investment A.** Every dollar invested now yields \$0.10 a year from now, and \$1.30 three years from now.

**Investment B.** Every dollar invested now yields \$0.20 a year from now and \$1.10 two years from now.

**Investment C.** Every dollar invested a year from now yields \$1.50 three years from now.

During each year leftover cash can be placed into money markets which yield 6% a year. The most that can be invested a single investment (A, B, or C) is \$50.

Formulate an LP to maximize the available cash three years from now.

Decision variables:  $x_A, x_B, x_C$ , amounts invested into Investments A, B, C, respectively  
 $y_0, y_1, y_2, y_3$  cash available/invested into money markets now, and in 1,2,3 years.

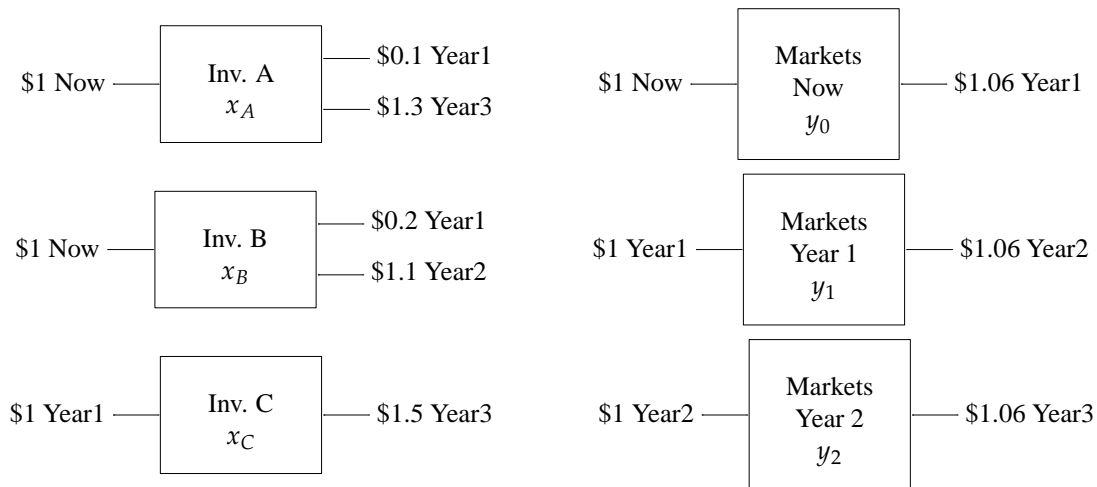
$$\begin{array}{rcl}
 \text{Max} & y_3 & \\
 \text{s.t.} & x_A + x_B & + y_0 = 100 \\
 & 0.1x_A + 0.2x_B - x_C & + 1.06y_0 = y_1 \\
 & & 1.1x_B + 1.06y_1 = y_2 \\
 & 1.3x_A & + 1.5x_C + 1.06y_2 = y_3 \\
 & x_A & \leq 50 \\
 & & x_B \leq 50 \\
 & & x_C \leq 50 \\
 & x_A, x_B, x_C, y_0, y_1, y_2, y_3 & \geq 0
 \end{array}$$

		Activities						
		Inv. A	Inv. B	Inv. C	Markets Now	Markets Year 1	Markets Year 2	External flow
Items	Now	-1	-1		-1			= -100
	Year1	0.1	0.2	-1	1.06	-1		= 0
	Year2		1.1			1.06	-1	= 0
	Year3	1.3		1.5			1.06	maximize

**Sign convention:** inputs have **negative** sign, outputs have **positive** signs.

External in-flow has **negative** sign, external out-flow has **positive** sign.

We have in-flow of \$100 cash “Now” which means we have  $-\$100$  on the right-hand side. No in-flow or out-flow of any other item.



$$\begin{array}{rcl}
 \text{Max} & 1.3x_A & + 1.5x_C & + 1.06y_2 \\
 \text{s.t.} & x_A + x_B & & + y_0 = 100 \\
 & 0.1x_A + 0.2x_B - x_C & + 1.06y_0 - y_1 & = 0 \\
 & & 1.1x_B & + 1.06y_1 - y_2 = 0 \\
 & & & y_0, y_1, y_2 \geq 0 \\
 & & & 0 \leq x_A, x_B, x_C \leq 50
 \end{array}$$

### Post office problem

Post office requires different numbers of full-time employees on different days. Each full time employee works 5 consecutive days (e.g. an employee may work from Monday to Friday or, say from Wednesday to Sunday). Post office wants to hire minimum number of employees that meet its daily requirements, which are as follows.

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
17	13	15	19	14	16	11

Let  $x_i$  denote the number of employees that **start working** in day  $i$  where  $i = 1, \dots, 7$  and work for 5 consecutive days from that day. How many workers work on Monday? Those that start on Monday, or Thursday, Friday, Saturday, or Sunday. Thus  $x_1 + x_4 + x_5 + x_6 + x_7$  should be at least 17.

Then the formulation is thus as follows:

$$\begin{array}{ll}
 \min & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\
 \text{s.t.} & x_1 + x_4 + x_5 + x_6 + x_7 \geq 17 \\
 & x_1 + x_2 + x_5 + x_6 + x_7 \geq 13 \\
 & x_1 + x_2 + x_3 + x_6 + x_7 \geq 15 \\
 & x_1 + x_2 + x_3 + x_4 + x_7 \geq 19 \\
 & x_1 + x_2 + x_3 + x_4 + x_5 \geq 14 \\
 & \phantom{x_1} + x_2 + x_3 + x_4 + x_5 + x_6 \geq 16 \\
 & \phantom{x_1} + x_3 + x_4 + x_5 + x_6 + x_7 \geq 11 \\
 & x_1, x_2, \dots, x_7 \geq 0
 \end{array}$$

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday	
Total	1	1	1	1	1	1	1	minimize
Monday	1			1	1	1	1	$\geq 17$
Tuesday	1	1			1	1	1	$\geq 13$
Wednesday	1	1	1			1	1	$\geq 15$
Thursday	1	1	1	1			1	$\geq 19$
Friday	1	1	1	1	1			$\geq 14$
Saturday		1	1	1	1	1		$\geq 16$
Sunday			1	1	1	1	1	$\geq 11$

### (Simple) Linear regression

Given a set of datapoints  $\{(1, 2), (3, 4), (4, 7)\}$  we want to find a line that most closely represents the datapoints. There are various ways to measure what it means "closely represent". We may, for instance, minimize the average distance (deviation) of the datapoints from the line, or minimize the sum of distances, or the sum of squares of distances, or minimize the maximum distance of a datapoint from the line. Here the distance can be either Euclidean distance, or vertical distance, or Manhattan distance (vertical+horizontal), or other.

We choose to minimize the maximum vertical distance of a point from the line. A general equation of a line with finite slope has form  $y = ax + c$  where  $a$  and  $c$  are parameters. For a point  $(p, q)$ , the vertical distance of the point from the line  $y = ax + c$  can be written as  $|q - ap - c|$ . Thus we want

**Problem:** Find constants  $a, c$  such that the largest of the three values  $|2 - a - c|, |4 - 3a - c|, |7 - 4a - c|$  is as small as possible.

$$\min \max \left\{ |2 - a - c|, |4 - 3a - c|, |7 - 4a - c| \right\}$$

We want to formulate it as a linear program. Issues: non-negativity, the absolute value, the min of max.

- the min of max:  $w \geq \max\{i_1, i_2, \dots, i_t\}$  if and only if  $w \geq i_1$  and  $w \geq i_2$  and  $\dots$  and  $w \geq i_t$

$$\begin{array}{ll} \text{Min} & w \\ \text{s.t.} & w \geq |2 - 1a - c| \\ & w \geq |4 - 3a - c| \\ & w \geq |7 - 4a - c| \end{array}$$

- absolute values:  $w \geq |i|$  if and only if  $w \geq i$  and  $w \geq -i$ .  
(in other words, the absolute value of  $i$  is at most  $w$  if and only if  $-w \leq i \leq w$ )

$$\begin{array}{ll} \text{Min} & w \\ \text{s.t.} & w \geq 2 - a - c \\ & w \geq -2 + a + c \\ & w \geq 4 - 3a - c \\ & w \geq -4 + 3a + c \\ & w \geq 7 - 4a - c \\ & w \geq -7 + 4a + c \end{array}$$

- unrestricted sign: write  $x = x^+ - x^-$  where  $x^+, x^- \geq 0$  are new variables

$$\begin{array}{ll} \text{Min} & w \\ \text{s.t.} & w \geq 2 - a^+ + a^- - c^+ + c^- \\ & w \geq -2 + a^+ - a^- + c^+ - c^- \\ & w \geq 4 - 3a^+ + 3a^- - c^+ + c^- \\ & w \geq -4 + 3a^+ - 3a^- + c^+ - c^- \\ & w \geq 7 - 4a^+ + 4a^- - c^+ + c^- \\ & w \geq -7 + 4a^+ - 4a^- + c^+ - c^- \\ & \text{where } a^+, a^-, c^+, c^-, w \geq 0 \end{array} \quad \begin{array}{ll} \text{Min} & w \\ \text{s.t.} & w + a^+ - a^- + c^+ - c^- \geq 2 \\ & w - a^+ + a^- - c^+ + c^- \geq -2 \\ & w + 3a^+ - 3a^- + c^+ - c^- \geq 4 \\ & w - 3a^+ + 3a^- - c^+ + c^- \geq -4 \\ & w + 4a^+ - 4a^- + c^+ - c^- \geq 7 \\ & w - 4a^+ + 4a^- - c^+ + c^- \geq -7 \\ & a^+, a^-, c^+, c^-, w \geq 0 \end{array}$$

## Note

The above formulation on the right is **standard** form of a minimization LP. We have already seen the standard form of a maximization problem; this is the same except that we minimize the objective function and the signs of inequalities switch (this is only done for convenience sake when we get to solving LPs).

## 2.2 Summary and further tricks

Let us summarize what we have learned so far.

- Linear Program (LP) is an optimization problem where
  - the **goal** is to maximize or minimize a **linear objective function**
  - over a set of **feasible solutions** – i.e. solution of a set of **linear inequalities** (forming the **feasible region**).
- **Standard form:** all inequalities are  $\leq$ -inequalities (or all are  $\geq$ -inequalities) and all variables are non-negative
  - to get a  $\leq$ -inequality from a  $\geq$ -inequality we multiply both sides by  $-1$  and reverse the sign (this gives us an equivalent problem)

$$x_1 - x_2 \leq 100 \quad \iff \quad -x_1 + x_2 \geq -100$$

- to get inequalities from an equation, we replace it by two identical inequalities, one with  $\leq$  and one with  $\geq$

$$x_1 - x_2 = 100 \quad \iff \quad \begin{array}{l} x_1 - x_2 \leq 100 \\ x_1 - x_2 \geq 100 \end{array}$$

→ each **unrestricted** variable (urs) is replaced by the **difference** of two new non-negative variables

$$\begin{array}{ccc} \dots + x_1 + \dots & \iff & \dots + (x_2 - x_3) + \dots \\ x_1 \text{ urs} & & x_2, x_3 \geq 0 \end{array}$$

→ a **non-positive** variable  $x_1 \leq 0$  is replaced by the **negative** of a new non-negative variable  $x_2$

$$\begin{array}{ccc} \dots + x_1 + \dots & \iff & \dots + (-x_2) + \dots \\ x_1 \leq 0 & & x_2 \geq 0 \end{array}$$

→ **absolute** value: we can replace **only** in certain situations

\* inequalities of type  $|f| \leq g$  where  $f$  and  $g$  are arbitrary expressions:

replace by two inequalities  $f \leq g$  and  $-g \leq f$

\* if  $|f|$  appears in the **objective function** and we are **minimizing** this function:

replace  $|f|$  in the objective function by a new variable  $x_1$  and add a constraint  $|f| \leq x_1$ .

(likewise if  $-|f|$  appears when maximizing)

→ **min of max**: if  $\max\{f_1, f_2, \dots, f_t\}$  in the **objective function** and we are **minimizing**, then replace this expression with a new variable  $x_1$  and add constraints  $f_i \leq x_1$  for each  $i = 1, \dots, t$ :

$$\begin{array}{ccc} \dots + \max\{f_1, \dots, f_t\} + \dots & \iff & \dots + x_1 + \dots \\ & & x_1 \text{ urs} \end{array} \quad \begin{array}{l} f_1 \leq x_1 \\ f_2 \leq x_1 \\ \vdots \\ f_t \leq x_1 \end{array}$$

→ **unrestricted expression**  $f$  can be written as a difference of two non-negative variables

$$\begin{array}{ccc} \dots + f + \dots & \iff & \dots + (x_2 - x_3) + \dots \\ & & x_2, x_3 \geq 0 \end{array}$$

Moreover, if we are **minimizing**, we can use  $+x_2$  and  $+x_3$  (**positive** multiples of  $x_2, x_3$ ) in the **objective function** (if maximizing, we can use negative multiples).

In an **optimal solution** the meaning of these new variables will be as follows:

- \* if  $f \geq 0$ , then  $x_2 = f$  and  $x_3 = 0$ ,
- \* if  $f < 0$ , then  $x_2 = 0$  and  $x_3 = -f$ .

In other words,  $x_2$  represents the **positive part** of  $f$ , and  $x_3$  the **negative part** of  $f$  (can you see why?). Note that this only guaranteed to hold for an optimal solution (but that will be enough for us).

**Exercise.** Try to justify for yourself why these restrictions are necessary.

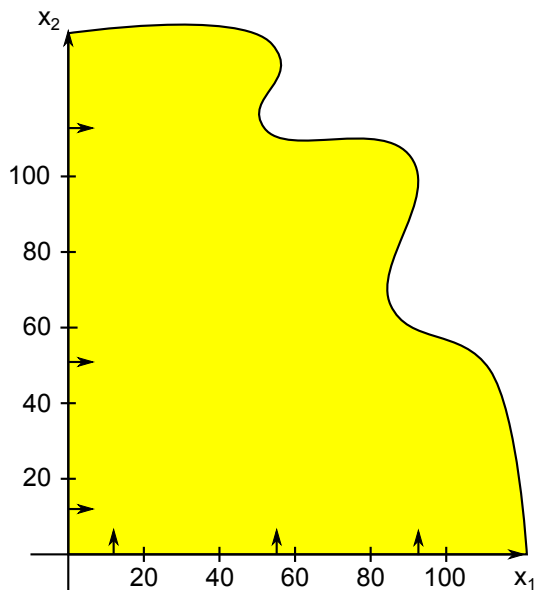
## Solving linear programs

### 3.1 Graphical method

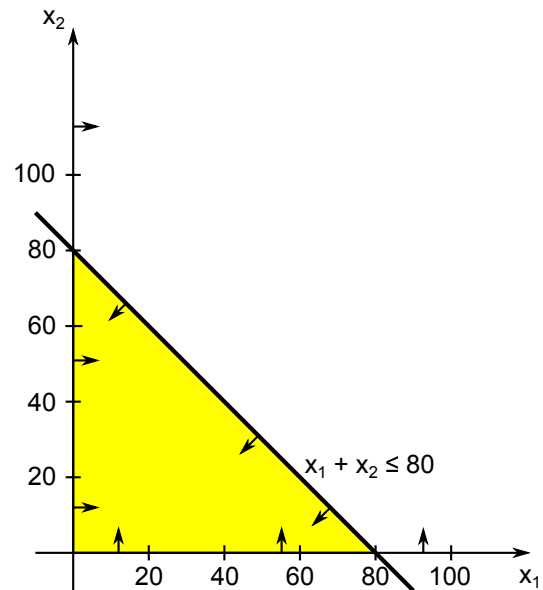
$$\begin{aligned} \text{Max } & 3x_1 + 2x_2 \\ & x_1 + x_2 \leq 80 \\ & 2x_1 + x_2 \leq 100 \\ & x_1 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

1. Find the feasible region.

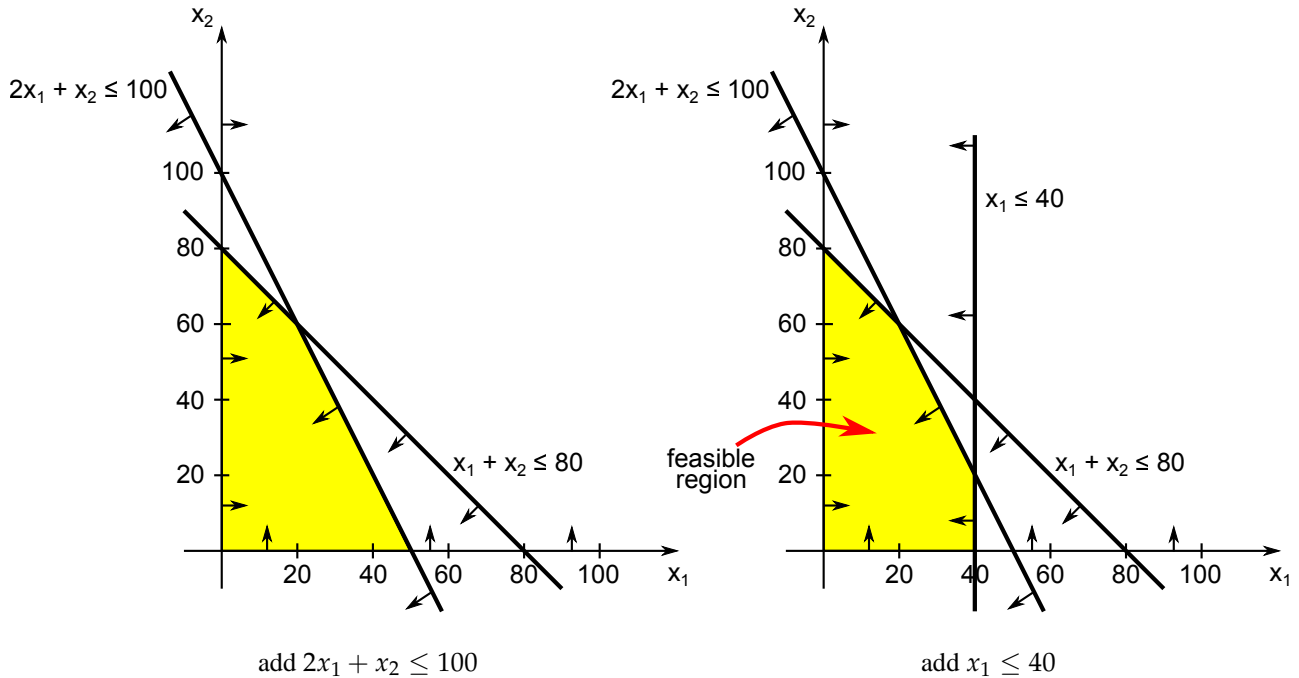
- Plot each constraint as an equation  $\equiv$  line in the plane
- Feasible points on one side of the line – plug in  $(0,0)$  to find out which



Start with  $x_1 \geq 0$  and  $x_2 \geq 0$



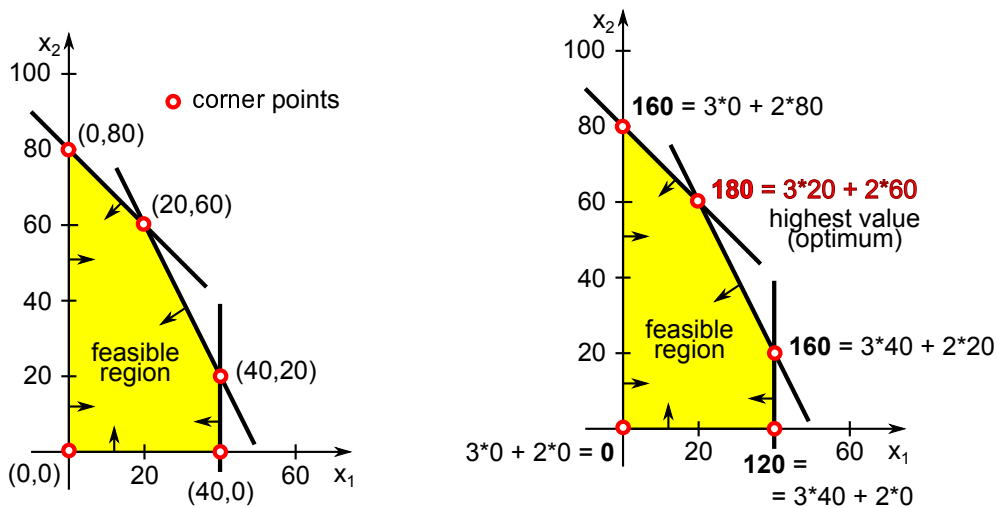
add  $x_1 + x_2 \leq 80$



A **corner** (extreme) point  $X$  of the region  $R \equiv$  every line through  $X$  intersects  $R$  in a segment whose one endpoint is  $X$ . Solving a linear program amounts to finding a best corner point by the following theorem.

**Theorem 1.** If a linear program has an **optimal** solution, then it also has an **optimal** solution that is a **corner point** of the feasible region.

**Exercise.** Try to find all corner points. Evaluate the objective function  $3x_1 + 2x_2$  at those points.



**Problem:** there may be too many corner points to check. There's a better way.

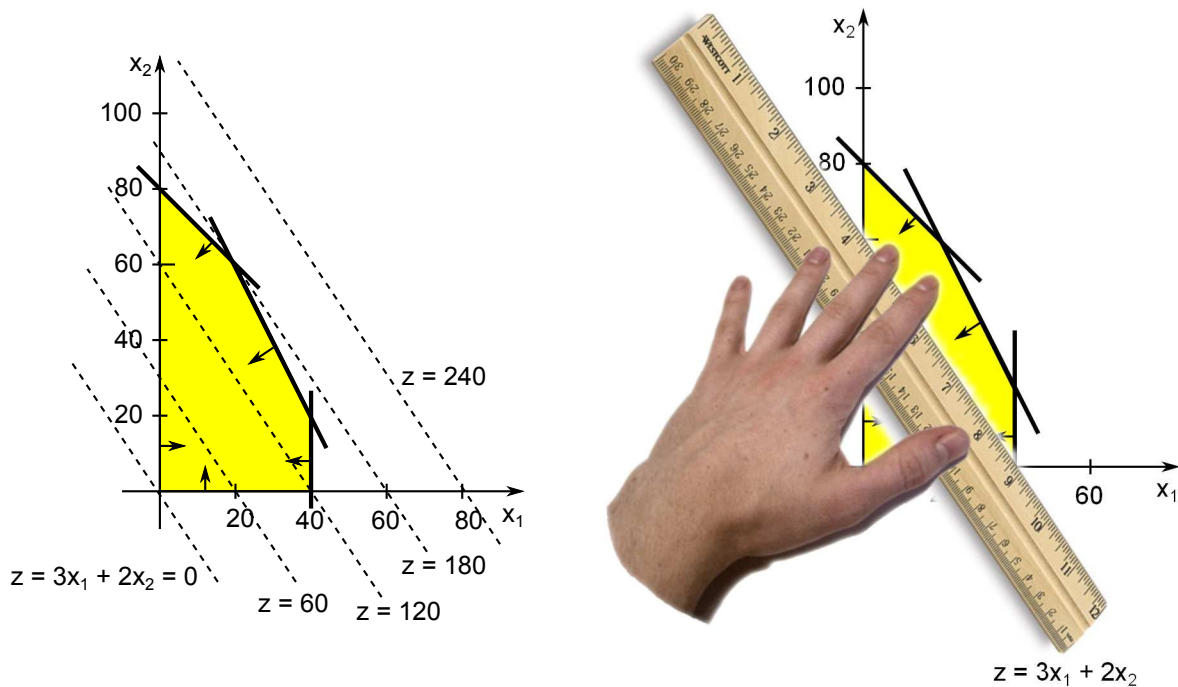
**Iso-value** line  $\equiv$  in all points on this line the objective function has the same value

For our objective  $3x_1 + 2x_2$  an iso-value line consists of points satisfying  $3x_1 + 2x_2 = z$  where  $z$  is some number.

**Graphical Method** (main steps):

1. Find the feasible region.
2. Plot an iso-value (isoprofit, isocost) line for some value.

3. Slide the line in the direction of increasing value until it only touches the region.
4. Read-off an optimal solution.



**Optimal solution** is  $(x_1, x_2) = (20, 60)$ .

Observe that this point is the intersection of two lines forming the boundary of the feasible region. Recall that lines we use to construct the feasible region come from inequalities (the points on the line satisfy the particular inequality with equality).

**Binding constraint**  $\equiv$  constraint satisfied with equality

For solution  $(20, 60)$ , the binding constraints are  $x_1 + x_2 \leq 80$  and  $2x_1 + x_2 \leq 100$  because  $20 + 60 = 80$  and  $2 \times 20 + 60 = 100$ . The constraint  $x_1 \leq 40$  is not binding because  $x_1 = 20 < 40$ .

The constraint is binding because changing it (a little) necessarily changes the optimality of the solution. Any change to the binding constraints either makes the solution not optimal or not feasible.

A constraint that is not binding can be changed (a little) without disturbing the optimality of the solution we found. Clearly we can change  $x_1 \leq 40$  to  $x_1 \leq 30$  and the solution  $(20, 60)$  is still optimal. We shall discuss this more in-depth when we learn about Sensitivity Analysis.

Finally, note that the above process always yields one of the following cases.

**Theorem 2.** *Every linear program has either*

- (i) a **unique** optimal solution, or
- (ii) multiple (**infinity**) optimal solutions, or
- (iii) is **infeasible** (has no feasible solution), or
- (iv) is **unbounded** (no feasible solution is maximal).



## 3.2 Fourier-Motzkin Elimination (FME)

A simple (but not yet most efficient) process to solve linear programs. Unlike the Graphical method, this process applies to arbitrary linear programs, but more efficient methods exist. The FME method

**finds a solution to a system of linear inequalities**

(much like Gaussian elimination from Linear algebra which finds a solution to a system of **linear equations**)

We shall discuss how this is done for  $\geq$ -inequalities and for **minimization** LPs. (Similarly it can be stated for  $\leq$ -inequalities and maximization LPs.) You can skip to the example below to get a better idea.

First, we need to adapt the method to solving linear programs. We need to incorporate the objective function as part of the inequalities. We **replace** the objective function by a **new** variable  $z$  and look for a solution to the inequalities such that  $z$  is smallest possible (explained how later).

0. **Objective function**  $c_1x_1 + c_2x_2 + \dots + c_nx_n$ : add a new constraint  $z \geq c_1x_1 + c_2x_2 + \dots + c_nx_n$

From this point, we assume that all we have is a system of  $\geq$ -inequalities with all variables on the left-hand side and a constant on the right-hand side. (We change  $\leq$ -inequalities to  $\geq$ -inequalities by multiplying by  $-1$ .) We proceed similarly as in Gaussian elimination. We try to eliminate variables one by one by **pivoting** a variable in **all inequalities** (not just one). Unlike Gaussian elimination, we are dealing with inequalities here and so we are **not allowed** to multiply by a negative constant when pivoting. This requires a more complex procedure to eliminate  $x_1$ .

1. **Normalize**  $x_1$ : if  $+cx_1$  or  $-cx_1$  where  $c > 0$  appears in an inequality, divide the inequality by  $c$ .

After normalizing, this gives us three types of inequalities: those with  $+x_1$  (call them **positive** inequalities), those with  $-x_1$  (call them **negative** inequalities), and those without  $x_1$ .

2. **Eliminate**  $x_1$ : consider each positive and each negative inequality and add them together to create a new inequality.

Note that we do this for **every pair** of such inequalities; each generates a new inequality without  $x_1$ . Taking all these generated inequalities and the inequalities that did not contain  $x_1$  in the first place gives us new problem, one without  $x_1$ . This new problem is **equivalent** to the original one.

3. **Repeat** this process eliminating  $x_2, x_3, \dots$ , in turn until only  $z$  remains to be eliminated.

4. **Solution**: determine the smallest value of  $z$  that satisfies the resulting inequalities.

5. **Back-substitution**: substitute the values in the reverse order of elimination to produce values of all eliminated variables.

In this process, when choice is possible for some variable, we can choose arbitrarily; any choice leads to a correct solution (for more, see the example below how this is done).

### Example

$$\begin{array}{ll} \min & 2x_1 + 2x_2 + 3x_3 \\ \text{s.t.} & x_1 + x_2 + x_3 \leq 2 \\ & 2x_1 + x_2 \leq 3 \\ & 2x_1 + x_2 + 3x_3 \geq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

1. make the objective function into a constraint  $z \geq \overbrace{2x_1 + 2x_2 + 3x_3}^{\text{objective function}}$  and change the objective to  $\min z$

$$\begin{array}{rcl}
 \min & & z \\
 \text{s.t.} & 2x_1 + 2x_2 + 3x_3 - z & \leq 0 \\
 & x_1 + x_2 + x_3 & \leq 2 \\
 & 2x_1 + x_2 & \leq 3 \\
 & 2x_1 + x_2 + 3x_3 & \geq 3 \\
 & x_1, x_2, x_3 & \geq 0
 \end{array}$$

2. change all inequalities to  $\geq$

$$\begin{array}{rcl}
 -2x_1 - 2x_2 - 3x_3 + z & \geq & 0 \\
 -x_1 - x_2 - x_3 & \geq & -2 \\
 -2x_1 - x_2 & \geq & -3 \\
 2x_1 + x_2 + 3x_3 & \geq & 3 \\
 x_1 & \geq & 0 \\
 & x_2 & \geq 0 \\
 & & x_3 \geq 0
 \end{array}$$

3. Eliminate  $x_1$

a) normalize = make the coefficients of  $x_1$  one of  $+1, -1, \text{ or } 0$

$$\begin{array}{rcl}
 -x_1 - x_2 - \frac{3}{2}x_3 + \frac{1}{2}z & \geq & 0 \\
 -x_1 - x_2 - x_3 & \geq & -2 \\
 -x_1 - \frac{1}{2}x_2 & \geq & -\frac{3}{2} \\
 x_1 + \frac{1}{2}x_2 + \frac{3}{2}x_3 & \geq & \frac{3}{2} \\
 x_1 & \geq & 0 \\
 & x_2 & \geq 0 \\
 & & x_3 \geq 0
 \end{array}$$

b) add inequalities = add each inequality with  $+x_1$  to every inequality with  $-x_1$ ; then remove all inequalities containing  $x_1$

$$\begin{array}{rcl}
 -\frac{1}{2}x_2 + \frac{1}{2}z & \geq & \frac{3}{2} \\
 -\frac{1}{2}x_2 + \frac{1}{2}x_3 & \geq & -\frac{1}{2} \\
 & \frac{3}{2}x_3 & \geq 0 \\
 -x_2 - \frac{3}{2}x_3 + \frac{1}{2}z & \geq & 0 \\
 -x_2 - x_3 & \geq & -2 \\
 -\frac{1}{2}x_2 & \geq & -\frac{3}{2} \\
 & x_2 & \geq 0 \\
 & & x_3 \geq 0
 \end{array}$$

Eliminate  $x_2$

$$\begin{array}{rcl}
 -x_2 + z & \geq & 3 \\
 -x_2 + x_3 & \geq & -1 \\
 -x_2 - \frac{3}{2}x_3 + \frac{1}{2}z & \geq & 0 \\
 -x_2 - x_3 & \geq & -2 \\
 -x_2 & \geq & -3 \\
 x_2 & \geq & 0 \\
 & \frac{3}{2}x_3 & \geq 0 \\
 & x_3 & \geq 0
 \end{array}
 \qquad
 \begin{array}{rcl}
 & z & \geq 3 \\
 & x_3 & \geq -1 \\
 -\frac{3}{2}x_3 + \frac{1}{2}z & \geq & 0 \\
 -x_3 & \geq & -2 \\
 & 0 & \geq -3 \\
 & \frac{3}{2}x_3 & \geq 0 \\
 & x_3 & \geq 0
 \end{array}$$

Eliminate  $x_3$

$$\begin{array}{rcl}
 -x_3 + \frac{1}{3}z & \geq & 0 \\
 -x_3 & \geq & -2 \\
 x_3 & \geq & -1 \\
 x_3 & \geq & 0 \\
 & z & \geq 3 \\
 & 0 & \geq -3
 \end{array}
 \qquad
 \begin{array}{rcl}
 \frac{1}{3}z & \geq & -1 \\
 0 & \geq & -3 \\
 \frac{1}{3}z & \geq & 0 \\
 0 & \geq & -2 \\
 z & \geq & 3 \\
 0 & \geq & -3
 \end{array}$$

Final list of inequalities

$$\begin{aligned} z &\geq -3 \\ z &\geq 0 \\ z &\geq 3 \\ 0 &\geq -3 \\ 0 &\geq -2 \end{aligned}$$

4. Choose smallest  $z$  that satisfies the inequalities,  $z = 3$

5. Back-substitution

$$\begin{array}{rcl} -x_3 + \frac{1}{3} \times 3 & \geq & 0 \\ -x_3 & \geq & -2 \\ x_3 & \geq & -1 \\ x_3 & \geq & 0 \end{array} \qquad \begin{array}{rcl} x_3 & \leq & 1 \\ x_3 & \leq & 2 \\ x_3 & \geq & -1 \\ x_3 & \geq & 0 \end{array} \qquad \begin{array}{l} 0 \leq x_3 \leq 1 \\ \text{Choose ANY value that} \\ \text{satisfies the inequalities} \end{array}$$

$$\boxed{x_3 = \frac{1}{2}}$$

$$\begin{array}{rcl} -x_2 & + & 3 \geq 3 \\ -x_2 + \frac{1}{2} & \geq & -1 \\ -x_2 - \frac{3}{2} \times \frac{1}{2} + \frac{1}{2} \times 3 & \geq & 0 \\ -x_2 - \frac{1}{2} & \geq & -2 \\ -x_2 & \geq & -3 \\ x_2 & \geq & 0 \end{array} \qquad \begin{array}{rcl} x_2 & \leq & 0 \\ x_2 & \leq & \frac{3}{2} \\ x_2 & \leq & \frac{3}{4} \\ x_2 & \leq & \frac{3}{2} \\ x_2 & \leq & 3 \\ x_2 & \geq & 0 \end{array} \qquad \begin{array}{l} 0 \leq x_2 \leq 0 \\ \text{Choose ANY value that} \\ \text{satisfies the inequalities (this} \\ \text{time only one)} \end{array}$$

$$\boxed{x_2 = 0}$$

$$\begin{array}{rcl} -x_1 - 0 - \frac{3}{2} \times \frac{1}{2} + \frac{1}{2} \times 3 & \geq & 0 \\ -x_1 - 0 - \frac{1}{2} & \geq & -2 \\ -x_1 - \frac{1}{2} \times 0 & \geq & -\frac{3}{2} \\ x_1 + \frac{1}{2} \times 0 + \frac{3}{2} \times \frac{1}{2} & \geq & \frac{3}{2} \\ x_1 & \geq & 0 \end{array} \qquad \begin{array}{rcl} x_1 & \leq & \frac{3}{4} \\ x_1 & \leq & \frac{3}{2} \\ x_1 & \leq & \frac{3}{2} \\ x_1 & \geq & \frac{3}{4} \\ x_1 & \geq & 0 \end{array} \qquad \begin{array}{l} \frac{3}{4} \leq x_1 \leq \frac{3}{4} \\ \text{Choose ANY value that} \\ \text{satisfies the inequalities} \\ \text{(again only one)} \end{array}$$

$$\boxed{x_1 = \frac{3}{4}}$$

$$\boxed{\text{Solution } x_1 = \frac{3}{4}, x_2 = 0, x_3 = \frac{1}{2} \text{ of value } z = 3}$$

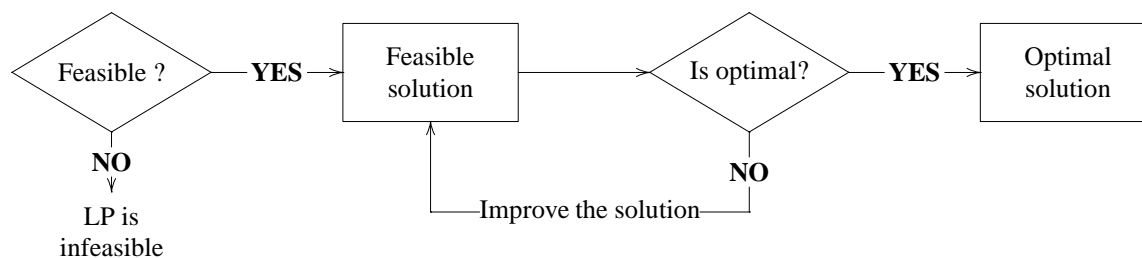
Notes:

- if at any point an inequality  $0x_1 + 0x_2 + 0x_3 + 0z \geq d$  is produced where  $d > 0$   
 $\longrightarrow$  no solution (**infeasible LP**)
- if the final system does not contain an inequality  $z \geq d$   
 $\longrightarrow$  no optimum (**unbounded LP**)

## Simplex method

The process consists of two steps

1. Find a **feasible** solution (or determine that **none exists**).
2. Improve the feasible solution to an **optimal** solution.



In many cases the first step is easy (for free; more on that later).

### 4.1 Canonical form

Linear program (LP) is in a **canonical form** if

- all constraints are **equations**
- all variables are **non-negative**

$$\begin{array}{rcllcl}
 \max & c_1x_1 & + & c_2x_2 & + & \dots & + & c_nx_n & & \\
 \text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\
 & \vdots & + & \vdots & & & & \vdots & & \vdots \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \\
 & & & & & & & x_1, x_2, \dots, x_n & \geq & 0
 \end{array}$$

#### Slack variables

To change an inequality to an equation, we add a **new non-negative** variable called a **slack** variable.

$$x_1 + x_2 \leq 80 \quad \longrightarrow \quad x_1 + x_2 + s_1 = 80$$

$$x_1 + x_2 \geq 80 \quad \longrightarrow \quad x_1 + x_2 - e_1 = 80$$

**Notes:**

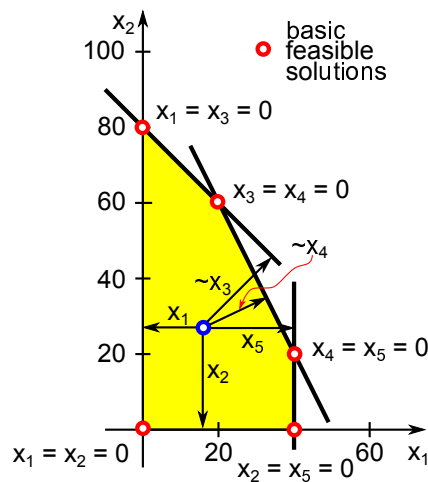
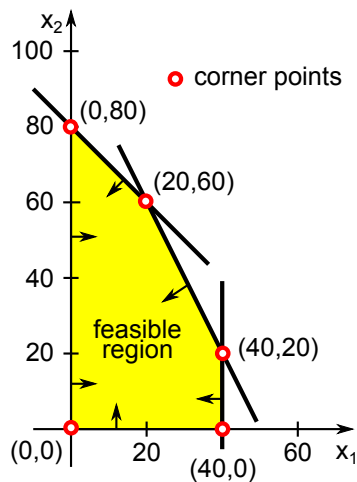
- the variable  $e_1$  is sometimes called an **excess** variable
- we can use  $x_i$  for slack variables (where  $i$  is a new index)

$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ & x_1 + x_2 \leq 80 \\ & 2x_1 + x_2 \leq 100 \\ & x_1 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$	$\longrightarrow$	$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ & x_1 + x_2 + x_3 = 80 \\ & 2x_1 + x_2 + x_4 = 100 \\ & x_1 + x_5 = 40 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$
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## 4.2 Simplex method by example

Consider the toyshop example from earlier lectures. Convert to equalities by adding *slack variables*

$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ & x_1 + x_2 \leq 80 \\ & 2x_1 + x_2 \leq 100 \\ & x_1 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$	$\longrightarrow$	$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ & x_1 + x_2 + x_3 = 80 \\ & 2x_1 + x_2 + x_4 = 100 \\ & x_1 + x_5 = 40 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$
--	-------------------	--



### Starting feasible solution

Set variables  $x_1, x_2$  to zero and set slack variables to the values on the right-hand side.

$$\rightarrow \text{yields a feasible solution } x_1 = x_2 = 0, x_3 = 80, x_4 = 100, x_5 = 40$$

Recall that the solution is feasible because all variables are **non-negative** and **satisfy** all equations.

(we get a feasible solution right away because the right-hand side is non-negative; this may not always work)

Note something **interesting**: in this feasible solution two variables (namely  $x_1, x_2$ ) are zero. Such a solution is called a **basic solution** of this problem, because the value of at least two variables is zero.

In a problem with  $n$  variables and  $m$  constraints, a solution where at least  $(n - m)$  variables are zero is a **basic solution**.

A basic solution that is also feasible is called a **basic feasible solution (BFS)**.

The importance of basic solutions is revealed by the following observation.

**Basic solutions** are precisely the **corner points** of the **feasible region**.

Recall that we have discussed that to find an optimal solution to an LP, it suffices to find a **best solution** among all **corner points**. The above tells us how to compute them – they are the **basic feasible solutions**.

A variable in a **basic solution** is called a **non-basic variable** if it is chosen to be zero.  
Otherwise, the variable is **basic**.

The basic variables we collectively call a **basis**.

## Dictionary

To conveniently deal with basic solutions, we use the so-called **dictionary**. A dictionary lists values of basic variables as a function of non-basic variables. The correspondence is obtained by expressing the basic variables from the initial set of equations. (We shall come back to this later; for now, have a look below.)

Express the slack variables from the individual equations.

$$\begin{array}{rcll} \max & 3x_1 + 2x_2 & & \\ & x_1 + x_2 + x_3 & = & 80 \\ & 2x_1 + x_2 & + x_4 & = 100 \\ & x_1 & & + x_5 = 40 \\ & & & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array} \quad \longrightarrow \quad \begin{array}{l} x_3 = 80 - x_1 - x_2 \\ x_4 = 100 - 2x_1 - x_2 \\ x_5 = 40 - x_1 \\ \hline z = 0 + 3x_1 + 2x_2 \end{array}$$

This is called a **dictionary**.

- $x_1, x_2$  independent (**non-basic**) variables
- $x_3, x_4, x_5$  dependent (**basic**) variables
- $\{x_3, x_4, x_5\}$  is a **basis**

set  $x_1 = x_2 = 0 \rightarrow$  the corresponding (feasible) solution is  $x_3 = 80, x_4 = 100, x_5 = 40$  with value  $z = 0$

## Improving the solution

Try to increase  $x_1$  from its current value 0 in hopes of improving the value of  $z$

try  $x_1 = 20, x_2 = 0$  and **substitute** into the dictionary to obtain the values of  $x_3, x_4, x_5$  and  $z$

$\rightarrow x_3 = 60, x_4 = 60, x_5 = 20$  with value  $z = 60 \rightarrow$  **feasible**

try again  $x_1 = 40, x_2 = 0 \rightarrow x_3 = 40, x_4 = 20, x_5 = 0$  with value  $z = 120 \rightarrow$  **feasible**

now try  $x_1 = 50, x_2 = 0 \rightarrow x_3 = 30, x_4 = 0, x_5 = -10 \rightarrow$  **not feasible**

How much we can increase  $x_1$  before a (dependent) variable becomes negative?

If  $x_1 = t$  and  $x_2 = 0$ , then the solution is feasible if

$$\left. \begin{array}{l} x_3 = 80 - t - 0 \geq 0 \\ x_4 = 100 - 2t - 0 \geq 0 \\ x_5 = 40 - t \geq 0 \end{array} \right\} \implies \begin{array}{l} t \leq 80 \\ t \leq 50 \\ t \leq 40 \end{array} \implies t \leq 40$$

Maximal value is  $x_1 = 40$  at which point the variable  $x_5$  becomes zero

$x_1$  is *incoming* variable and  $x_5$  is *outgoing* variable

(we say that  $x_1$  **enters** the dictionary/basis, and  $x_5$  **leaves** the dictionary/basis)

## Ratio test

The above analysis can be streamlined into the following simple “ratio” test.

$$x_3 : \frac{80}{1} = 80$$

$$x_4 : \frac{100}{2} = 50$$

$$x_5 : \frac{40}{1} = 40$$

$  \begin{array}{r}  x_3 = 80 - x_1 - x_2 \\  x_4 = 100 - 2x_1 - x_2 \\  x_5 = 40 - x_1 \\  \hline  z = 0 + 3x_1 + 2x_2  \end{array}  $	ratio for $x_4$ : $\frac{100}{2} = 50$
---	---

**(watch-out:** we only consider this ratio because the coefficient of  $x_1$  is negative  $(-2)$ ... more on that in the later steps)

Minimum achieved with  $x_5 \implies$  outgoing variable

Express  $x_1$  from the equation for  $x_5$

$$x_5 = 40 - x_1 \implies x_1 = 40 - x_5$$

Substitute  $x_1$  to all other equations  $\implies$  new feasible dictionary

$  \begin{array}{r}  x_1 = (40 - x_5) \\  x_3 = 80 - (40 - x_5) - x_2 \\  x_4 = 100 - 2(40 - x_5) - x_2 \\  \hline  z = 0 + 3(40 - x_5) + 2x_2  \end{array}  $	$\implies$	$  \begin{array}{r}  x_1 = 40 - x_5 \\  x_3 = 40 - x_2 + x_5 \\  x_4 = 20 - x_2 + 2x_5 \\  \hline  z = 120 + 2x_2 - 3x_5  \end{array}  $
--	------------	--

now  $x_2, x_5$  are independent variables and  $x_1, x_3, x_4$  are dependent

$\rightarrow \{x_1, x_3, x_4\}$  is a basis

we repeat: we increase  $x_2 \rightarrow$  incoming variable, ratio test:

$x_1$  : does not contain  $x_2 \rightarrow$  no constraint

$$x_2 : \frac{40}{1} = 40$$

$$x_4 : \frac{20}{1} = 20$$

minimum achieved for  $x_4 \rightarrow$  outgoing variable

$$x_4 = 20 - x_2 + 2x_5 \implies x_2 = 20 - x_4 + 2x_5$$

$  \begin{array}{r}  x_1 = 40 - x_5 \\  x_2 = (20 - x_4 + 2x_5) \\  x_3 = 40 - (20 - x_4 + 2x_5) + x_5 \\  \hline  z = 120 + 2(20 - x_4 + 2x_5) - 3x_5  \end{array}  $	$\implies$	$  \begin{array}{r}  x_1 = 40 - x_5 \\  x_2 = 20 - x_4 + 2x_5 \\  x_3 = 20 + x_4 - x_5 \\  \hline  z = 160 - 2x_4 + x_5  \end{array}  $
--	------------	---

$x_5$  incoming variable, ratio test:

$$x_1 : \frac{40}{1} = 40$$

$x_2$  : positive coefficient  $\rightarrow$  no constraint

$$x_3 : \frac{20}{1} = 20$$

minimum achieved for  $x_3 \rightarrow$  outgoing variable

$$x_3 = 20 + x_4 - x_5 \implies x_5 = 20 + x_4 - x_3$$

$  \begin{array}{r}  x_1 = 40 - (20 + x_4 - x_3) \\  x_2 = 20 - x_4 + 2(20 + x_4 - x_3) \\  x_5 = (20 + x_4 - x_3) \\  \hline  z = 160 - 2x_4 + (20 + x_4 - x_3)  \end{array}  $	$\implies$	$  \begin{array}{r}  x_1 = 20 + x_3 - x_4 \\  x_2 = 60 - 2x_3 + x_4 \\  x_5 = 20 - x_3 + x_4 \\  \hline  z = 180 - x_3 - x_4  \end{array}  $
--	------------	--

no more improvement possible  $\implies$  **optimal solution**

$$x_1 = 20, x_2 = 60, x_3 = 0, x_4 = 0, x_5 = 20 \text{ of value } z = 180$$

Why? setting  $x_3, x_4$  to any non-zero values results in a smaller value of  $z$

Each dictionary is **equivalent** to the original system (the two have the same set of solutions)

## Simplex algorithm

**Preparation:** find a starting feasible solution/dictionary

1. Convert to the canonical form (constraints are equalities) by adding slack variables  $x_{n+1}, \dots, x_{n+m}$
2. Construct a starting dictionary - express slack variables and objective function  $z$
3. If the resulting dictionary is feasible, then we are done with preparation  
If not, try to find a feasible dictionary using the **Phase I. method** (next lecture).

**Simplex step (maximization LP):** try to improve the solution

1. **(Optimality test):** If **no variable** appears with a **positive** coefficient in the equation for  $z$   
→ STOP, current solution is **optimal**
  - set non-basic variables to zero
  - read off the values of the basic variables and the objective function  $z$   
→ Hint: the values are the constant terms in respective equations
  - report this (optimal) solution
2. Else pick a variable  $x_i$  having positive coefficient in the equation for  $z$   
 $x_i \equiv$  *incoming* variable
3. Ratio test: in the dictionary, find an equation for a variable  $x_j$  in which
  - $x_i$  appears with a negative coefficient  $-a$
  - the ratio  $\frac{b}{a}$  is smallest possible  
(where  $b$  is the constant term in the equation for  $x_j$ )
4. If no such such  $x_j$  exists → stop, no optimal solution, report that **LP is unbounded**
5. Else  $x_j \equiv$  *outgoing* variable → construct a new dictionary by *pivoting*:
  - express  $x_i$  from the equation for  $x_j$ ,
  - add this as a new equation,
  - remove the equation for  $x_j$ ,
  - substitute  $x_i$  to all other equations (including the one for  $z$ )
6. Repeat from 1.

### Questions:

- which variable to choose as incoming, which as outgoing
- is this guaranteed to terminate in a finite number of steps
- how to convert other LP formulations to the standard form
- how to find a starting dictionary
- how do we find alternative optimal solutions



### 4.3 Two phase Simplex method

**canonical form** = equations, non-negative variables

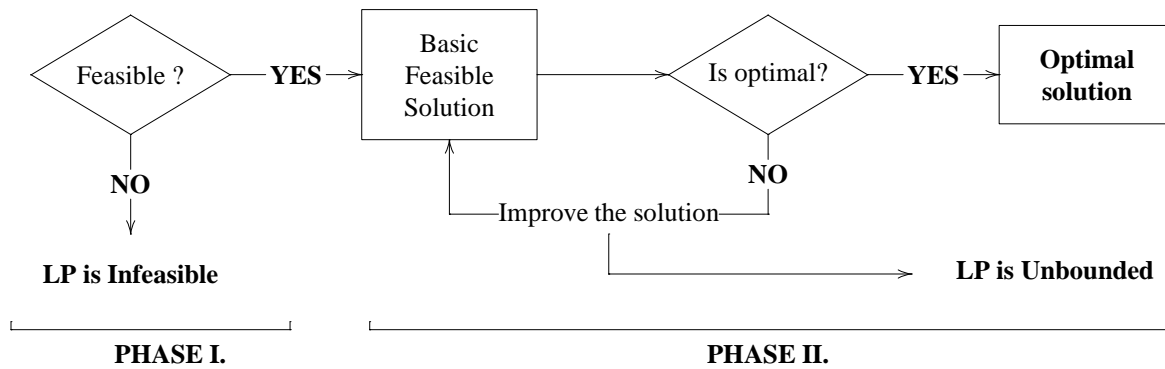
$n$  = number of variables

$m$  = number of equations

**basic solution** = at least  $(n - m)$  variables are zero

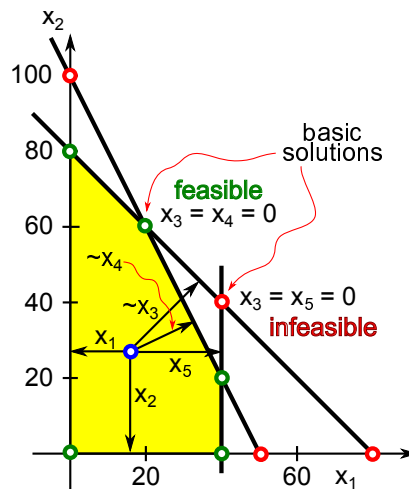
**basic solutions** = **dictionaries**

**basic feasible solutions** = **corner/extreme points** = **feasible dictionaries**



$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 80 \\ & 2x_1 + x_2 \leq 100 \\ & x_1 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 80 \\ & 2x_1 + x_2 + x_4 = 100 \\ & x_1 + x_5 = 40 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$



Feasible dictionary:  $x_3 = x_4 = 0$

$$\begin{aligned} x_1 &= 20 + x_3 - x_4 \\ x_2 &= 60 - 2x_3 + x_4 \\ x_5 &= 20 - x_3 + x_4 \\ \hline z &= 180 - x_3 - x_4 \end{aligned}$$

Infeasible dictionary:  $x_3 = x_5 = 0$

$$\begin{aligned} x_1 &= 40 - x_5 \\ x_2 &= 40 - x_3 + x_5 \\ x_4 &= -20 + x_3 + x_5 \\ \hline z &= 200 - 2x_3 - x_5 \end{aligned}$$

(it is infeasible since  $x_4 = -20$ )

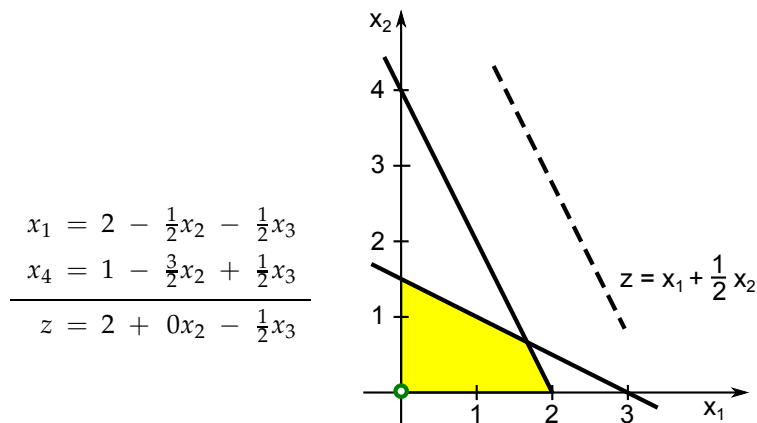
### 4.4 Special cases

**Alternative solutions**

$$\begin{aligned} \max \quad & x_1 + \frac{1}{2}x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} x_3 &= 4 - 2x_1 - x_2 \\ x_4 &= 3 - x_1 - 2x_2 \\ \hline z &= x_1 + \frac{1}{2}x_2 \end{aligned}$$

Pivoting:  $x_1$  enters, ratio test:  $x_3 : \frac{4}{2} = 2, x_4 : \frac{3}{1} = 3 \rightarrow$  thus  $x_3$  leaves:  $x_1 = 2 - \frac{1}{2}x_2 - \frac{1}{2}x_3$



Optimal solution (all coefficients non-positive)  $x_1 = 2, x_2 = 0, x_3 = 0, x_4 = 1, z = 2$

Note that  $x_2$  appears with zero coefficient in the expression for  $z$

→ increasing  $x_2$  is possible, but does not affect the value of  $z$

we pivot again:  $x_2$  enters, ratio test  $x_1 : \frac{2}{1/2} = 4, x_4 : \frac{1}{3/2} = 2/3$  → thus  $x_4$  leaves

$$\begin{array}{r} x_1 = \frac{5}{3} - \frac{2}{3}x_3 + \frac{1}{3}x_4 \\ x_2 = \frac{2}{3} + \frac{1}{3}x_3 - \frac{2}{3}x_4 \\ \hline z = 2 - \frac{1}{2}x_3 + 0x_4 \end{array}$$

Again an optimal solution  $x_1 = \frac{5}{3}, x_2 = \frac{2}{3}, x_3 = 0, x_4 = 0, z = 2$  → same value

What if we pivot again (on  $x_4$ ) ?

## Unbounded LP

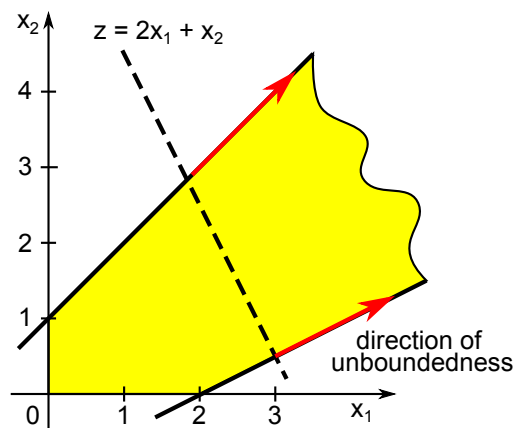
$$\begin{array}{r} \max \quad 2x_1 + x_2 \\ \text{s.t.} \quad -x_1 + x_2 \leq 1 \\ \quad \quad x_1 - 2x_2 \leq 2 \\ \quad \quad x_1, x_2 \geq 0 \end{array} \quad \begin{array}{r} x_3 = 1 + x_1 - x_2 \\ x_4 = 2 - x_1 + 2x_2 \\ \hline z = 2x_1 + x_2 \end{array}$$

Pivoting:  $x_1$  enters,  $x_4$  leaves (the only choice),  $x_1 = 2 + 2x_2 - x_4$

$$\begin{array}{r} x_1 = 2 + 2x_2 - x_4 \\ x_3 = 3 + x_2 - x_4 \\ \hline z = 4 + 5x_2 - 2x_4 \end{array}$$

for  $x_2 = x_4 = 0$ , we have  $x_1 = 2, x_3 = 3, z = 4$

→ a feasible solution



What if we now increase  $x_2$ ? no positive value of  $x_2$  makes one of  $x_1, x_3$  negative

→ we can make  $x_2$  arbitrarily large and thus make  $z$  arbitrarily large → **unbounded LP**

**direction of unboundedness:** set  $x_2 = t, x_4 = 0$  →  $x_1 = 2 + 2t, x_3 = 3 + t, z = 4 + 5t$

for increasing  $t$  → gives a sequence of feasible solution of increasing value

**Degeneracy**

$$\begin{array}{r} x_4 = 1 \qquad \qquad \qquad - 2x_3 \\ x_5 = 3 - 2x_1 + 4x_2 - 6x_3 \\ x_6 = 2 + x_1 - 3x_2 - 4x_3 \\ \hline z = \qquad 2x_1 - x_2 + 8x_3 \end{array}$$

Pivoting:  $x_3$  enters, ratio test:  $x_4 : \frac{1}{2} = 1/2$ ,  $x_5 : \frac{3}{6} = 1/2$ ,  $x_6 : \frac{2}{4} = 1/2 \rightarrow$  any of  $x_4, x_5, x_6$  can be chosen

$\rightarrow$  we choose  $x_4$  to leave,  $x_3 = \frac{1}{2} - \frac{1}{2}x_4$

$$\begin{array}{r} x_3 = \frac{1}{2} \qquad \qquad \qquad - \frac{1}{2}x_4 \\ x_5 = \qquad - 2x_1 + 4x_2 + 3x_4 \\ x_6 = \qquad \qquad x_1 - 3x_2 + 2x_4 \\ \hline z = 4 + 2x_1 - x_2 - 4x_4 \end{array}$$

setting  $x_1 = x_2 = x_4 = 0$  yields  $x_3 = \frac{1}{2}$ ,  $x_5 = 0$ ,  $x_6 = 0$

now  $x_1$  enters, and  $x_5$  leaves (the only choice),  $x_1 = 2x_2 - \frac{3}{2}x_4 - \frac{1}{2}x_5$

$$\begin{array}{r} x_1 = \qquad 2x_2 + \frac{3}{2}x_4 - \frac{1}{2}x_5 \\ x_3 = \frac{1}{2} \qquad \qquad - \frac{1}{2}x_4 \\ x_6 = \qquad - x_2 + \frac{7}{2}x_4 - \frac{1}{2}x_5 \\ \hline z = 4 + 3x_2 - x_4 - x_5 \end{array}$$

setting  $x_2 = x_4 = x_5 = 0$  yields  $x_1 = 0$ ,  $x_3 = \frac{1}{2}$ ,  $x_6 = 0$

$\rightarrow$  **same solution** as before

if some basic variable is zero, then the basic solution is **degenerate**

This happens, for instance, if there is more than one choice for an outgoing variable (the ones not chosen will be zero in the subsequent dictionary)

**Problem:** several dictionaries may correspond to the same (degenerate) solution

The simplex rule may cycle, it is possible to go back to the same dictionary if we are not careful enough when choosing the incoming/outgoing variables

**Bland's rule** From possible options, choose an incoming (outgoing) variable  $x_k$  with smallest subscript  $k$ .

Simplex method using Bland's rule is guaranteed to terminate in a finite number of steps.

Alternative: **lexicographic rule** – choose as outgoing variable one whose row is lexicographically smallest (when divided by the constant term) – the coefficients in the objective function are guaranteed to strictly increase lexicographically

**4.5 Phase I.**

$$\begin{array}{r} \max \quad x_1 - x_2 + x_3 \\ \text{s.t.} \quad 2x_1 - x_2 + 2x_3 \leq 4 \\ \quad \quad 2x_1 - 3x_2 + x_3 \leq -5 \\ \quad \quad -x_1 + x_2 - 2x_3 \leq -1 \\ \quad \quad \quad \quad x_1, x_2, x_3 \geq 0 \end{array} \quad \begin{array}{r} x_4 = 4 - 2x_1 + x_2 - 2x_3 \\ x_5 = -5 - 2x_1 + 3x_2 - x_3 \\ x_6 = -1 + x_1 - x_2 + 2x_3 \\ \hline z = \qquad \quad x_1 - x_2 + x_3 \end{array}$$

If we choose the starting basis to be the **slack variables**, then the resulting dictionary is **not feasible**:

$\rightarrow$  we let  $x_1 = x_2 = x_3 = 0$ , we get  $x_4 = 4$ ,  $x_5 = -5$ ,  $x_6 = -1 \rightarrow$  **not feasible** because  $x_5 < 0$

We need to find a starting feasible dictionary for Phase II. To do this, we **solve** a different problem.

**Intuition:** we want to get a starting feasible solution where all variables  $x_1, x_2, x_3$  are zero.

**Option 1**

Add new **artificial** variables to each inequality as follows:

$$\begin{array}{ll}
 x_1 + x_2 \leq -100 & \longrightarrow x_1 + x_2 - a_1 \leq -100 \\
 x_1 + x_2 \geq 100 & \longrightarrow x_1 + x_2 + a_1 \geq 100 \\
 x_1 + x_2 = 100 & \longrightarrow x_1 + x_2 + a_1 = 100 \\
 x_1 + x_2 = -100 & \longrightarrow x_1 + x_2 - a_1 = -100 \\
 \\
 \left. \begin{array}{l}
 x_1 + x_2 \leq 100 \\
 x_1 + x_2 \geq -100 \\
 x_1 + x_2 = 0
 \end{array} \right\} & \longrightarrow \text{no change}
 \end{array}$$

**New objective function:** minimize the **sum** of **all artificial** variables  $a_1 + a_2 + \dots + a_m$

**Observe** that setting all variables  $x_i$  to be 0 allows us to choose non-negative values for the artificial variables ( $a_1 = 100$  in the above) to obtain a **starting feasible solution** for this **new problem**.

$$\begin{array}{ll}
 \max & x_1 - x_2 + x_3 & \min & a_2 + a_3 \\
 \text{s.t.} & 2x_1 - x_2 + 2x_3 \leq 4 & \text{s.t.} & 2x_1 - x_2 + 2x_3 \leq 4 \\
 & 2x_1 - 3x_2 + x_3 \leq -5 & & 2x_1 - 3x_2 + x_3 - a_2 \leq -5 \\
 & -x_1 + x_2 - 2x_3 \leq -1 & & -x_1 + x_2 - 2x_3 - a_3 \leq -1 \\
 & x_1, x_2, x_3 \geq 0 & & x_1, x_2, x_3, a_2, a_3 \geq 0
 \end{array}$$

Notice that if we now set  $x_1 = x_2 = x_3 = 0$  and  $a_2 = 5$  and  $a_3 = 1$ , this satisfies all inequalities (is **feasible**). After adding the slack variables, we produce the corresponding **feasible dictionary** as follows. Since we want the **maximization form**, we also re write the objective function as  $\max w = -a_2 - a_3$ . Therefore

$$\begin{array}{ll}
 \max w = & -a_2 - a_3 \\
 \text{s.t.} & 2x_1 - x_2 + 2x_3 + x_4 = 4 \\
 & 2x_1 - 3x_2 + x_3 - a_2 + x_5 = -5 \\
 & -x_1 + x_2 - 2x_3 - a_3 + x_6 = -1 \\
 & x_1, x_2, x_3, a_2, a_3, x_4, x_5, x_6 \geq 0
 \end{array}$$

If the optimal solution to this problem has **negative** value  $w$ , then the initial LP is **Infeasible**.  
Otherwise, we produce a **starting** feasible dictionary for Phase II from the optimal dictionary of Phase I.

To get a starting (Phase I.) feasible dictionary, we take as a basis **all artificial** variables ( $a_2, a_3$ ) and add to that **slack variables** of equations that do not have artificial variables (1st equation, add  $x_4$ ).

$$\begin{array}{l}
 x_4 = 4 - 2x_1 + x_2 - 2x_3 \\
 a_2 = 5 + 2x_1 - 3x_2 + x_3 + x_5 \\
 a_3 = 1 - x_1 + x_2 - 2x_3 + x_6
 \end{array}$$

The final step is the objective function  $w = -a_2 - a_3$  which we have to write in terms of **non-basic** variables (so far it is not since we chose  $a_2$  and  $a_3$  to be basic). We substitute from the above equations:

$$w = -a_2 - a_3 = -\overbrace{(5 + 2x_1 - 3x_2 + x_3 + x_5)}^{a_2} - \overbrace{(1 - x_1 + x_2 - 2x_3 + x_6)}^{a_3} = -6 - x_1 + 2x_2 + x_3 - x_5 - x_6$$

The resulting starting feasible dictionary it then as follows:

$$\begin{array}{l}
 x_4 = 4 - 2x_1 + x_2 - 2x_3 \\
 a_2 = 5 + 2x_1 - 3x_2 + x_3 + x_5 \\
 a_3 = 1 - x_1 + x_2 - 2x_3 + x_6 \\
 \hline
 w = -6 - x_1 + 2x_2 + x_3 - x_5 - x_6
 \end{array}$$

**Option 2**

(for  $\leq$  inequalities): Introduce **one** new artificial variable  $x_0$  and a new objective  $w = -x_0$

$$\begin{array}{ll} \max & -x_0 \\ \text{s.t.} & 2x_1 - x_2 + 2x_3 - x_0 \leq 4 \\ & 2x_1 - 3x_2 + x_3 - x_0 \leq -5 \\ & -x_1 + x_2 - 2x_3 - x_0 \leq -1 \\ & x_0, x_1, x_2, x_3 \geq 0 \end{array} \qquad \begin{array}{ll} \max & -x_0 \\ \text{s.t.} & 2x_1 - x_2 + 2x_3 - x_0 + x_4 = 4 \\ & 2x_1 - 3x_2 + x_3 - x_0 + x_5 = -5 \\ & -x_1 + x_2 - 2x_3 - x_0 + x_6 = -1 \\ & x_0, x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

It is easy to get a starting feasible solution for this problem – a starting feasible basis is as follows:

- take all slack variables ( $x_4, x_5, x_6$ )
- consider the inequality whose right-hand side is most negative (in this case 2nd inequality)
- this inequality has an associated slack variable ( $x_5$ ), remove this variable from our set  $\rightarrow \{x_4, x_6\}$
- add  $x_0$  in place of the removed variable  $\rightarrow \{x_0, x_4, x_6\}$

This is guaranteed to be a feasible basis (in this new problem).

$$\begin{array}{r} x_0 = 5 + 2x_1 - 3x_2 + x_3 + x_5 \\ x_4 = 9 - 2x_2 - x_3 + x_5 \\ x_6 = 4 + 3x_1 - 4x_2 + 3x_3 + x_5 \\ \hline w = -5 - 2x_1 + 3x_2 - x_3 - x_5 \end{array}$$

**Example**

Let us solve the above problem.

$x_2$  enters, ratio test:  $x_0 : \frac{5}{3}, x_4 : \frac{9}{2}, x_6 : \frac{4}{4} = 1 \rightarrow$  thus  $x_6$  leaves,  $x_2 = 1 + \frac{3}{4}x_1 + \frac{3}{4}x_3 + \frac{1}{4}x_5 - \frac{1}{4}x_6$

$$\begin{array}{r} x_0 = 2 - \frac{1}{4}x_1 - \frac{5}{4}x_3 + \frac{1}{4}x_5 + \frac{3}{4}x_6 \\ x_2 = 1 + \frac{3}{4}x_1 + \frac{3}{4}x_3 + \frac{1}{4}x_5 - \frac{1}{4}x_6 \\ x_4 = 7 - \frac{3}{2}x_1 - \frac{5}{2}x_3 + \frac{1}{2}x_5 + \frac{1}{2}x_6 \\ \hline w = -2 + \frac{1}{4}x_1 + \frac{5}{4}x_3 - \frac{1}{4}x_5 - \frac{3}{4}x_6 \end{array}$$

letting  $x_1 = x_3 = x_5 = x_6 = 0$  yields  $x_0 = 2, x_2 = 1, x_4 = 7 \rightarrow$  feasible solution of value  $w = -2$ .

now  $x_3$  enters, ratio test:  $x_0 : \frac{2}{5/4} = \frac{8}{5}, x_4 : \frac{7}{5/2} = \frac{14}{5} \rightarrow$  thus  $x_0$  leaves,  $x_3 = \frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6 - \frac{4}{5}x_0$ .

$$\begin{array}{r} x_2 = \frac{11}{5} + \frac{3}{5}x_1 + \frac{2}{5}x_5 + \frac{1}{5}x_6 - \frac{3}{5}x_0 \\ x_3 = \frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6 - \frac{4}{5}x_0 \\ x_4 = 3 - x_1 - x_6 + 2x_0 \\ \hline w = -x_0 \end{array}$$

Feasible solution  $x_0 = x_1 = x_5 = x_6 = 0, x_2 = \frac{11}{5}, x_3 = \frac{8}{5}, x_4 = 3$  of value  $w = 0$

Since  $x_0 = 0$ , the solution is also feasible in the original problem.

Now drop the variable  $x_0$  and remove the auxiliary objective  $w$

$$\begin{aligned}x_2 &= \frac{11}{5} + \frac{3}{5}x_1 + \frac{2}{5}x_5 + \frac{1}{5}x_6 \\x_3 &= \frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6 \\x_4 &= 3 - x_1 - x_6\end{aligned}$$

Finally, introduce the original objective  $z = x_1 - x_2 + x_3$

Note that  $x_2$  and  $x_3$  appear in  $z$  but are not non-basic variables of the dictionary

→ we must substitute them using the dictionary

$$z = x_1 - x_2 + x_3 = x_1 - \overbrace{\left(\frac{11}{5} + \frac{3}{5}x_1 + \frac{2}{5}x_5 + \frac{1}{5}x_6\right)}^{x_2} + \overbrace{\left(\frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6\right)}^{x_3} = -\frac{3}{5} + \frac{1}{5}x_1 - \frac{1}{5}x_5 + \frac{2}{5}x_6$$

Thus the resulting starting feasible dictionary for the original problem is as follows:

$$\begin{array}{r}x_2 = \frac{11}{5} + \frac{3}{5}x_1 + \frac{2}{5}x_5 + \frac{1}{5}x_6 \\x_3 = \frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6 \\x_4 = 3 - x_1 - x_6 \\ \hline z = -\frac{3}{5} + \frac{1}{5}x_1 - \frac{1}{5}x_5 + \frac{2}{5}x_6\end{array}$$

## Linear Algebra Review

**scalar** = a number, could be **real** like  $\pi = 3.14\dots$ , **rational** (fraction) like  $\frac{3}{4}$ , **integer** (whole) like 5 or  $-6$   
 (scalars “change scale” in proportion to their value)

**vector** = a sequence of numbers, for example  $\mathbf{x} = (3, 1, 0, 2)$

sometimes we write  $\mathbf{x} = \begin{bmatrix} 3 & 1 & 0 & 2 \end{bmatrix}$  and say it is a **row vector**,

or we write  $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}$  and say it is a **column vector**

**multiplying** a vector by a scalar (“scaling”)

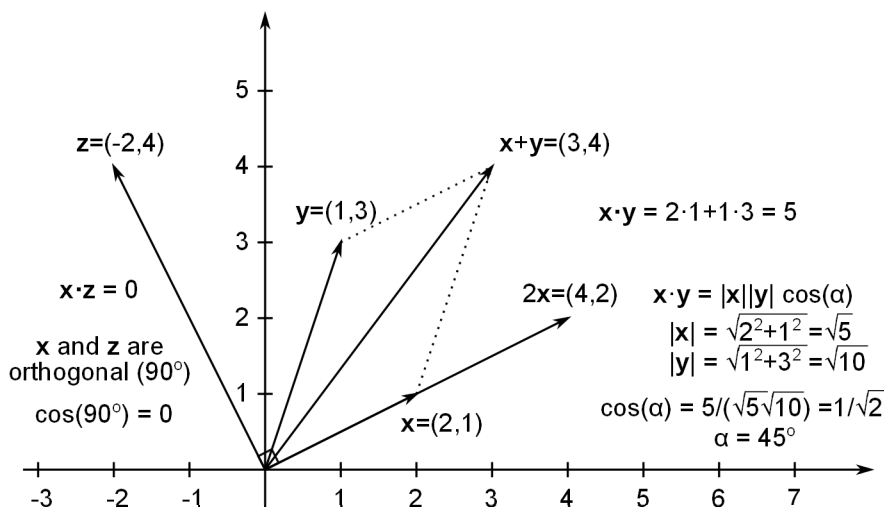
$$\mathbf{x} = (x_1, x_2, \dots, x_n) \quad a \text{ is a scalar} \quad a \cdot \mathbf{x} = (ax_1, ax_2, \dots, ax_n)$$

For example if  $\mathbf{x} = (3, 1, 0, 2)$ , then  $5 \cdot \mathbf{x} = (5 \cdot 3, 5 \cdot 1, 5 \cdot 0, 5 \cdot 2) = (15, 5, 0, 10)$

**adding vectors**

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \quad \mathbf{y} = (y_1, y_2, \dots, y_n) \quad \mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

For example if  $\mathbf{x} = (3, 1, 0, 2)$  and  $\mathbf{y} = (0, -1, 2, 1)$ , then  $\mathbf{x} + \mathbf{y} = (3 + 0, 1 + (-1), 0 + 2, 2 + 1) = (3, 0, 2, 3)$



**multiplying vectors = scalar product** (“dot” product) of vectors

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \quad \mathbf{y} = (y_1, y_2, \dots, y_n) \quad \mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

For example if  $\mathbf{x} = (3, 1, 0, 2)$  and  $\mathbf{y} = (0, -1, 2, 1)$ , then  $\mathbf{x} \cdot \mathbf{y} = 3 \cdot 0 + 1 \cdot (-1) + 0 \cdot 2 + 2 \cdot 1 = 1$

**scalar product** = corresponds to the **angle** between the vectors

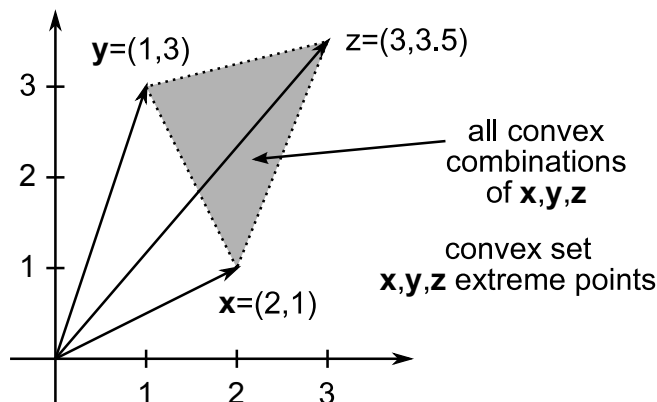
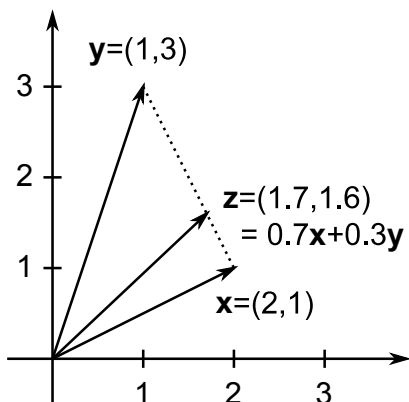
(more precisely, scalar product is **proportional** to the **cosine** of the angle — for instance, the scalar product equals **zero** if and only if the two vector are **orthogonal** (perpendicular) to each other — the angle between them is  $90^\circ$ )

**linear combination** of vectors  $\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^m$  is

$$a_1\mathbf{x}^1 + a_2\mathbf{x}^2 + \dots + a_m\mathbf{x}^m$$

where  $a_1, a_2, \dots, a_m$  are numbers (scalars). If each  $a_i$  is between 0 and 1, and if  $a_1 + a_2 + \dots + a_m = 1$ , then this is called a **convex combination**.

For example, if  $\mathbf{x} = (3, 1, 0, 2)$  and  $\mathbf{y} = (0, -1, 2, 1)$ , then  $0.2\mathbf{x} + 0.8\mathbf{y} = (0.6, 0.2, 0, 0.4) + (0, -0.8, 1.6, 0.8) = (0.6, -0.6, 1.6, 1.2)$  is a linear combination of  $\mathbf{x}$  and  $\mathbf{y}$ . Moreover, it is a convex combination, since the coefficients are 0.2 and 0.8 (both between 0 and 1) and  $0.2 + 0.8 = 1$ .



**convex set** = is a set of vectors such that whenever we take a convex combination of vectors from this set, then this convex combination also belongs to the set

**extreme point** of a convex set = cannot be written down as a convex combination of other vectors

(every convex set is the set of all convex combinations (**convex hull**) of its extreme points)

**matrix** = 2-dimensional array of numbers, for example  $\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 3 & 2 & 4 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 4 & 1 & 2 \end{bmatrix}$

$m \times n$  matrix has  $m$  rows and  $n$  columns, entries of matrix  $\mathbf{A}$  are  $a_{ij}$  where  $i$  is row and  $j$  is column

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \underbrace{\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}}_{i\text{-th row of } \mathbf{A}} \quad \left. \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \right\} j\text{-th column of } \mathbf{A}$$

**multiplying by a scalar**

matrix  $\mathbf{A}$  with entries  $a_{ij}$ , then  $\mathbf{B} = k \cdot \mathbf{A}$  is a matrix  $\mathbf{B}$  with entries  $b_{ij} = k \cdot a_{ij}$

$$2 \cdot \begin{bmatrix} 1 & 0 & 3 & 1 \\ 3 & 2 & 4 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 6 & 2 \\ 6 & 4 & 8 & 0 \\ 4 & 6 & 0 & 2 \\ 0 & 8 & 2 & 4 \end{bmatrix}$$



**adding matrices** of the same size  $m \times n$

adding **A** with entries  $a_{ij}$  to matrix **B** with entries  $b_{ij}$  is a matrix **C** with entries  $c_{ij} = a_{ij} + b_{ij}$

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 3 & 2 & 4 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 4 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 4 & 3 \\ 2 & 0 & 1 & 3 \\ 0 & 2 & 1 & 4 \\ 3 & 0 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 7 & 4 \\ 5 & 2 & 5 & 3 \\ 2 & 5 & 1 & 5 \\ 3 & 4 & 4 & 6 \end{bmatrix}$$

**multiplying matrices:** matrix **A** of size  $m \times n$  multiplied by matrix **B** of size  $n \times k$  results in a matrix

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B} \text{ of size } m \times k \text{ with entries } c_{ij} \text{ where } c_{ij} = b_{j1}a_{1i} + b_{j2}a_{2i} + \dots + b_{jn}a_{ni}$$

$c_{ij}$  is the **scalar product** of  $i$ -th row of **A** with  $j$ -th column of **B**

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 3 & 2 & 4 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 4 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 4 & 3 \\ 2 & 0 & 1 & 3 \\ 0 & 2 & 1 & 4 \\ 3 & 0 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 10 & 19 \\ 13 & 11 & 18 & 31 \\ 15 & 2 & 14 & 19 \\ 14 & 2 & 11 & 24 \end{bmatrix} \quad (2, 3, 0, 1) \cdot (1, 0, 2, 0) = 2$$

Note that  $\mathbf{A} \cdot \mathbf{B}$  is **not** the same as  $\mathbf{B} \cdot \mathbf{A}$

→ except for this, matrix addition and multiplication obey exactly the same laws as numbers

→ from now on **vector** with  $m$  entries is to be treated as a  $m \times 1$  matrix (column)

→ for all practical intents and purposes, we can deal with matrices and vectors just like we deal with numbers

**multiplying matrix** by a **vector** = just like multiplying two matrices

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 3 & 2 & 4 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 4 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 2 \\ 2 \end{bmatrix}$$

**transpose of a matrix:** rotate an  $m \times n$  matrix **A** along its main diagonal, the resulting  $n \times m$  matrix  $\mathbf{A}^T$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 3 & 2 & 4 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \quad \mathbf{A}^T = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix} \quad \mathbf{x}^T = [ 1 \ 0 \ 3 \ 1 ]$$

Note that  $(\mathbf{A}^T)^T = \mathbf{A}$  and  $(\mathbf{A} \cdot \mathbf{B})^T = \mathbf{B}^T \cdot \mathbf{A}^T$

## 5.1 Systems of linear equations

A system of linear equations has the following form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

Using the matrix notation we can simply write it as  $\mathbf{Ax} = \mathbf{b}$  where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

**Basis** of solutions to the system  $\mathbf{Ax} = \mathbf{b}$

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 3 & 2 & 4 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Let us multiply (from the left) both sides of the equation by this matrix:  $\begin{bmatrix} -4 & 3 & -2 \\ 8/3 & -2 & 5/3 \\ 5/3 & -1 & 2/3 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} -4 & 3 & -2 \\ 8/3 & -2 & 5/3 \\ 5/3 & -1 & 2/3 \end{bmatrix}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 3 & 1 \\ 3 & 2 & 4 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -4 & 3 & -2 \\ 8/3 & -2 & 5/3 \\ 5/3 & -1 & 2/3 \end{bmatrix}} \cdot \underbrace{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 13/3 \\ 0 & 0 & 1 & 7/3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -10/3 \\ -4/3 \end{bmatrix}$$

This operation does not change the solutions to this system (if we multiply with a **non-singular** matrix)

We can expand it back to the system of linear equations

$$\begin{array}{rcl} x_1 & - 6x_4 & = 5 \\ x_2 & + \frac{13}{3}x_4 & = -\frac{10}{3} \\ x_3 & + \frac{7}{3}x_4 & = -\frac{4}{3} \end{array} \quad \rightarrow \rightarrow \quad \begin{array}{rcl} x_1 & = & 5 + 6x_4 \\ x_2 & = & -\frac{10}{3} - \frac{13}{3}x_4 \\ x_3 & = & -\frac{4}{3} - \frac{7}{3}x_4 \end{array}$$

The system on the right is in a **dictionary** form.

We can read-off a solution by setting  $x_4$  to some number and calculating the values of  $x_1, x_2, x_3$  from the equations. In particular, we can set  $x_4 = 0$  in which case  $x_1 = 5$ ,  $x_2 = -\frac{10}{3}$ , and  $x_3 = -\frac{4}{3}$ .

**How did we choose the matrix to multiply?** We chose the **inverse** matrix of the first three columns.

This is so that the first three columns will be turned to the **identity** matrix  $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let us write  $\mathbf{B}$  for the matrix of first three columns:  $\mathbf{B} = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 2 & 4 \\ 2 & 3 & 0 \end{bmatrix}$

**inverse matrix** of a matrix  $\mathbf{B}$  is a matrix  $\mathbf{B}^{-1}$  such that  $\mathbf{B}^{-1} \cdot \mathbf{B} = \mathbf{I}$

(the inverse matrix may not always exist – we shall ignore this issue here)

**How do we obtain an inverse matrix?** we perform elementary row operations on the following matrix

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 3 & 2 & 4 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array} \right]$$

$\underbrace{\hspace{1.5cm}}_{\mathbf{B}} \quad \underbrace{\hspace{1.5cm}}_{\mathbf{I}}$

### elementary row operations

- multiply a row by a non-zero number
- add a row to another row
- exchange two rows

We call  $\mathbf{B}$  the **basis matrix**.

Let us try a different basis. Choose the 2nd, 3rd, and 4th columns of  $\mathbf{A}$  corresponding to variables  $x_2, x_3, x_4$ . (With a slight abuse of notation, we say that  $\{x_2, x_3, x_4\}$  is our **basis**, and  $x_2, x_3, x_4$  are **basic variables**.)

$$\mathbf{B} = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \text{attach the identity matrix:} \quad \left[ \begin{array}{ccc|ccc} 0 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

elementary row operations: multiply the 3rd row by  $-2/3$  then add to the 2nd row

$$\left[ \begin{array}{ccc|ccc} 0 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ -2 & 0 & -\frac{2}{3} & 0 & 0 & -\frac{2}{3} \end{array} \right] \quad \left[ \begin{array}{ccc|ccc} 0 & 3 & 1 & 1 & 0 & 0 \\ 0 & 4 & -\frac{2}{3} & 0 & 1 & -\frac{2}{3} \\ -2 & 0 & -\frac{2}{3} & 0 & 0 & -\frac{2}{3} \end{array} \right]$$

multiply the 1st row by  $-4/3$  and then add to the 2nd row

$$\left[ \begin{array}{ccc|ccc} 0 & -4 & -\frac{4}{3} & -\frac{4}{3} & 0 & 0 \\ 0 & 4 & -\frac{2}{3} & 0 & 1 & -\frac{2}{3} \\ -2 & 0 & -\frac{2}{3} & 0 & 0 & -\frac{2}{3} \end{array} \right] \quad \left[ \begin{array}{ccc|ccc} 0 & -4 & -\frac{4}{3} & -\frac{4}{3} & 0 & 0 \\ 0 & 0 & -2 & -\frac{4}{3} & 1 & -\frac{2}{3} \\ -2 & 0 & -\frac{2}{3} & 0 & 0 & -\frac{2}{3} \end{array} \right]$$

multiply the 1st, 2nd, 3rd row by  $-1/4$ ,  $1/6$ ,  $-1/2$  respectively; then add 2nd row to 1st and 3rd row

$$\left[ \begin{array}{ccc|ccc} 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{9} & \frac{1}{6} & -\frac{1}{9} \\ 1 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{array} \right] \quad \left[ \begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{9} & \frac{1}{6} & -\frac{1}{9} \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{9} & \frac{1}{6} & -\frac{1}{9} \\ 1 & 0 & 0 & -\frac{2}{9} & \frac{1}{6} & \frac{2}{9} \end{array} \right]$$

multiply the 2nd row by  $-1/2$  and swap the order of rows to get the identity matrix on the left

$$\left[ \begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{9} & \frac{1}{6} & -\frac{1}{9} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{2} & \frac{1}{3} \\ 1 & 0 & 0 & -\frac{2}{9} & \frac{1}{6} & \frac{2}{9} \end{array} \right] \quad \underbrace{\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{9} & \frac{1}{6} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{1}{9} & \frac{1}{6} & -\frac{1}{9} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{2} & \frac{1}{3} \end{array} \right]}_{\mathbf{I} \quad \mathbf{B}^{-1}}$$

Going back to our original system  $\mathbf{Ax} = \mathbf{b}$ , we multiply by  $\mathbf{B}^{-1}$  from the left

$$\mathbf{B}^{-1}\mathbf{Ax} = \mathbf{B}^{-1}\mathbf{b}$$

$$\underbrace{\begin{bmatrix} -\frac{2}{9} & \frac{1}{6} & \frac{2}{9} \\ \frac{1}{9} & \frac{1}{6} & -\frac{1}{9} \\ \frac{2}{3} & -\frac{1}{2} & \frac{1}{3} \end{bmatrix}}_{\mathbf{B}^{-1}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 3 & 1 \\ 3 & 2 & 4 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}}_{\mathbf{A}} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{2}{9} & \frac{1}{6} & \frac{2}{9} \\ \frac{1}{9} & \frac{1}{6} & -\frac{1}{9} \\ \frac{2}{3} & -\frac{1}{2} & \frac{1}{3} \end{bmatrix}}_{\mathbf{B}^{-1}\mathbf{b}} \cdot \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{13}{18} & 1 & 0 & 0 \\ \frac{7}{18} & 0 & 1 & 0 \\ -\frac{1}{6} & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{5}{18} \\ \frac{11}{18} \\ -\frac{5}{6} \end{bmatrix}$$

$$\begin{array}{rcl} \frac{13}{18}x_1 + x_2 & = & \frac{5}{18} \\ \frac{7}{18}x_1 + x_3 & = & \frac{11}{18} \\ -\frac{1}{6}x_1 + x_4 & = & -\frac{5}{6} \end{array} \quad \rightarrow \rightarrow \quad \begin{array}{rcl} x_2 & = & \frac{5}{18} - \frac{13}{18}x_1 \\ x_3 & = & \frac{11}{18} - \frac{7}{18}x_1 \\ x_4 & = & -\frac{5}{6} + \frac{1}{6}x_1 \end{array}$$

## 5.2 Summary

Consider the maximization problem

$$\begin{array}{ll} \max & \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N \\ \text{subject to} & \left[ \begin{array}{c|c} \mathbf{B} & \mathbf{N} \end{array} \right] \cdot \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} \mathbf{b} \end{bmatrix} \\ & \mathbf{x} \geq 0 \end{array}$$

1. choose basic variables, let  $\mathbf{x}_B$  denote the **vector** of basic variables
2. let  $\mathbf{B}$  denote the **basis** matrix formed by taking the columns of  $\mathbf{A}$  corresponding to the basic variables
3. let  $\mathbf{c}_B$  denote the vector of the coefficients of  $\mathbf{c}$  of the basic variables
4. let  $\mathbf{x}_N$  denote the vector of the remaining (non-basic) variables, and let  $\mathbf{c}_N$  denote the vector of the corresponding coefficients of  $\mathbf{c}$
5. let  $\mathbf{N}$  denote the columns of  $\mathbf{A}$  corresponding to the non-basic variables in  $\mathbf{x}_N$

Then (assuming  $\mathbf{B}^{-1}$  exists) we can rewrite

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{Bx}_B + \mathbf{Nx}_N &= \mathbf{b} \\ \mathbf{B}^{-1}(\mathbf{Bx}_B + \mathbf{Nx}_N) &= \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{Bx}_B + \mathbf{B}^{-1}\mathbf{Nx}_N &= \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{x}_B + \mathbf{B}^{-1}\mathbf{Nx}_N &= \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{x}_B &= \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{Nx}_N \end{aligned}$$

Now we can substitute  $\mathbf{x}_B$  to the objective function:

$$z = \mathbf{c}^T \mathbf{x} = \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N = \mathbf{c}_B^T (\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{Nx}_N) + \mathbf{c}_N^T \mathbf{x}_N = \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{N}) \mathbf{x}_N$$

We put it together to obtain the corresponding dictionary:

$$\begin{array}{l} \mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{Nx}_N \\ \hline z = \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{N}) \mathbf{x}_N \end{array}$$

From this we immediately see that the corresponding **basic solution** when  $\mathbf{x}_N = 0$  is given as

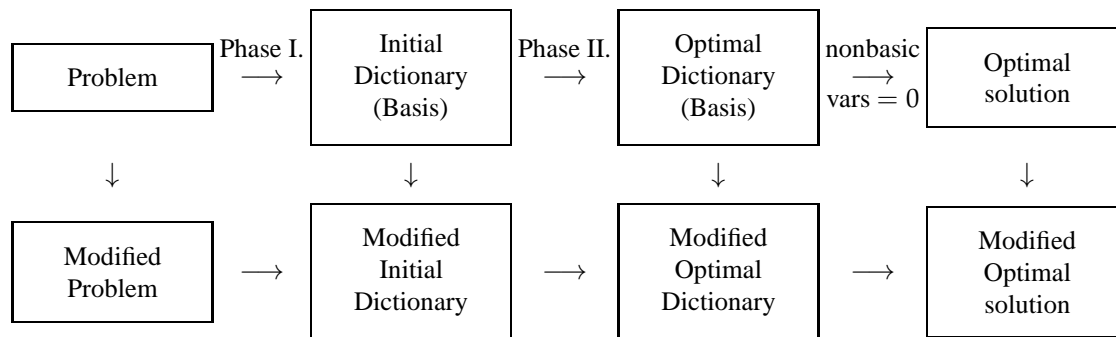
$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \quad \text{with the value of the objective function} \quad z = \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{b}$$

The non-basic variables in the objective function are  $(\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{N}) \mathbf{x}_N$ . Their coefficients tell us whether or not this solution is optimal. In other words, the solution is optimal(maximal) if  $(\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{N}) \leq 0$ .

## Sensitivity Analysis

Also called *post-optimality analysis*: determining what effect (if any) do changes to the input problem have on the optimality (feasibility) of a solution to the problem.

**Motivation:** often coefficients/values in a mathematical formulation are only best estimates; we need to know how much room for an error do we have, how sensitive the solution is to the quality of these estimates.



Modified problem is obtained by:

- changing the objective function
- changing the right-hand side (rhs) of a constraint
- adding a variable/activity
- adding a constraint

For what changes is the original optimal solution also optimal in the modified problem?

For what changes is it feasible?

For what changes is the optimal basis also optimal in the modified problem?

How do we recompute modified optimal solution from optimal basis?

How do we recompute modified optimal dictionary?

### Key concepts:

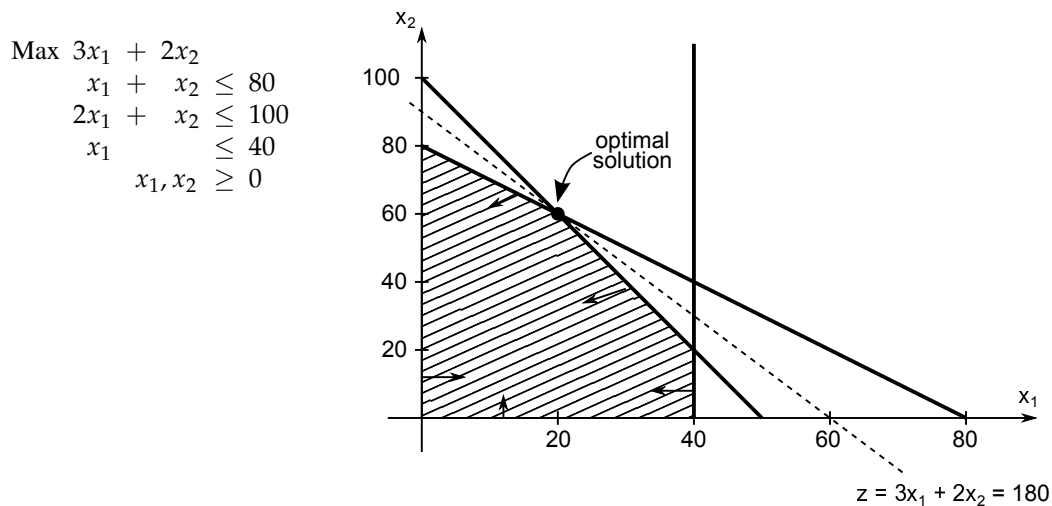
- *shadow prices* = a mechanism to assign prices to *items* (rows)
- *reduced costs* = costs of *activities* (columns) in terms of shadow prices



$$\text{basic variables: } \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 80 \\ 100 \\ 40 \end{pmatrix} = (-80 + 100, 160 - 100, 80 - 100 + 40) = (20, 60, 20)$$

$$\text{shadow prices: } \boldsymbol{\pi} = \mathbf{c}_B \mathbf{B}^{-1} = (3, 2, 0) \begin{pmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = (-3 + 4 + 0, 3 - 2 + 0, 0 + 0 + 0) = (1, 1, 0)$$

$$\text{reduced costs: } \mathbf{c}_N - \mathbf{c}_B \mathbf{B}^{-1} \mathbf{N} = \mathbf{c}_N - \boldsymbol{\pi} \mathbf{N} = (0, 0) - (1, 1, 0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = (0, 0) - (1, 1) = (-1, -1)$$

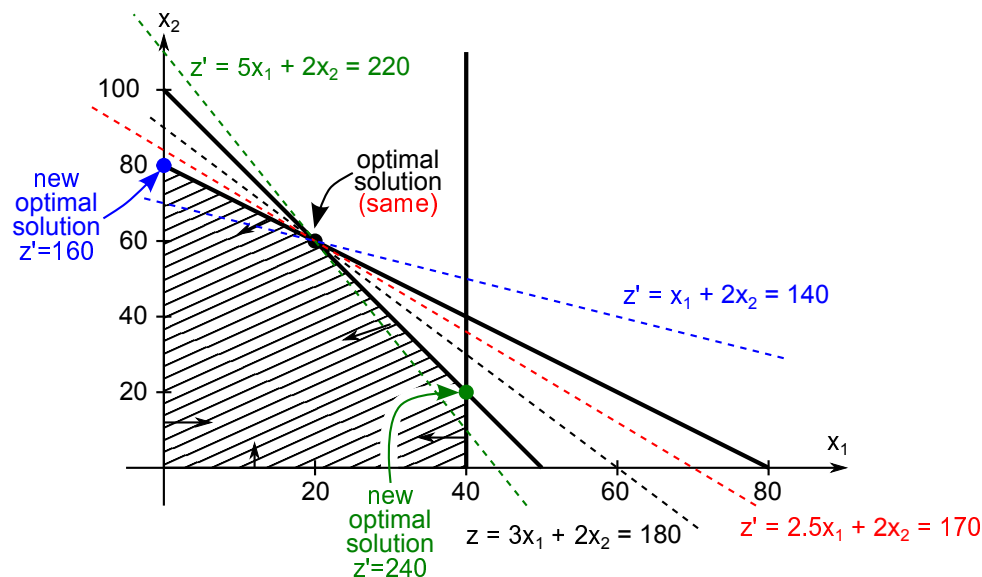


## 6.1 Changing the objective function

optimal solution  $x_1 = 20, x_2 = 60$

change the coefficient  $c_1 = 3$  in  $z$  to  $2.5 \rightarrow z' = 2.5x_1 + 2x_2$

$\rightarrow$  the solution  $x_1 = 20, x_2 = 60$  still optimal, value  $z' = 170$



change  $c_1 = 3$  to  $1 \rightarrow z' = x_1 + 2x_2$

$\rightarrow$  the solution  $x_1 = 20, x_2 = 60$  **not optimal**, value  $z' = 140$

$\rightarrow$  better solution  $x_1 = 0, x_2 = 80$ , value  $z' = 160 > 140$

change  $c_1 = 3$  to  $5 \rightarrow z' = 5x_1 + 2x_2$

$\rightarrow$  the solution  $x_1 = 20, x_2 = 60$  **not optimal**, value  $z' = 220$

$\rightarrow$  better solution  $x_1 = 40, x_2 = 20$ , value  $z' = 240 > 220$

problem formulation	initial (feasible) dictionary	optimal dictionary
$\begin{aligned} \text{Max } & 3x_1 + 2x_2 \\ & x_1 + x_2 \leq 80 \\ & 2x_1 + x_2 \leq 100 \\ & x_1 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$	$\begin{aligned} x_3 &= 80 - x_1 - x_2 \\ x_4 &= 100 - 2x_1 - x_2 \\ x_5 &= 40 - x_1 \\ \hline z &= 0 + 3x_1 + 2x_2 \end{aligned}$	$\begin{aligned} x_1 &= 20 + x_3 - x_4 \\ x_2 &= 60 - 2x_3 + x_4 \\ x_5 &= 20 - x_3 + x_4 \\ \hline z &= 180 - x_3 - x_4 \end{aligned}$

From any dictionary (basic vars above the line) we can always express the objective function below the line

$$z = 3x_1 + 2x_2 = 3(\overbrace{20 + x_3 - x_4}^{x_1}) + 2(\overbrace{60 - 2x_3 + x_4}^{x_2}) = 60 + 3x_3 - 3x_4 + 120 - 4x_3 + 2x_4 = 180 - x_3 - x_4$$

This allows us to easily obtain a dictionary for the modified problem corresponding to  $x_1 = 20, x_2 = 60$

$c_1 = 2.5$	$c_1 = 1$	$c_1 = 5$
$\begin{aligned} x_1 &= 20 + x_3 - x_4 \\ x_2 &= 60 - 2x_3 + x_4 \\ x_5 &= 20 - x_3 + x_4 \\ \hline z' &= 2.5x_1 + 2x_2 \\ &= 50 + 2.5x_3 - 2.5x_4 \\ &\quad + 120 - 4x_3 + 2x_4 \\ z' &= 170 - 1.5x_3 - 0.5x_4 \end{aligned}$	$\begin{aligned} x_1 &= 20 + x_3 - x_4 \\ x_2 &= 60 - 2x_3 + x_4 \\ x_5 &= 20 - x_3 + x_4 \\ \hline z' &= x_1 + 2x_2 \\ &= 20 + x_3 - x_4 \\ &\quad + 120 - 4x_3 + 2x_4 \\ z' &= 140 - 3x_3 + x_4 \end{aligned}$	$\begin{aligned} x_1 &= 20 + x_3 - x_4 \\ x_2 &= 60 - 2x_3 + x_4 \\ x_5 &= 20 - x_3 + x_4 \\ \hline z' &= 5x_1 + 2x_2 \\ &= 100 + 5x_3 - 5x_4 \\ &\quad + 120 - 4x_3 + 2x_4 \\ z' &= 220 + x_3 - 3x_4 \end{aligned}$
<b>optimal</b>	<b>not optimal</b>	<b>not optimal</b>

### Coefficient ranging

we want to find the range for the coefficient  $c_1$  for which the optimal solution remains optimal

$$\begin{aligned} z' &= c_1x_1 + 2x_2 = c_1(\overbrace{20 + x_3 - x_4}^{x_1}) + 2(\overbrace{60 - 2x_3 + x_4}^{x_2}) = \\ &= 20c_1 + c_1x_3 - c_1x_4 + 120 - 4x_3 + 2x_4 \\ &= (20c_1 + 120) + \underbrace{(c_1 - 4)x_3}_{\leq 0} + \underbrace{(2 - c_1)x_4}_{\leq 0} \end{aligned}$$

It is **optimal** if all coefficients are non-positive  $\rightarrow c_1 - 4 \leq 0$  and  $2 - c_1 \leq 0 \implies \boxed{2 \leq c_1 \leq 4}$

Same for 2nd coefficient  $c_2$ : 
$$z' = 3x_1 + c_2x_2 = 3(\overbrace{20 + x_3 - x_4}^{x_1}) + c_2(\overbrace{60 - 2x_3 + x_4}^{x_2}) =$$

$$= 60 + 3x_3 - 3x_4 + 60c_2 - 2c_2x_3 + c_2x_4$$

$$= (60c_2 + 60) + \underbrace{(3 - 2c_2)x_3}_{\leq 0} + \underbrace{(c_2 - 3)x_4}_{\leq 0}$$

$\rightarrow 3 - 2c_2 \leq 0$  and  $c_2 - 3 \leq 0 \implies \boxed{1.5 \leq c_2 \leq 3}$

### General formula

Changing the coefficient  $c_i$  of  $x_i$  in  $z$  to  $(c_i + \Delta) \rightarrow$  the value of  $z$  changes by  $\Delta x_i$

$$z' = \underbrace{\mathbf{c}_B \mathbf{x}_B + \mathbf{c}_N \mathbf{x}_N}_z + \Delta x_i = z + \Delta x_i = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} + (\mathbf{c}_N - \boldsymbol{\pi} \mathbf{N}) \mathbf{x}_N + \Delta x_i$$



**Non-basic variable:** If  $x_i$  is  $j$ -th non-basic variable, then only the coefficient of  $x_i$  changes in  $z$ , it is increased by  $\Delta$ , we must sure it remains non-positive.

The original coefficient of  $x_i$  is the reduced cost  $\bar{c}_i$  of  $x_i$ , the  $j$ -th coefficient in  $\mathbf{c}_N - \boldsymbol{\pi}\mathbf{N}$ .

$$\bar{c}_i + \Delta \leq 0 \rightarrow \text{solve for } \Delta \rightarrow \boxed{\Delta \leq -\bar{c}_i}$$

**Basic variable:** If  $x_i$  is  $j$ -th basic variable, then all coefficients of  $z$  change; the coefficients change by  $-\Delta\mathbf{a}$  where  $\mathbf{a}$  is the  $j$ -th row of  $\mathbf{B}^{-1}\mathbf{N}$ . The resulting coefficients must remain non-positive.

The original coefficients of variables in  $z$  are the reduced costs  $\mathbf{c}_N - \boldsymbol{\pi}\mathbf{N}$ .

$$(\mathbf{c}_N - \boldsymbol{\pi}\mathbf{N}) - \Delta\mathbf{a} \leq 0 \rightarrow \text{solve for } \Delta$$

## 6.2 Changing the right-hand side value

<p>Max <math>3x_1 + 2x_2</math>  <math>x_1 + x_2 \leq 80</math>  <math>2x_1 + x_2 \leq 100</math>  <math>x_1 \leq 40</math>  <math>x_1, x_2 \geq 0</math></p>	<p>optimal dictionary</p> $\begin{array}{r} x_1 = 20 + x_3 - x_4 \\ x_2 = 60 - 2x_3 + x_4 \\ x_5 = 20 - x_3 + x_4 \\ \hline z = 180 - x_3 - x_4 \end{array}$	<p>optimal basis <math>\{x_1, x_2, x_5\}</math></p>
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change the coefficient  $b_1 = 80$  on the rhs of the 1st constraint to 70  $\rightarrow x_1 + x_2 \leq 70$

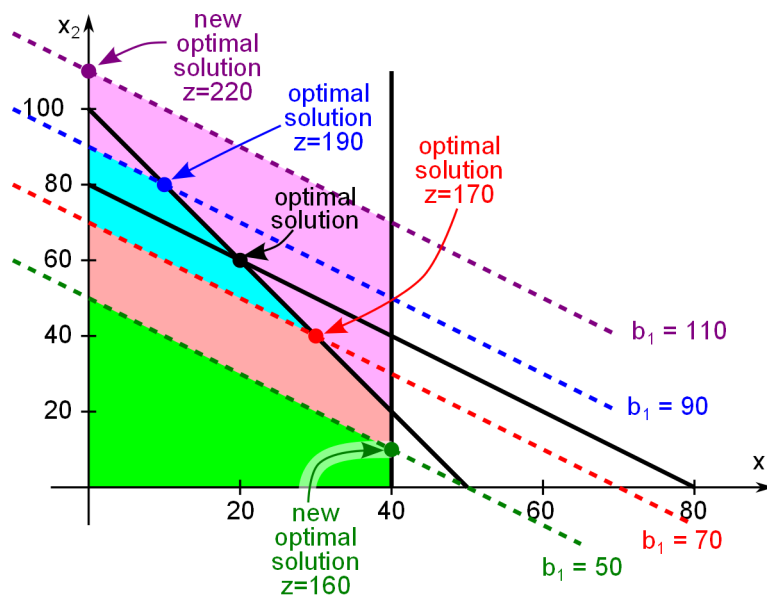
$\rightarrow$  the point  $x_1 = 20, x_2 = 60$  **not feasible**

$\rightarrow$  new optimum  $x_1 = 30, x_2 = 40$  but **same basis**  $\{x_1, x_2, x_5\}$

change  $b_1$  to 50  $\rightarrow x_1 + x_2 \leq 50$

$\rightarrow$  the point  $x_1 = 20, x_2 = 60$  **not feasible**

$\rightarrow$  new optimum  $x_1 = 10, x_2 = 40$  **new basis**  $\{x_1, x_2, x_4\}$



change  $b_1$  to 90  $\rightarrow x_1 + x_2 \leq 90$

$\rightarrow$  the point  $x_1 = 20, x_2 = 60$  feasible but **not optimal**

$\rightarrow$  new optimum  $x_1 = 10, x_2 = 80$  **same basis**

change  $b_1$  to 110  $\rightarrow x_1 + x_2 \leq 110$

$\rightarrow$  the point  $x_1 = 20, x_2 = 60$  feasible but **not optimal**

$\rightarrow$  new optimum  $x_1 = 0, x_2 = 110$  **new basis**  $\{x_2, x_4, x_5\}$

**Coefficient ranging**

We want to find the values of  $b_1$  for which the optimal basis remains optimal (optimal solution may not)

initial dictionary	final dictionary
$x_3 = 80 - x_1 - x_2$	$x_1 = 20 + x_3 - x_4$
$x_4 = 100 - 2x_1 - x_2$	$x_2 = 60 - 2x_3 + x_4$
$x_5 = 40 - x_1$	$x_5 = 20 - x_3 + x_4$
$z = 0 + 3x_1 + 2x_2$	$z = 180 - x_3 - x_4$

Changing  $b_1 = 80$  to  $80 + \Delta_1$  where  $\Delta_1$  can be positive or negative. How does it change the final dictionary?

$x_3 = (80 + \Delta_1) - x_1 - x_2$	$(x_3 - \Delta_1) = 80 - x_1 - x_2$	$x'_3 = 80 - x_1 - x_2$
$x_4 = 100 - 2x_1 - x_2$	$x_4 = 100 - 2x_1 - x_2$	$x_4 = 100 - 2x_1 - x_2$
$x_5 = 40 - x_1$	$x_5 = 40 - x_1$	$x_5 = 40 - x_1$
$z = 0 + 3x_1 + 2x_2$	$z = 0 + 3x_1 + 2x_2$	$z = 0 + 3x_1 + 2x_2$

where  $x'_3 = x_3 - \Delta_1$

**same dictionary** except that  $x_3$  is now  $x'_3$  → following the same pivoting steps

we pivot to the basis  $\{x_1, x_2, x_5\}$  and must reach the **same** final dictionary (with  $x'_3$  in place of  $x_3$ )

$x_1 = 20 + x'_3 - x_4$	$x_1 = 20 + (x_3 - \Delta_1) - x_4$	$x_1 = (20 - \Delta_1) + x_3 - x_4$
$x_2 = 60 - 2x'_3 + x_4$	$x_2 = 60 - 2(x_3 - \Delta_1) + x_4$	$x_2 = (60 + 2\Delta_1) - 2x_3 + x_4$
$x_5 = 20 - x'_3 + x_4$	$x_5 = 20 - (x_3 - \Delta_1) + x_4$	$x_5 = (20 + \Delta_1) - x_3 + x_4$
$z = 180 - x'_3 - x_4$	$z = 180 - (x_3 - \Delta_1) - x_4$	$z = (180 + \Delta_1) - x_3 - x_4$
		final modified dictionary

When is this dictionary optimal? When it is **feasible**, since all coefficients in  $z$  are non-positive

It is feasible, if  $x_1, x_2, x_5$  are non-negative. Setting the non-basic variables  $x_3 = x_4 = 0$ , we obtain

$$\left. \begin{array}{l} x_1 = 20 - \Delta_1 \geq 0 \\ x_2 = 60 + 2\Delta_1 \geq 0 \\ x_5 = 20 + \Delta_1 \geq 0 \end{array} \right\} \rightarrow \left. \begin{array}{l} \Delta_1 \leq 20 \\ \Delta_1 \geq -30 \\ \Delta_1 \geq -20 \end{array} \right\} \boxed{-20 \leq \Delta_1 \leq 20}$$

Try  $\Delta_1 = 10 \rightarrow b_1 = 90, x_1 = 10, x_2 = 80, z = 190 \rightarrow$  **exactly** as we saw before

$$\Delta_1 = -10 \rightarrow b_1 = 70, x_1 = 30, x_2 = 40, z = 170$$

Similarly for  $b_2 = 100$  the coeff of the 2nd constraint  $\rightarrow b_2 = 100 + \Delta_2 \Rightarrow$  substitute  $x'_4 = x_4 - \Delta_2$

$x_1 = 20 + x_3 - x'_4$	$x_1 = 20 + x_3 - (x_4 - \Delta_2)$	$x_1 = (20 + \Delta_2) + x_3 - x_4$
$x_2 = 60 - 2x_3 + x'_4$	$x_2 = 60 - 2x_3 + (x_4 - \Delta_2)$	$x_2 = (60 - \Delta_2) - 2x_3 + x_4$
$x_5 = 20 - x_3 + x'_4$	$x_5 = 20 - x_3 + (x_4 - \Delta_2)$	$x_5 = (20 - \Delta_2) - x_3 + x_4$
$z = 180 - x_3 - x'_4$	$z = 180 - x_3 - (x_4 - \Delta_2)$	$z = (180 + \Delta_2) - x_3 - x_4$

$$\text{optimal if feasible} \rightarrow \left. \begin{array}{l} x_1 = 20 + \Delta_2 \geq 0 \\ x_2 = 60 - \Delta_2 \geq 0 \\ x_5 = 20 - \Delta_2 \geq 0 \end{array} \right\} \rightarrow \left. \begin{array}{l} \Delta_2 \geq -20 \\ \Delta_2 \leq 60 \\ \Delta_2 \leq 20 \end{array} \right\} \boxed{-20 \leq \Delta_2 \leq 20}$$

Finally, changing  $b_3 = 40$ , the rhs of the 3rd constraint  $\rightarrow b_3 = 40 + \Delta_3 \Rightarrow$  substitute  $x'_5 = x_5 - \Delta_3$

$x_1 = 20 + x_3 - x_4$	$x_1 = 20 + x_3 - x_4$	$x_1 = 20 + x_3 - x_4$
$x_2 = 60 - 2x_3 + x_4$	$x_2 = 60 - 2x_3 + x_4$	$x_2 = 60 - 2x_3 + x_4$
$x'_5 = 20 - x_3 + x_4$	$(x_5 - \Delta_3) = 20 - x_3 + x_4$	$x_5 = (20 + \Delta_3) - x_3 + x_4$
$z = 180 - x_3 - x_4$	$z = 180 - x_3 - x_4$	$z = 180 - x_3 - x_4$

$$\text{optimal if feasible} \rightarrow (20 + \Delta_3) \geq 0 \rightarrow \Delta_3 \geq -20 \Rightarrow \boxed{-20 \leq \Delta_3 \leq \infty}$$

## General formula

Changing the rhs coefficient  $b_i$  to  $b_i + \Delta$ . Let  $\mathbf{d}$  denote the  $i$ -**the column of**  $B^{-1}$ . The dictionary changes:

$$\begin{aligned} \mathbf{x}_B &= \mathbf{B}^{-1}\mathbf{b} + \Delta\mathbf{d} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \\ z &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} + \mathbf{c}_B\Delta\mathbf{d} + (\mathbf{c}_N - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N \end{aligned}$$

We find for what values of  $\Delta$  the values of  $\mathbf{x}_B$  are non-negative (if the non-basic variables  $\mathbf{x}_N$  are set to 0).

$$\mathbf{x}_B = \underbrace{\mathbf{B}^{-1}\mathbf{b}}_{\text{old values}} + \Delta\mathbf{d} \geq 0$$

## Shadow prices

Let us summarize:

changing $b_1 = 80$ to $80 + \Delta_1$	changing $b_2 = 100$ to $100 + \Delta_2$	changing $b_3 = 40$ to $40 + \Delta_3$
$x_1 = 20 - \Delta_1$	$x_1 = 20 + \Delta_2$	$x_1 = 20$
$x_2 = 60 + 2\Delta_1$	$x_2 = 60 - \Delta_2$	$x_2 = 60$
$x_5 = 20 + \Delta_1$	$x_5 = 20 - \Delta_2$	$x_5 = 20 + \Delta_3$
<hr/> $z = 180 + \Delta_1$	<hr/> $z = 180 + \Delta_2$	<hr/> $z = 180$
for $-20 \leq \Delta_1 \leq 20$	for $-20 \leq \Delta_2 \leq 20$	for $-20 \leq \Delta_3 \leq \infty$

if the change is within the above bounds on  $\Delta_1, \Delta_2, \Delta_3$ , then:

- increasing the value of  $b_1$  by **1 unit** increases the objective (profit) by \$1  
(decreasing) (decreases)
  - increasing/decreasing  $b_2$  by **1 unit** increases/decreases the profit by \$1
  - increasing/decreasing  $b_3$  by **1 unit** increases/decreases the profit by \$0
- } shadow prices

**Economic interpretation:** if instead of production we sell/rent all capacity (80 hours in the carving shop, 100 hours in the finishing shop, 20 units of remaining demand) at shadow prices (\$1/hour, \$1/hour, \$0/unit), we obtain the same profit ( $80 \times \$1 + 100 \times \$1 + 20 \times \$0 = \$180$ ).

In fact, if we sell at \$1/hour up to 20 hours of capacity in the carving shop (since  $\Delta_1 \geq -20$ ) **or** we sell at \$1 up to 20 hours of capacity in the finishing shop (since  $\Delta_2 \geq -20$ ), and optimally produce toys using the remaining capacity, then our profit does not change (remains \$180). Similarly, if we obtain additional capacity at \$1/hour up to 20 hours in one of the shops (since  $\Delta_1 \leq 20$  and  $\Delta_2 \leq 20$ ), then profit remains the same.

Thus if we can obtain additional shop capacity at less than \$1/hour, then it pays to do it (up to 20 hours) and produce more toys using the additional capacity (yields higher profit). On the other hand, if we can sell shop capacity at more than \$1/hour, then it pays to do it instead of using it for production.

## 6.3 Detailed example

In the following LP, optimal solution is achieved for basis  $\{x_2, x_3, x_4\}$

$$\begin{aligned} \max z &= 2x_1 - 3x_2 + x_3 - x_5 \\ x_1 - x_2 + x_3 + x_4 &= 4 \\ -2x_1 + x_2 - x_3 + x_5 &= 1 \\ 2x_2 + x_3 + x_4 - x_5 &= 9 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Find ranges for individual coefficients of the objective function and for rhs coefficients.

## Using matrices

**1. Basis**  $\{x_2, x_3, x_4\}$  → split the coefficients to the basis matrix  $\mathbf{B}$  and non-basic matrix  $\mathbf{N}$

→ calculate the inverse of  $\mathbf{B}$ , denoted  $\mathbf{B}^{-1}$  (recall from linear algebra how to do this)

$$\mathbf{B} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad \mathbf{B}^{-1} = \begin{pmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{3} & -1 & \frac{1}{3} \\ 1 & 1 & 0 \end{pmatrix} \quad \mathbf{N} = \begin{pmatrix} 1 & 0 \\ -2 & -1 \\ 0 & -1 \end{pmatrix} \quad \begin{array}{l} \mathbf{c}_B = (-3, 1, 0) \\ \mathbf{c}_N = (2, -1) \end{array}$$

$$\boldsymbol{\pi} = \mathbf{c}_B \mathbf{B}^{-1} = (-3, 1, 0) \begin{pmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{3} & -1 & \frac{1}{3} \\ 1 & 1 & 0 \end{pmatrix} = \left(\frac{2}{3}, -1, -\frac{2}{3}\right)$$

$$\mathbf{c}_N - \mathbf{c}_B \mathbf{B}^{-1} \mathbf{N} = \mathbf{c}_N - \boldsymbol{\pi} \mathbf{N} = \left[ (2, -1) - \left(\frac{2}{3}, -1, -\frac{2}{3}\right) \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 0 & -1 \end{pmatrix} \right] = \left(-\frac{2}{3}, -\frac{2}{3}\right)$$

$$\mathbf{B}^{-1} \mathbf{N} = \begin{pmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{3} & -1 & \frac{1}{3} \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} \\ \frac{5}{3} & -\frac{4}{3} \\ -1 & 1 \end{pmatrix}$$

$$\mathbf{B}^{-1} \mathbf{b} = \begin{pmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{3} & -1 & \frac{1}{3} \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{2}{3} \\ 5 \end{pmatrix}$$

## 2. Changing the right-hand side

$$b_1: \text{ change from } 4 \text{ to } 4 + \Delta, \text{ the 1st column of } \mathbf{B}^{-1} \text{ is } \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

$$\begin{array}{ll} x_2 = \frac{5}{3} - \frac{1}{3}\Delta \geq 0 & \Delta \leq 5 \\ x_3 = \frac{2}{3} - \frac{1}{3}\Delta \geq 0 & \rightarrow \Delta \leq 2 \\ x_4 = 5 + \Delta \geq 0 & -5 \leq \Delta \end{array} \quad \rightarrow \boxed{-5 \leq \Delta \leq 2}$$

$$b_2: \text{ change from } 1 \text{ to } 1 + \Delta, \text{ the 2nd column of } \mathbf{B}^{-1} \text{ is } \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{array}{ll} x_2 = \frac{5}{3} + 0\Delta \geq 0 & 0 \leq \frac{5}{3} \\ x_3 = \frac{2}{3} - \Delta \geq 0 & \rightarrow \Delta \leq \frac{2}{3} \\ x_4 = 5 + \Delta \geq 0 & -5 \leq \Delta \end{array} \quad \rightarrow \boxed{-5 \leq \Delta \leq \frac{2}{3}}$$

$$b_3: \text{ change from } 9 \text{ to } 9 + \Delta, \text{ the 3rd column of } \mathbf{B}^{-1} \text{ is } \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}$$

$$\begin{array}{ll} x_2 = \frac{5}{3} + \frac{1}{3}\Delta \geq 0 & -5 \leq \Delta \\ x_3 = \frac{2}{3} + \frac{1}{3}\Delta \geq 0 & \rightarrow -2 \leq \Delta \\ x_4 = 5 + 0\Delta \geq 0 & 0 \leq 5 \end{array} \quad \rightarrow \boxed{-2 \leq \Delta}$$

## 3. Changing the objective function coefficients

$x_1$ : change from 2 to  $2 + \Delta$  →  $z$  changes by  $\Delta x_1$

$x_1$  is non-basic with reduced cost  $-\frac{2}{3}$  (1st coeff in  $\mathbf{c}_N - \boldsymbol{\pi} \mathbf{N}$ ) → coefficient of  $x_1$  in  $z$  is  $-\frac{2}{3}$

only the coefficient of  $x_1$  changes in  $z$  → the new coeff is  $-\frac{2}{3} + \Delta$  → must be non-positive

$$-\frac{2}{3} + \Delta \leq 0 \quad \rightarrow \quad \boxed{\Delta \leq \frac{2}{3}}$$

$x_2$  : change from  $-3$  to  $-3 + \Delta \rightarrow z$  changes by  $\Delta x_2$

$x_2$  is 1st basic variable, the coefficients of  $(x_1, x_5)$  in  $z$  change by  $-\Delta \mathbf{a}$  where  $\mathbf{a}$  is the 1st row of  $\mathbf{B}^{-1}\mathbf{N}$   
 coefficients of  $(x_1, x_5)$  in  $z$  are the reduced costs  $\mathbf{c}_N - \boldsymbol{\pi}\mathbf{N} = (-\frac{2}{3}, -\frac{2}{3})$  and  $\mathbf{a} = (-\frac{1}{3}, -\frac{1}{3})$

the resulting coefficients must be non-positive (in order for the basis to remain optimal)

$$\begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} - \Delta \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{array}{l} -\frac{2}{3} + \frac{1}{3}\Delta \leq 0 \\ -\frac{2}{3} + \frac{1}{3}\Delta \leq 0 \end{array} \rightarrow \boxed{\Delta \leq 2}$$

$x_3$  : change from  $1$  to  $1 + \Delta \rightarrow z$  changes by  $\Delta x_3$ , the 2nd basic variable, the 2nd row of  $\mathbf{B}^{-1}\mathbf{N} = (\frac{5}{3}, -\frac{4}{3})$

$$\begin{array}{l} -\frac{2}{3} - \frac{5}{3}\Delta \leq 0 \\ 0 \end{array} \rightarrow \Delta \geq -\frac{2}{5} \rightarrow \begin{array}{l} -\frac{2}{3} + \frac{4}{3}\Delta \leq 0 \\ 0 \end{array} \rightarrow \Delta \leq \frac{1}{2} \rightarrow \boxed{-\frac{2}{5} \leq \Delta \leq \frac{1}{2}}$$

$x_4$  : change from  $0$  to  $0 + \Delta \rightarrow z$  changes by  $\Delta x_4$ , the 3rd basic variable, the 3rd row of  $\mathbf{B}^{-1}\mathbf{N} = (-1, 1)$

$$\begin{array}{l} -\frac{2}{3} + \Delta \leq 0 \\ -\frac{2}{3} - \Delta \leq 0 \end{array} \rightarrow \begin{array}{l} \Delta \leq \frac{2}{3} \\ \Delta \geq -\frac{2}{3} \end{array} \rightarrow \boxed{-\frac{2}{3} \leq \Delta \leq \frac{2}{3}}$$

$x_5$  : changes from  $-1$  to  $-1 + \Delta \rightarrow z$  changes by  $\Delta x_5$ , non-basic variable, only coeff  $-\frac{2}{3}$  of  $x_5$  changes by  $\Delta$

$$-\frac{2}{3} + \Delta \leq 0 \rightarrow \boxed{\Delta \leq \frac{2}{3}}$$

## Using dictionaries (optional)

### 1. Construct corresponding dictionary

Change rhs to (artificial) variables  $b_1, b_2, b_3$  and pivot to the basis  $\{x_2, x_3, x_4\}$

$$\begin{array}{rcl} x_1 - x_2 + x_3 + x_4 & = & b_1 \\ -2x_1 + x_2 - x_3 + x_5 & = & b_2 \\ 2x_2 + x_3 + x_4 - x_5 & = & b_3 \end{array} \quad \begin{array}{l} \boxed{x_2 = -b_1 + x_1 + x_3 + x_4} \\ -2x_1 + (-b_1 + x_1 + x_3 + x_4) - x_3 + x_5 = b_2 \\ 2(-b_1 + x_1 + x_3 + x_4) + x_3 + x_4 - x_5 = b_3 \end{array}$$

$$\boxed{x_4 = b_1 + b_2 + x_1 - x_5} \quad \Leftrightarrow \quad \begin{array}{rcl} -x_1 + x_4 + x_5 & = & b_1 + b_2 \\ 2x_1 + 3x_3 + 3x_4 - x_5 & = & 2b_1 + b_3 \end{array}$$

$$2x_1 + 3x_3 + 3(b_1 + b_2 + x_1 - x_5) - x_5 = 2b_1 + b_3$$

$$5x_1 + 3x_3 - 4x_5 = -b_1 - 3b_2 + b_3 \quad \boxed{x_3 = -\frac{1}{3}b_1 - b_2 + \frac{1}{3}b_3 - \frac{5}{3}x_1 + \frac{4}{3}x_5}$$

substitute back to  $x_4$  and then to  $x_2$ :

$$x_4 = b_1 + b_2 + x_1 - x_5 \quad x_2 = -b_1 + x_1 + x_3 + x_4 = -b_1 + x_1 + (-\frac{1}{3}b_1 - b_2 + \frac{1}{3}b_3 - \frac{5}{3}x_1 + \frac{4}{3}x_5) + (b_1 + b_2 + x_1 - x_5) = -\frac{1}{3}b_1 + \frac{1}{3}b_3 + \frac{1}{3}x_1 + \frac{1}{3}x_5$$

$$\text{substitute to } z \rightarrow z = 2x_1 - 3x_2 + x_3 - x_5 = 2x_1 - 3(-\frac{1}{3}b_1 + \frac{1}{3}b_3 + \frac{1}{3}x_1 + \frac{1}{3}x_5) + (-\frac{1}{3}b_1 - b_2 + \frac{1}{3}b_3 - \frac{5}{3}x_1 + \frac{4}{3}x_5) - x_5 = \frac{2}{3}b_1 - b_2 - \frac{2}{3}b_3 - \frac{2}{3}x_1 - \frac{2}{3}x_5$$

Resulting dictionary:

$$\boxed{\begin{array}{rcl} x_2 & = & -\frac{1}{3}b_1 + \frac{1}{3}b_3 + \frac{1}{3}x_1 + \frac{1}{3}x_5 \\ x_3 & = & -\frac{1}{3}b_1 - b_2 + \frac{1}{3}b_3 - \frac{5}{3}x_1 + \frac{4}{3}x_5 \\ x_4 & = & b_1 + b_2 + x_1 - x_5 \\ \hline z & = & \frac{2}{3}b_1 - b_2 - \frac{2}{3}b_3 - \frac{2}{3}x_1 - \frac{2}{3}x_5 \end{array}}$$

### 2. Changing the right-hand side coefficients

$b_1$  : change  $b_1 = 4$  to  $4 + \Delta$ , and let  $b_2 = 1$  and  $b_3 = 9$  the same. Check if the resulting values of basic variables are non-negative (remember to set the non-basic variables to zero).

$$\begin{aligned} x_2 &= -\frac{1}{3}(4 + \Delta) + \frac{9}{3} \geq 0 && \Delta \leq 5 \\ x_3 &= -\frac{1}{3}(4 + \Delta) - 1 + \frac{9}{3} \geq 0 && \rightarrow \Delta \leq 2 \quad \rightarrow \boxed{-5 \leq \Delta \leq 2} \\ x_4 &= (4 + \Delta) + 1 \geq 0 && -5 \leq \Delta \end{aligned}$$

$b_2$  : change  $b_2 = 1$  to  $1 + \Delta$ , let  $b_1 = 4$  and  $b_3 = 9$

$$\begin{aligned} x_2 &= -\frac{4}{3} + \frac{9}{3} \geq 0 && 0 \leq \frac{5}{3} \\ x_3 &= -\frac{4}{3} - (1 + \Delta) + \frac{9}{3} \geq 0 && \rightarrow \Delta \leq \frac{2}{3} \quad \rightarrow \boxed{-5 \leq \Delta \leq \frac{2}{3}} \\ x_4 &= 4 + (1 + \Delta) \geq 0 && -5 \leq \Delta \end{aligned}$$

$b_3$  : change  $b_3 = 9$  to  $9 + \Delta$ , let  $b_1 = 4$  and  $b_2 = 1$

$$\begin{aligned} x_2 &= -\frac{4}{3} + \frac{1}{3}(9 + \Delta) \geq 0 && -5 \leq \Delta \\ x_3 &= -\frac{4}{3} - 1 + \frac{1}{3}(9 + \Delta) \geq 0 && \rightarrow -2 \leq \Delta \quad \rightarrow \boxed{-2 \leq \Delta} \\ x_4 &= 4 + 1 \geq 0 && 0 \leq 5 \end{aligned}$$

### 3. Changing the objective function coefficients

set the rhs back to  $b_1 = 4$ ,  $b_2 = 1$ ,  $b_3 = 9$  and express  $z = \frac{8}{3} - 1 - \frac{18}{3} - \frac{2}{3}x_1 - \frac{2}{3}x_5 = -\frac{13}{3} - \frac{2}{3}x_1 - \frac{2}{3}x_5$   
 $\rightarrow$  changing the coefficient of  $x_i$  in  $z$  by  $\Delta$  changes  $z$  by exactly  $\Delta x_i$

$$\begin{aligned} x_1: \text{ change } 2x_1 \text{ to } (2 + \Delta)x_1 &\quad \rightarrow \quad z' = -\frac{13}{3} - \frac{2}{3}x_1 - \frac{2}{3}x_5 + \Delta x_1 = -\frac{13}{3} + \underbrace{\left(-\frac{2}{3} + \Delta\right)}_{\leq 0} x_1 - \frac{2}{3}x_5 \\ \text{Thus } -\frac{2}{3} + \Delta \leq 0 &\quad \rightarrow \quad \boxed{\Delta \leq \frac{2}{3}} \end{aligned}$$

$$x_2: \text{ change } -3x_2 \text{ to } (-3 + \Delta)x_2 \quad \rightarrow \quad z' = -\frac{13}{3} - \frac{2}{3}x_1 - \frac{2}{3}x_5 + \Delta x_2$$

substitute  $x_2$  from the dictionary

$$\begin{aligned} z' &= -\frac{13}{3} - \frac{2}{3}x_1 - \frac{2}{3}x_5 + \Delta\left(\frac{5}{3} + \frac{1}{3}x_1 + \frac{1}{3}x_5\right) = \left(-\frac{13}{3} + \Delta\frac{5}{3}\right) + \underbrace{\left(-\frac{2}{3} + \frac{1}{3}\Delta\right)}_{\leq 0} x_1 + \underbrace{\left(-\frac{2}{3} + \frac{1}{3}\Delta\right)}_{\leq 0} x_5 \\ \text{Thus } -\frac{2}{3} + \frac{1}{3}\Delta \leq 0 &\quad \rightarrow \quad \boxed{\Delta \leq 2} \end{aligned}$$

$$\begin{aligned} x_3: \text{ change } 1x_3 \text{ to } (1 + \Delta)x_3 &\quad \rightarrow \quad z' = -\frac{13}{3} - \frac{2}{3}x_1 - \frac{2}{3}x_5 + \Delta x_3 = \\ &= -\frac{13}{3} - \frac{2}{3}x_1 - \frac{2}{3}x_5 + \Delta\left(\frac{2}{3} - \frac{5}{3}x_1 + \frac{4}{3}x_5\right) = \left(-\frac{13}{3} + \frac{2}{3}\Delta\right) + \underbrace{\left(-\frac{2}{3} - \frac{5}{3}\Delta\right)}_{\leq 0} x_1 + \underbrace{\left(-\frac{2}{3} + \frac{4}{3}\Delta\right)}_{\leq 0} x_5 \\ -\frac{2}{3} - \frac{5}{3}\Delta \leq 0 &\quad \rightarrow \quad -\frac{2}{5} \leq \Delta \\ -\frac{2}{3} + \frac{4}{3}\Delta \leq 0 &\quad \rightarrow \quad \Delta \leq \frac{1}{2} \quad \rightarrow \quad \boxed{-\frac{2}{5} \leq \Delta \leq \frac{1}{2}} \end{aligned}$$

$$\begin{aligned} x_4: \text{ change } 0x_4 \text{ to } (0 + \Delta)x_4 &\quad \rightarrow \quad z' = -\frac{13}{3} - \frac{2}{3}x_1 - \frac{2}{3}x_5 + \Delta x_4 = \\ &= -\frac{13}{3} - \frac{2}{3}x_1 - \frac{2}{3}x_5 + \Delta(5 + x_1 - x_5) = \left(-\frac{13}{3} + 5\Delta\right) + \underbrace{\left(-\frac{2}{3} + \Delta\right)}_{\leq 0} x_1 + \underbrace{\left(-\frac{2}{3} - \Delta\right)}_{\leq 0} x_5 \\ -\frac{2}{3} + \Delta \leq 0 &\quad \Delta \leq \frac{2}{3} \\ -\frac{2}{3} - \Delta \leq 0 &\quad \rightarrow \quad -\frac{2}{3} \leq \Delta \quad \rightarrow \quad \boxed{-\frac{2}{3} \leq \Delta \leq \frac{2}{3}} \end{aligned}$$

$$\begin{aligned} x_5: \text{ change } -x_5 \text{ to } (-1 + \Delta)x_5 &\quad \rightarrow \quad z' = -\frac{13}{3} - \frac{2}{3}x_1 - \frac{2}{3}x_5 + \Delta x_5 = -\frac{13}{3} - \frac{2}{3}x_1 + \underbrace{\left(-\frac{2}{3} + \Delta\right)}_{\leq 0} x_5 \\ -\frac{2}{3} + \Delta \leq 0 &\quad \rightarrow \quad \boxed{\Delta \leq \frac{2}{3}} \end{aligned}$$

## 6.4 Adding a variable/activity

Toy cars:  $\frac{1}{2}$  hour carving, 1 hour finishing, sale price \$1

$x_6$  = the number of cars produced

$$\begin{array}{rcl}
 \text{Max } 3x_1 + 2x_2 & + & x_6 \\
 x_1 + x_2 + x_3 & + & \frac{1}{2}x_6 = 80 \\
 2x_1 + x_2 + x_4 & & x_6 = 100 \\
 x_1 & + & x_5 = 40 \\
 & & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
 \end{array}$$

$$\begin{array}{rcl}
 x_3 = 80 - x_1 - x_2 - \frac{1}{2}x_6 & & x_1 = 20 + x_3 - x_4 + ?x_6 \\
 x_4 = 100 - 2x_1 - x_2 - x_6 & \implies & x_2 = 60 - 2x_3 + x_4 + ?x_6 \\
 x_5 = 40 - x_1 & & x_5 = 20 - x_3 + x_4 + ?x_6 \\
 \hline
 z = 0 + 3x_1 + 2x_2 + x_6 & & z = 180 - x_3 - x_4 + ?x_6
 \end{array}$$

We make  $x_6$  non-basic  $\rightarrow$  produce no cars  $x_6 = 0 \rightarrow$  previously optimal solution remains feasible

Is it also optimal? In other words, does it pay to produce cars? *Price out* the new variable

- **pricing out** an activity/variable/column  $\equiv$  evaluating the **cost** of producing one unit of product (activity)  $x_j$  in terms of current shadow prices
- **reduced cost**  $\equiv$  net contribution of one unit of  $x_j$  to the value of the objective function (profit)

$$= \underbrace{\text{revenue from one unit of } x_j}_{\text{(the coefficient of } x_j \text{ in } z)} - \underbrace{\text{the production cost in shadow prices}}_{\text{(multiply the coeffs in } x_j\text{-th column by the respective shadow prices and then sum up)}}$$

$\rightarrow$  if the reduced cost is positive, then it pays to produce  $x_j$ ;

$\rightarrow$  if it is negative, then we are better-off producing other items (which is equivalent to selling all available resources at shadow prices)

Shadow prices:  $\pi_1 = \$1, \pi_2 = \$1, \pi_3 = \$0$

$\rightarrow$  one hour in each shop costs \$1 while each unit of remaining demand (3rd constraint) costs \$0

$$\text{Price out } x_6 : \underbrace{\pi_1 \times \frac{1}{2}}_{\text{cost of carving}} + \underbrace{\pi_2 \times 1}_{\text{cost of finishing}} + \pi_3 \times 0 = \$1 \times \frac{1}{2} + \$1 \times 1 + \$0 \times 0 = \$1.50$$

$$\text{Reduced cost of } x_6 : \underbrace{\$1}_{\text{sale price}} - \underbrace{\$1.50}_{\text{production cost}} = -\$0.50$$

**Answer:** it does not pay to produce toy cars, we would be losing \$.50 for each unit (instead of producing other products – shadow prices reflect the value of current production)  $\rightarrow$  current solution remains optimal

**Note:** notice that reduced costs are **precisely** the coefficients of non-basic variables in the last row of dictionary, the z-value. If they are all non-positive, then we know the solution is optimal! Put differently, if reduced costs of all non-basic variables (i.e. the coefficients in z) are not positive, then we do not make more by producing any of those items (making one of them basic and thus non-zero) and so the solution is optimal.

Conclusion from this is that  $-0.5$  is the coefficient of  $x_6$  in the modified final dictionary

$$\begin{array}{rcl}
 x_1 = 20 + x_3 - x_4 + ?x_6 \\
 x_2 = 60 - 2x_3 + x_4 + ?x_6 \\
 x_5 = 20 - x_3 + x_4 + ?x_6 \\
 \hline
 z = 180 - x_3 - x_4 - 0.5x_6
 \end{array}$$

To determine that the solution is optimal, we **do not** need to know all the other missing coefficients! More on that later...

## 6.5 Adding a constraint

packaging 150 pieces of cardboard, 1 piece per each toy soldier, 2 pieces per each toy train

new constraint:  $x_1 + 2x_2 \leq 150$

introduce a slack variable  $x_6$ :  $x_6 = 150 - x_1 - 2x_2$

substitute from the final dictionary:  $x_6 = 150 - (20 + x_3 - x_4) - 2(60 - 2x_3 + x_4) = 10 + 3x_3 - x_4$

Adding this to the final dictionary yields a feasible dictionary (since the value 10 in the constraint is non-negative)  $\rightarrow$  optimal solution remains optimal after adding the constraint

$x_1 = 20 + x_3 - x_4$	What if we only have 130 units of cardboard?
$x_2 = 60 - 2x_3 + x_4$	Then the original optimal solution becomes infeasible in the modified problem (the constant term works out to $-10$ instead of $10$ ) and we need to recalculate (or use the Dual Simplex Method)
$x_5 = 20 - x_3 + x_4$	
$x_6 = 10 + 3x_3 - x_4$	
$z = 180 - x_3 - x_4$	

Conclusion: so long as we have at least 140 units of cardboard we don't need to change the production plan

## 6.6 Modifying the left-hand side of a constraint

Equipment shortages cause that toy soldiers require 3 hours in the finishing shop (instead of 2).

Original problem	Initial dictionary	Final dictionary
Max $3x_1 + 2x_2$	$x_3 = 80 - x_1 - x_2$	$x_1 = 20 + x_3 - x_4$
$x_1 + x_2 \leq 80$	$x_4 = 100 - 2x_1 - x_2$	$x_2 = 60 - 2x_3 + x_4$
$2x_1 + x_2 \leq 100$	$x_5 = 40 - x_1$	$x_5 = 20 - x_3 + x_4$
$x_1 \leq 40$	$z = 0 + 3x_1 + 2x_2$	$z = 180 - x_3 - x_4$
$x_1, x_2 \geq 0$		
Modified problem	Modified Initial dictionary	
Max $3x_1 + 2x_2$	$x_3 = 80 - x_1 - x_2$	
$x_1 + x_2 \leq 80$	$x_4 = 100 - 3x_1 - x_2$	$\rightarrow$ ?
$3x_1 + x_2 \leq 100$	$x_5 = 40 - x_1$	
$x_1 \leq 40$	$z = 0 + 3x_1 + 2x_2$	
$x_1, x_2 \geq 0$		

Rearrange the terms in the dictionary and substitute  $x'_4 = x_4 + x_1$  in the final dictionary ( $x'_4$  in place of  $x_4$ )

$x_3 = 80 - x_1 - x_2$	$x_1 = 20 + x_3 - x'_4$	$x_1 = 20 + x_3 - (x_4 + x_1)$
$(x_4 + x_1) = 100 - 2x_1 - x_2$	$x_2 = 60 - 2x_3 + x'_4$	$x_2 = 60 - 2x_3 + (x_4 + x_1)$
$x_5 = 40 - x_1$	$x_5 = 20 - x_3 + x'_4$	$x_5 = 20 - x_3 + (x_4 + x_1)$
$z = 0 + 3x_1 + 2x_2$	$z = 180 - x_3 - x'_4$	$z = 180 - x_3 - (x_4 + x_1)$

Eliminate  $x_1$  from the rhs by solving for  $x_1$  from the first equation, and then substituting:

$x_1 = 20 + x_3 - (x_4 + x_1)$	$x_2 = 60 - 2x_3 + (x_4 + x_1) = 60 - 2x_3 + x_4$	$x_1 = 10 + \frac{1}{2}x_3 - \frac{1}{2}x_4$
$2x_1 = 20 + x_3 - x_4$	$+ (10 + \frac{1}{2}x_3 - \frac{1}{2}x_4) = 70 - \frac{3}{2}x_3 + \frac{1}{2}x_4$	$x_2 = 70 - \frac{3}{2}x_3 + \frac{1}{2}x_4$
$x_1 = 10 + \frac{1}{2}x_3 - \frac{1}{2}x_4$	$x_5 = 20 - x_3 + (x_4 + x_1) = 20 - x_3 + x_4$	$x_5 = 30 - \frac{1}{2}x_3 + \frac{1}{2}x_4$
	$+ (10 + \frac{1}{2}x_3 - \frac{1}{2}x_4) = 30 - \frac{1}{2}x_3 + \frac{1}{2}x_4$	
	$z = 180 - x_3 - (x_4 + x_1) = 180 - x_3 - x_4$	$z = 170 - \frac{1}{2}x_3 - \frac{1}{2}x_4$
	$- (10 + \frac{1}{2}x_3 - \frac{1}{2}x_4) = 170 - \frac{3}{2}x_3 - \frac{1}{2}x_4$	

Conclusion: the optimal solution remains optimal.

What if we buy new machine that shortens the finishing process of toy soldiers to 1.5 hour (instead of 2) ? Then current optimal solution will no longer be optimal ( $+x_4$  will appear in the  $z$  value).

(Curiously, what happens if it shortens to **exactly** 1 hour? Then the finishing constraint is redundant.)



## 7.1 Pricing interpretation

Back to our usual manufacturing LP problem. For the sake of illustration, we drop the 3rd constraint, and consider the items as *blocks of wood* and *cans of paint* (instead of shop hours).

Manufacturer	Market
Max $3x_1 + 2x_2$	Prices:
$x_1 + x_2 \leq 80$ [wood]	$y_1 =$ price (in \$) of one block of wood
$2x_1 + x_2 \leq 100$ [paint]	$y_2 =$ price (in \$) of one can of paint
$x_1, x_2 \geq 0$	

Manufacturer owns 80 blocks of wood and 100 cans of paint. He can sell his stock at market prices or buy additional stock at market prices. He can also produce and sell goods (toys) using the available stock.

What is his best strategy (assuming everything produced will be sold)?

★ Selling stock generates a profit of  $80y_1 + 100y_2$ .

★ If the cost (in market prices) of producing  $x_1$  *toy soldiers* is strictly less than the sale price, i.e. if

$$y_1 + 2y_2 < 3$$

then there is **no limit** on the profit of manufacturer. He can generate arbitrarily large profit by buying additional stock to produce toy soldiers in arbitrary amounts.

**Why?** The manufacturer can produce  $x_1$  toy soldiers by purchasing  $x_1$  blocks of wood, and  $2x_1$  additional cans of paint. He pays  $x_1(y_1 + 2y_2)$  and makes  $3x_1$  in sales. Net profit is then  $x_1(3 - y_1 - 2y_2)$ . Now, if  $y_1 + 2y_2 < 3$ , say if  $y_1 + 2y_2 \leq 2.9$ , then the net profit is then  $x_1(3 - y_1 - 2y_2) \geq (3 - 2.9) = 0.1x_1$ . So making arbitrarily many  $x_1$  toy soldiers generates a profit of  $0.1x_1$  (arbitrarily high).

★ Similarly, **no limit** on the profit if the cost of producing  $x_2$  *toy trains* is less than the sale price, i.e. if

$$y_1 + y_2 < 2$$

★ Market prices are **non-negative**.

Market (the competition) will not allow the manufacturer to make arbitrarily large profit. It will set its prices so that the manufacturer makes as little as possible. The market is thus solving the following:

$$\left. \begin{array}{l} \text{Min } 80y_1 + 100y_2 \\ y_1 + 2y_2 \geq 3 \quad [\text{toy soldiers}] \\ y_1 + y_2 \geq 2 \quad [\text{toy trains}] \\ y_1, y_2 \geq 0 \end{array} \right\} \text{Dual of the manufacturing problem}$$

### Estimating the optimal value

$$\begin{aligned} \text{Max } & 3x_1 + 2x_2 \\ & x_1 + x_2 \leq 80 \quad [\text{wood}] \\ & 2x_1 + x_2 \leq 100 \quad [\text{paint}] \\ & x_1, x_2 \geq 0 \end{aligned}$$

Before solving the LP, the manufacturer wishes to get a quick rough estimate (upper bound) on the value of the optimal solution. For instance, the objective function is  $3x_1 + 2x_2$  which is certainly less than  $3x_1 + 3x_2$ , since the variables  $x_1, x_2$  are non-negative. We can rewrite this as  $3(x_1 + x_2)$  and we notice that  $x_1 + x_2 \leq 80$  by the first constraint. Together we have:

$$z = 3x_1 + 2x_2 \leq 3x_1 + 3x_2 \leq 3(x_1 + x_2) \leq 3 \times 80 = \$240$$

Conclusion is that every production plan will generate no more than \$240, i.e., the value of any feasible solution (including the optimal one) is not more than 240. Likewise we can write:

$$z = 3x_1 + 2x_2 \leq 4x_1 + 2x_2 \leq 2(2x_1 + x_2) \leq 2 \times 100 = \$200$$

since  $2x_1 + x_2 \leq 100$  by the 2nd constraint. We can also combine constraints for an even better estimate:

$$z = 3x_1 + 2x_2 \leq (x_1 + x_2) + (2x_1 + x_2) \leq 80 + 100 = \$180$$

In general, we consider  $y_1 \geq 0, y_2 \geq 0$  and take  $y_1$  times the 1st constraint +  $y_2$  times the 2nd constraint.

$$y_1(x_1 + x_2) + y_2(2x_1 + x_2) \leq 80y_1 + 100y_2$$

We can rewrite this expression by collecting coefficients of  $x_1$  and  $x_2$ :

$$(y_1 + 2y_2)x_1 + (y_1 + y_2)x_2 \leq 80y_1 + 100y_2$$

In this expression, if the **coefficient** of  $x_1$  is **at least 3** and the coefficient of  $x_2$  is **at least 2**, i.e., if

$$y_1 + 2y_2 \geq 3$$

$$y_1 + y_2 \geq 2$$

then, just like before, we obtain an upper bound on the value of  $z = 3x_1 + 2x_2$ :

$$z = 3x_1 + 2x_2 \leq (y_1 + 2y_2)x_1 + (y_1 + y_2)x_2 = y_1(x_1 + x_2) + y_2(2x_1 + x_2) \leq 80y_1 + 100y_2$$

If we want the best possible upper bound, we want this expression be as small as possible.

$$\left. \begin{aligned} \text{Min } & 80y_1 + 100y_2 \\ & y_1 + 2y_2 \geq 3 \\ & y_1 + y_2 \geq 2 \\ & y_1, y_2 \geq 0 \end{aligned} \right\} \text{The Dual problem}$$

The original problem is then called the **Primal** problem.

$$\begin{array}{ll} \text{Primal} & \begin{aligned} \text{Max } & 3x_1 + 2x_2 \\ & x_1 + x_2 \leq 80 \\ & 2x_1 + x_2 \leq 100 \\ & x_1, x_2 \geq 0 \end{aligned} \\ & \text{Dual} \end{array} \quad \begin{aligned} \text{Min } & 80y_1 + 100y_2 \\ & y_1 + 2y_2 \geq 3 \\ & y_1 + y_2 \geq 2 \\ & y_1, y_2 \geq 0 \end{aligned}$$

### Matrix formulation

In general, for maximization problem with  $\leq$  inequalities, the dual is obtained simply by

- transposing (flipping around the diagonal) the matrix  $\mathbf{A}$ ,
- swapping vectors  $\mathbf{b}$  and  $\mathbf{c}$ ,
- switching the inequalities to  $\geq$ , and
- changing max to min.

$$\begin{array}{ll} \text{max } & \mathbf{c}^T \mathbf{x} \\ & \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array} \quad \begin{array}{ll} \text{min } & \mathbf{b}^T \mathbf{y} \\ & \mathbf{A}^T \mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{array}$$

## 7.2 Duality Theorems and Feasibility

**Theorem 3 (Weak Duality Theorem).** If  $\mathbf{x}$  is any feasible solution of the primal and  $\mathbf{y}$  is any feasible solution of the dual, then

$$\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}$$

In other words, the value of **any** feasible solution to the **dual** yields an upper bound on the value of any feasible solution (including the optimal) to the **primal**.

$$\mathbf{c}^T \mathbf{x} \leq (\mathbf{A}^T \mathbf{y})^T \mathbf{x} = (\mathbf{y}^T \mathbf{A}) \mathbf{x} = \mathbf{y}^T (\mathbf{A} \mathbf{x}) \leq \mathbf{y}^T \mathbf{b} = \mathbf{b}^T \mathbf{y}$$

Consequently, if **primal** is *unbounded*, then **dual** must be *infeasible* and likewise, if **dual** is *unbounded*, then **primal** must be *infeasible*. Note that it is possible that both **primal** and **dual** are *infeasible*. But if both are *feasible*, then neither of them is *unbounded*.

		primal			
		infeasible	feasible bounded	unbounded	
d u a l	infeasible	✓	✗	✓	✓ possible
	feasible bounded	✗	✓	✗	✗ impossible
	unbounded	✓	✗	✗	

### Strong Duality and Complementary Slackness

**Theorem 4 (Strong Duality Theorem).** If  $\mathbf{x}$  is an **optimal** solution to the primal and  $\mathbf{y}$  is an **optimal** solution to the dual, then

$$\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{y}$$

Moreover,  $\mathbf{y}^T (\underbrace{\mathbf{b} - \mathbf{A} \mathbf{x}}_{\text{slack in primal}}) = 0$  and  $\mathbf{x}^T (\underbrace{\mathbf{A}^T \mathbf{y} - \mathbf{c}}_{\text{slack in dual}}) = 0$ .

In simple terms: whenever a constraint is **not tight** (has a positive slack) in the **primal**, then the **dual** variable corresponding to this constraint must be 0. Conversely, if a **primal** variable is strictly positive, then the corresponding **dual** constraint must be tight (slack is zero).

This can be seen as follows (note that  $\mathbf{x} \geq 0$ ,  $\mathbf{y} \geq 0$ ,  $\mathbf{A} \mathbf{x} \geq \mathbf{b}$  and  $\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$ )

$$\begin{aligned} 0 &\leq \mathbf{y}^T (\mathbf{b} - \mathbf{A} \mathbf{x}) = \mathbf{y}^T \mathbf{b} - \mathbf{y}^T \mathbf{A} \mathbf{x} = \mathbf{b}^T \mathbf{y} - (\mathbf{A}^T \mathbf{y})^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y} - \mathbf{c}^T \mathbf{x} = 0 \\ 0 &\leq \mathbf{x}^T (\mathbf{A}^T \mathbf{y} - \mathbf{c}) = (\mathbf{y}^T \mathbf{A} - \mathbf{c}^T) \mathbf{x} = \mathbf{y}^T (\mathbf{A} \mathbf{x}) - \mathbf{c}^T \mathbf{x} \leq \mathbf{y}^T \mathbf{b} - \mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{y} - \mathbf{c}^T \mathbf{x} = 0 \end{aligned}$$

## 7.3 General LPs

If LP contains equalities or unrestricted variables, these can be also handled with ease. In particular,

**equality** constraint corresponds to an **unrestricted** variable, and **vice-versa**.

**Why?** Notice that when we produced an upper bound, we considered only non-negative  $y_1 \geq 0$ , since multiplying the  $\leq$  constraint by a negative value changes the sign to  $\geq$  and thus the upper bound becomes a lower bound instead. However, if the constraint was an equality (i.e. if we had  $x_1 + x_2 = 80$  instead), we could allow negative  $y_1$  as well and still produce an upper bound. For instance, we could write

$$3x_1 + 2x_2 \leq 5x_1 + 2x_2 = (-1) \times \underbrace{(x_1 + x_2)}_{=80} + 3 \times \underbrace{(2x_1 + x_2)}_{\leq 100} \leq -80 + 3 \times 100 = \$220$$

So we would make  $y_1$  unrestricted.

Conversely, if some variable in our problem, say  $x_1$ , were unrestricted in sign (could be negative as well), then we could **not** conclude that  $3x_1 + 2x_2 \leq 4x_1 + 2x_2$  holds for all feasible solutions, as we did in our 2nd estimate; namely if  $x_1$  is **negative**, then this is **not true** (it is actually  $>$  rather than  $\leq$ ). However, if  $x_1$  is unrestricted but  $x_2 \geq 0$ , we could still conclude that  $3x_1 + 2x_2 \leq 3x_1 + 2x_2$ , since the coefficient of  $x_1$  is not changing in this expression. In our general expression, we had  $(y_1 + 2y_2)x_1$  and we demanded that the coefficient  $y_1 + 2y_2$  of  $x_1$  is at least 3 for the

upper bound to work. If  $x_1$  is **unrestricted**, we can simply insist that the coefficient  $y_1 + 2y_2$  **equals** 3 to make the upper bound work.

The same way we can conclude that

$\geq$  constraint corresponds to an **non-positive** variable, and **vice-versa**.

Primal (Max)	Dual (Min)
$i$ -th constraint $\leq$	variable $y_i \geq 0$
$i$ -th constraint $\geq$	variable $y_i \leq 0$
$i$ -th constraint $=$	variable $y_i$ unrestricted
$x_i \geq 0$	$i$ -th constraint $\geq$
$x_i \leq 0$	$i$ -th constraint $\leq$
$x_i$ unrestricted	$i$ -th constraint $=$

<p>Max <math>3x_1 + 2x_2 + x_3</math>  <math>x_1 + x_2 + \frac{1}{2}x_3 \leq 80</math>  <math>2x_1 + x_2 + x_3 = 100</math>  <math>x_1 + x_3 \geq 40</math>  <math>x_1</math> unrestricted  <math>x_2 \leq 0</math>  <math>x_3 \geq 0</math></p> <p style="text-align: center;"><b>Primal</b></p>	<p>Min <math>80y_1 + 100x_2 + 40x_3</math>  <math>y_1 + 2y_2 + y_3 = 3</math>  <math>y_1 + y_2 \leq 2</math>  <math>\frac{1}{2}y_1 + y_2 + y_3 \geq 1</math>  <math>y_1 \geq 0</math>  <math>y_2</math> unrestricted  <math>y_3 \leq 0</math></p> <p style="text-align: center;"><b>Dual</b></p>
---	--

## 7.4 Complementary slackness

$$\begin{aligned}
 \max \quad & 6x_1 + x_2 - x_3 - x_4 \\
 & x_1 + 2x_2 + x_3 + x_4 \leq 5 \\
 & 3x_1 + x_2 - x_3 \leq 8 \\
 & x_2 + x_3 + x_4 = 1 \\
 & x_2, x_3, x_4 \geq 0 \\
 & x_1 \text{ unrestricted}
 \end{aligned}$$

We wish to check if one of the following assignments is an optimal solution.

- a)  $x_1 = 2, x_2 = 1, x_3 = 0, x_4 = 0$
- b)  $x_1 = 3, x_2 = 0, x_3 = 1, x_4 = 0$

To this end, we use **Complementary Slackness**. Let us discuss the theory first.

### Theory

As usual, let  $\mathbf{x}$  denote the vector of variables, let  $\mathbf{c}$  be the vector of coefficients of variables of the objective function, let  $\mathbf{A}$  be the coefficient matrix of the left-hand side of our constraints, and let  $\mathbf{b}$  be the vector of the right-hand side of the constraints. Let  $\mathbf{y}$  be the variables of the dual.

$$\begin{array}{ll}
 \max \quad \mathbf{c}^T \mathbf{x} & \min \quad \mathbf{b}^T \mathbf{y} \\
 \mathbf{A} \mathbf{x} \leq \mathbf{b} \quad \text{PRIMAL} & \mathbf{A}^T \mathbf{y} \geq \mathbf{c} \quad \text{DUAL} \\
 \mathbf{x} \geq 0 & \mathbf{y} \geq 0
 \end{array}$$

We say that vectors  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{y} = (y_1, \dots, y_m)$  are **complementary** if

$$\mathbf{y}^T (\underbrace{\mathbf{b} - \mathbf{A} \mathbf{x}}_{\text{slack in primal}}) = 0 \quad \text{and} \quad \mathbf{x}^T (\underbrace{\mathbf{A}^T \mathbf{y} - \mathbf{c}}_{\text{slack in dual}}) = 0$$

In other words,

- whenever  $y_i > 0$ , then  $\mathbf{x}$  satisfies the  $i$ -th constraint with equality (“the constraint is **tight**”)
- whenever  $x_i > 0$ , then  $\mathbf{y}$  satisfies the  $i$ -th constraint of the dual with equality

**Exercise.** Show that the shadow prices  $\boldsymbol{\pi}$  defined by (a basic solution)  $\mathbf{x}$  are always complementary to  $\mathbf{x}$ .

Recall that **Strong Duality** this says that if  $\mathbf{x}$  is an optimal solution to the primal and  $\mathbf{y}$  is an optimal solution to the dual, then  $\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{y}$ . In fact, more is true.

**Complementary Slackness** (and some consequences): Assume that  $\mathbf{x}$  is an optimal solution to the primal.

- If  $\mathbf{y}$  is an **optimal** solution to the dual, then  $\mathbf{x}$  and  $\mathbf{y}$  are **complementary**.
- If  $\mathbf{y}$  is a **feasible** solution in the dual and is **complementary** to  $\mathbf{x}$ , then  $\mathbf{y}$  is **optimal** in the dual.
- There **exists** a **feasible** solution  $\mathbf{y}$  to the dual such that  $\mathbf{x}$  and  $\mathbf{y}$  are **complementary**.

Notice that the last bullet follows from our observation about shadow prices. Another consequence of this is:

If  $\mathbf{x}$  is a basic feasible primal solution and  $\boldsymbol{\pi}$  are the corresponding shadow prices, then  $\mathbf{x}$  is **optimal** if and only if  $\boldsymbol{\pi}$  is a **feasible** solution of the dual

If we have equalities,  $\geq$ -inequalities, unrestricted or non-positive variables, everything works just the same.

## Back to example

To check if the provided solutions are optimal, we need the dual.

$$\begin{array}{ll}
 \max & 6x_1 + x_2 - x_3 - x_4 \\
 & x_1 + 2x_2 + x_3 + x_4 \leq 5 \\
 & 3x_1 + x_2 - x_3 \leq 8 \\
 & x_2 + x_3 + x_4 = 1 \\
 & x_2, x_3, x_4 \geq 0 \\
 & x_1 \text{ unrestricted} \\
 \min & 5y_1 + 8y_2 + y_3 \\
 & y_1 + 3y_2 = 6 \\
 & 2y_1 + y_2 + y_3 \geq 1 \\
 & y_1 - y_2 + y_3 \geq -1 \\
 & y_1 + y_3 \geq -1 \\
 & y_1, y_2 \geq 0 \\
 & y_3 \text{ unrestricted}
 \end{array} \quad \text{DUAL}$$

- a)  $x_1 = 2, x_2 = 1, x_3 = 0, x_4 = 0 \rightarrow$  assume  $\mathbf{x}$  is optimal, then
- $\rightarrow$  there are  $y_1, y_2, y_3$  such that  $\mathbf{y} = (y_1, y_2, y_3)$  is feasible in the dual and complementary to  $\mathbf{x}$
  - check 1<sup>st</sup> primal constraint:  $x_1 + 2x_2 + x_3 + x_4 = 2 + 2 + 0 + 0 = 4 < 5$  **not tight**
  - $\rightarrow$  therefore  $y_1$  must be 0 because  $\mathbf{y}$  is complementary to  $\mathbf{x}$
  - check 2<sup>nd</sup> primal constraint:  $3x_1 + x_2 - x_3 = 6 + 1 - 0 = 7 < 8$  **not tight**
  - $\rightarrow$  therefore  $y_2$  must be 0 because  $\mathbf{y}$  is complementary to  $\mathbf{x}$
  - Knowing this, check the 1<sup>st</sup> dual constraint:  $y_1 + 3y_2 = 0 + 0 = 0 \neq 6$
  - $\rightarrow$  this shows that  $(y_1, y_2, y_3)$  **not feasible** in the dual, but we assume that it is.
- This means that our assumptions were wrong and so  $(x_1, x_2, x_3, x_4)$  is **not optimal**.

- b)  $x_1 = 3, x_2 = 0, x_3 = 1, x_4 = 0 \rightarrow$  again assume that  $\mathbf{x}$  is optimal, then
- $\rightarrow$  there are  $y_1, y_2, y_3$  such that  $\mathbf{y} = (y_1, y_2, y_3)$  is feasible in the dual and complementary to  $\mathbf{x}$
  - check 1<sup>st</sup> primal constraint:  $x_1 + 2x_2 + x_3 + x_4 = 3 + 0 + 1 + 0 = 4 < 5$  **not tight**  $\rightarrow y_1 = 0$
  - check 2<sup>nd</sup> primal constraint:  $3x_1 + x_2 - x_3 = 9 + 0 - 1 = 8$  **tight**
  - check 3<sup>rd</sup> primal constraint:  $x_1 + x_2 + x_3 = 1$  **tight**
  - check sign constraints:  $x_2, x_3, x_4 \geq 0 \rightarrow$  we conclude that  $(x_1, x_2, x_3, x_4)$  is **feasible** in the primal

Now we look at values in  $\mathbf{x}$  with respect to the dual constraints:

$x_1$  is unrestricted  $\rightarrow$  1<sup>st</sup> dual constraint  $y_1 + 3y_2 = 6$  is (always) **tight**

$x_3 > 0$  we deduce  $\rightarrow$  3<sup>rd</sup> dual constraint must be **tight**:  $y_1 - y_2 + y_3 = -1$

$$\begin{aligned} \text{Together we have} \quad y_1 &= 0 \\ y_1 + 3y_2 &= 6 \\ y_1 - y_2 + y_3 &= -1 \end{aligned}$$

This has a unique solution  $y_1 = 0, y_2 = 2, y_3 = 1$ . By construction, this solution is **complementary** to  $\mathbf{x}$ .

The last step is to **check** if  $\mathbf{y}$  is also **feasible** in the dual. We already checked 1<sup>st</sup> and 3<sup>rd</sup> dual constraint.

check 2<sup>nd</sup> dual constraint:  $2y_1 + y_2 + y_3 = 0 + 2 + 1 = 3 \geq 1 \rightarrow$  the constraint is **satisfied**

check 4<sup>th</sup> dual constraint:  $y_1 + y_3 = 0 + 1 \geq -1$  the constraint is **satisfied**

check sign restrictions:  $y_1 = 0 \geq 0$  and  $y_2 = 2 \geq 0 \rightarrow$  sign restrictions **satisfied**

$\rightarrow$  this shows that  $(y_1, y_2, y_3)$  indeed a **feasible** solution to the dual.

From this we can conclude that  $(x_1, x_2, x_3, x_4)$  is indeed **optimal**.

## Summary

- given  $\mathbf{x}$ , check if  $\mathbf{x}$  is feasible
- then find which variables  $y_i$  should be 0
- then find which dual constraints should be tight
- this yields a system of equations
- solve the system
- verify that the solution is feasible in the dual

If all these steps succeed, then the given  $\mathbf{x}$  is indeed optimal; otherwise, it is not.

**Question:** what happens if  $\mathbf{x}$  is feasible but not a basic solution ?

## Review

$\begin{aligned} \text{Max } 3x_1 + 2x_2 \\ x_1 + x_2 &\leq 80 \\ 2x_1 + x_2 &\leq 100 \\ x_1 &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$ <p>Original problem</p>	$\begin{aligned} x_3 &= 80 - x_1 - x_2 \\ x_4 &= 100 - 2x_1 - x_2 \\ x_5 &= 40 - x_1 \\ \hline z &= 0 + 3x_1 + 2x_2 \end{aligned}$ <p>Initial dictionary</p>	$\begin{aligned} x_1 &= 20 + x_3 - x_4 \\ x_2 &= 60 - 2x_3 + x_4 \\ x_5 &= 20 - x_3 + x_4 \\ \hline z &= 180 - x_3 - x_4 \end{aligned}$ <p>Optimal dictionary</p>
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Add a new activity: *toy cars*,  $\frac{1}{2}$ h carving, 1h finishing, 1 unit towards demand limit, \$1 price  $\rightarrow x_6 = \#cars$

$\begin{aligned} \text{Max } 3x_1 + 2x_2 + x_6 \\ x_1 + x_2 + \frac{1}{2}x_6 &\leq 80 \\ 2x_1 + x_2 + x_6 &\leq 100 \\ x_1 + x_6 &\leq 40 \\ x_1, x_2, x_6 &\geq 0 \end{aligned}$ <p>Original problem</p>	$\begin{aligned} x_3 &= 80 - x_1 - x_2 - \frac{1}{2}x_6 \\ x_4 &= 100 - 2x_1 - x_2 - x_6 \\ x_5 &= 40 - x_1 - x_6 \\ \hline z &= 0 + 3x_1 + 2x_2 + x_6 \end{aligned}$ <p>Initial dictionary</p>	<p style="text-align: center;">?</p> <p>Optimal dictionary</p>
---	---	--

in the optimal dictionary  $\rightarrow$  make  $x_6$  *non-basic* (no production of cars)  $\rightarrow$  **feasible** modified dictionary

$\begin{aligned} x_3 + \frac{1}{2}x_6 &= 80 - x_1 - x_2 \\ x_4 + x_6 &= 100 - 2x_1 - x_2 \\ x_5 + x_6 &= 40 - x_1 \\ \hline z - x_6 &= 0 + 3x_1 + 2x_2 \end{aligned}$	$\begin{aligned} x'_3 &= 80 - x_1 - x_2 \\ x'_4 &= 100 - 2x_1 - x_2 \\ x'_5 &= 40 - x_1 \\ \hline z' &= 0 + 3x_1 + 2x_2 \end{aligned}$	$\begin{aligned} x_1 &= 20 + x'_3 - x'_4 \\ x_2 &= 60 - 2x'_3 + x'_4 \\ x_5 &= 20 - x'_3 + x'_4 \\ \hline z' &= 180 - x'_3 - x'_4 \end{aligned}$
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substitute:  $x'_3 = x_3 + \frac{1}{2}x_6$ ,  $x'_4 = x_4 + x_6$ ,  $x'_5 = x_5 + x_6$ ,  $z' = z - x_6$

$$\begin{array}{rcl}
 x_1 & = & 20 + (x_3 + \frac{1}{2}x_6) - (x_4 + x_6) \\
 x_2 & = & 60 - 2(x_3 + \frac{1}{2}x_6) + (x_4 + x_6) \\
 x_5 + x_6 & = & 20 - (x_3 + \frac{1}{2}x_6) + (x_4 + x_6) \\
 \hline
 z - x_6 & = & 180 - (x_3 + \frac{1}{2}x_6) - (x_4 + x_6)
 \end{array}
 \qquad
 \begin{array}{rcl}
 x_1 & = & 20 + x_3 - x_4 - \frac{1}{2}x_6 \\
 x_2 & = & 60 - 2x_3 + x_4 \\
 x_5 & = & 20 - x_3 + x_4 - \frac{1}{2}x_6 \\
 \hline
 z & = & 180 - x_3 - x_4 - \frac{1}{2}x_6
 \end{array}$$

Add a new constraint: *packaging*, 250 units of cardboard, 3/soldier, 4/train, 1/car  $\rightarrow x_7 = \text{slack}$

$$\begin{array}{rcl}
 \text{Max } 3x_1 + 2x_2 + x_6 & & \\
 x_1 + x_2 + \frac{1}{2}x_6 & \leq & 80 \\
 2x_1 + x_2 + x_6 & \leq & 100 \\
 x_1 & + & x_6 \leq 40 \\
 3x_1 + 4x_2 + x_6 & \leq & 250 \\
 x_1, x_2, x_6 & \geq & 0
 \end{array}
 \qquad
 \begin{array}{rcl}
 x_3 & = & 80 - x_1 - x_2 - \frac{1}{2}x_6 \\
 x_4 & = & 100 - 2x_1 - x_2 - x_6 \\
 x_5 & = & 40 - x_1 - x_6 \\
 x_7 & = & 250 - 3x_1 - 4x_2 - x_6 \\
 \hline
 z & = & 0 + 3x_1 + 2x_2 + x_6
 \end{array}$$

in the optimal dictionary  $\rightarrow$  make  $x_7$  **basic**  $\rightarrow$  express  $x_7$  using the dictionary  $\rightarrow$  new dictionary

$$x_7 = 250 - 3x_1 - 4x_2 - x_6 = 250 - 3(20 + x_3 - x_4 - \frac{1}{2}x_6) - 4(60 - 2x_3 + x_4) - x_6 = -50 + 5x_3 - x_4 + \frac{1}{2}x_6$$

$$\begin{array}{rcl}
 x_1 & = & 20 + x_3 - x_4 - \frac{1}{2}x_6 \\
 x_2 & = & 60 - 2x_3 + x_4 \\
 x_5 & = & 20 - x_3 + x_4 - \frac{1}{2}x_6 \\
 \hline
 z & = & 180 - x_3 - x_4 - \frac{1}{2}x_6
 \end{array}
 \qquad
 \rightarrow
 \qquad
 \begin{array}{rcl}
 x_1 & = & 20 + x_3 - x_4 - \frac{1}{2}x_6 \\
 x_2 & = & 60 - 2x_3 + x_4 \\
 x_5 & = & 20 - x_3 + x_4 - \frac{1}{2}x_6 \\
 x_7 & = & -50 + 5x_3 - x_4 + \frac{1}{2}x_6 \\
 \hline
 z & = & 180 - x_3 - x_4 - \frac{1}{2}x_6
 \end{array}$$

If the resulting dictionary is feasible, then it is also optimal (we don't change  $z$ , all coeffs still non-positive)

However, the resulting dictionary may be **infeasible** if some basic variable is negative (here  $x_7 < 0$ )

$\rightarrow$  to recover optimal solution, we use **Dual Simplex Method**.

## Other Simplex Methods

### 8.1 Dual Simplex Method

- we use when the dictionary is **infeasible** but **dually feasible**
- same as Simplex on the Dual **without** dealing with the Dual directly
- useful when adding new constraints to quickly recover optimal solution

**Recall:** for every (basic) solution of the Primal, we have **shadow prices** that we can assign to each item (constraint) in such a way that the total value of items in shadow prices is **exactly** the value of the solution

<b>P</b>	Max	$3x_1 + 2x_2$	$x_1 = 40$	$- x_5$	Solution: $x_1 = 40, x_2 = 0, x_3 = 40,$
<b>R</b>		$x_1 + x_2 \leq 80$	$x_3 = 40 - x_2 + x_5$		$x_4 = 20, x_5 = 0$ of value $z = 120$
<b>I</b>		$2x_1 + x_2 \leq 100$	$x_4 = 20 - x_2 + 2x_5$		Shadow prices: $\pi_1 = 0, \pi_2 = 0, \pi_3 = 3$
<b>M</b>		$x_1 \leq 40$	$z = 120 + 2x_2 - 3x_5$		$80\pi_1 + 100\pi_2 + 40\pi_3 = 120 = 3x_1 + 2x_2$
<b>A</b>		$x_1, x_2 \geq 0$			
<b>L</b>					

**Note:** the above shadow prices  $\rightarrow$  an **infeasible** solution in Dual

<b>D</b>	Min	$80y_1 + 100y_2 + 40y_3$	$80\pi_1 + 100\pi_2 + 40\pi_3 = 120$	the same value as Primal
<b>U</b>		$y_1 + 2y_2 + y_3 \geq 3$	$\pi_1 + 2\pi_2 + \pi_3 = 3$	$\geq 3$
<b>A</b>		$y_1 + y_2 \geq 2$	$\pi_1 + \pi_2 = 0$	$\not\geq 2$
<b>L</b>		$y_1, y_2, y_3 \geq 0$	$\pi_1 = 0 \geq 0, \pi_2 = 0 \geq 0, \pi_3 = 3 \geq 0$	

Shadow prices corresponding to a non-optimal **feasible** solution of the Primal are **infeasible** in the Dual

Shadow prices corresponding to an **optimal** solution of the Primal are **optimal** in the Dual.

Shadow prices for an **infeasible** solution of the Primal **may or may not** be **feasible** in the Dual.

A solution of the Primal is **dually feasible** if the corresponding shadow prices are **feasible** in the dual.

### Dual Simplex Algorithm

We **maintain** that the current solution is **dually feasible** but may itself be **infeasible**

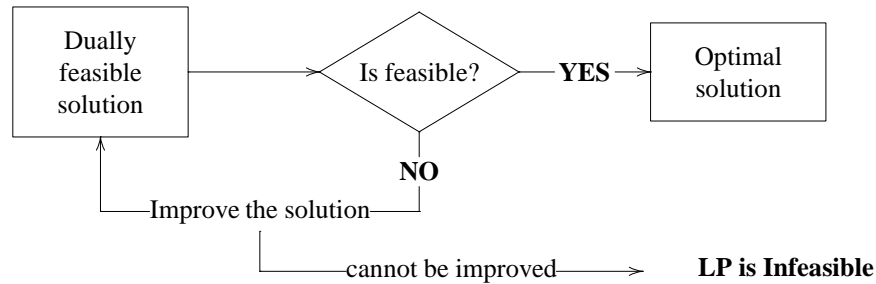
$$\begin{aligned}
 x_1 &= 20 + x_3 - x_4 - \frac{1}{2}x_6 \\
 x_2 &= 60 - 2x_3 + x_4 \\
 x_5 &= 20 - x_3 + x_4 - \frac{1}{2}x_6 \\
 x_7 &= -50 + 5x_3 - x_4 + \frac{1}{2}x_6 \\
 \hline
 z &= 180 - x_3 - x_4 - \frac{1}{2}x_6 \quad \leftarrow \text{is dually feasible}
 \end{aligned}$$

How do we know that the solution is **dually feasible**?

... if all coefficients of variables in  $z$  are **non-positive**

(Why? Because the coeffs of slack variables are shadow prices while the coeffs of non-slack variables are reduced costs)





Since  $x_7 < 0$ , the solution is **infeasible**. We want to make it **feasible**. In order to do that, we need to **increase** the value of  $x_7$ . We make  $x_7$  non-basic (value = 0).

We can do this by increasing one of the **non-basic** variables ( $x_3$  or  $x_4$  or  $x_6$ ), i.e., by making one of the non-basic variables **basic**. Clearly,  $x_4$  is not good for this since increasing  $x_4$  makes  $x_7$  only **more** negative ( $x_4$  appears in the equation for  $x_7$  with negative coefficient).

So we must increase  $x_3$  or  $x_6$ . But we also must make sure that this results in **dually feasible** dictionary, i.e., the resulting coefficients of non-basic variables in  $z$  are non-positive.

If  $x_6$  enters the basis, then  $x_7 = -50 + 5x_3 - x_4 + \frac{1}{2}x_6 \rightarrow \frac{1}{2}x_6 = 50 - 5x_3 + x_4 + x_7$  and so

$$z = 180 - x_3 - x_4 - \frac{1}{2}x_6 = 180 - x_3 - x_4 - \frac{1}{2} \times 2(50 - 5x_3 + x_4 + x_7) = 130 + 4x_3 - 2x_4 - x_7$$

If  $x_3$  enters the basis, then  $x_7 = -50 + 5x_3 - x_4 + \frac{1}{2}x_6 \rightarrow 5x_3 = 50 + x_4 - \frac{1}{2}x_6 + x_7$  and so

$$z = 180 - x_3 - x_4 - \frac{1}{2}x_6 = 180 - 1 \times \frac{1}{5}(50 + x_4 - \frac{1}{2}x_6 + x_7) - x_4 - \frac{1}{2}x_6 = 170 - \frac{6}{5}x_4 - \frac{2}{5}x_6 - \frac{1}{5}x_7$$

When  $x_6$  enters the basis, the solution is **not** dually feasible, while it **is** when  $x_3$  enters the basis. How do we find out? Both substitutions are a result of **adding** to  $z$  some multiple  $\Delta$  of the equation for  $x_7$ .

$$z = 180 - x_3 - x_4 - \frac{1}{2}x_6 - \Delta \overbrace{(50 - 5x_3 + x_4 - \frac{1}{2}x_6 + x_7)}^0$$

$$= (180 - 50\Delta) + \underbrace{(-1 + 5\Delta)}_{\leq 0}x_3 + \underbrace{(-1 - \Delta)}_{\leq 0}x_4 + \underbrace{(-\frac{1}{2} + \frac{1}{2}\Delta)}_{\leq 0}x_6 + \underbrace{(-\Delta)}_{\leq 0}x_7$$

$$-1 + 5\Delta \leq 0$$

$$-1 - \Delta \leq 0$$

$$-\frac{1}{2} + \frac{1}{2}\Delta \leq 0$$

$$0$$

$$-\Delta \leq 0$$

$$\Delta \leq \frac{1}{5}$$

$$-1 \leq \Delta$$

$$\Delta \leq 1$$

$$0 \leq \Delta$$

$$\boxed{0 \leq \Delta \leq \frac{1}{5}}$$

for  $\Delta = \frac{1}{5}$  the bound is tight for  $x_3$

$\rightarrow x_3$  disappears from  $z \rightarrow$  enters the basis

### Ratio test

Simplified procedure: compare coefficients of non-basic variables in  $z$  and in  $x_7$ , choose smallest ratio

$x_3$  : coeff  $-1x_3$  in  $z$  and  $5x_3$  in  $x_7$ , ratio  $\frac{1}{5} = 0.2$

$x_4$  : coeff  $-1x_4$  in  $z$  and  $-1x_4$  in  $x_7$ , *no constraint*  
(negative coeff in  $x_7$ )

$x_6$  : coeff  $-\frac{1}{2}x_6$  in  $z$  and  $\frac{1}{2}x_6$  in  $x_7$ , ratio  $\frac{\frac{1}{2}}{\frac{1}{2}} = 1$

$x_1 =$	$20 + x_3 - x_4 - \frac{1}{2}x_6$
$x_2 =$	$60 - 2x_3 + x_4$
$x_5 =$	$20 - x_3 + x_4 - \frac{1}{2}x_6$
$x_7 =$	$-50 + 5x_3 - x_4 + \frac{1}{2}x_6$
$z =$	$180 - 1x_3 - x_4 - \frac{1}{2}x_6$

ratio for  $x_3$ :

$$\frac{1}{5} = 0.2$$

(again **watch-out**: we only consider this ratio because the coefficient of  $x_3$  is positive)

Smallest ratio $\frac{1}{5}$ for $x_3 \rightarrow x_3$ enters  $x_7 = -50 + 5x_3 - x_4 + \frac{1}{2}x_6$ $\rightarrow x_3 = 10 + \frac{1}{5}x_4 - \frac{1}{10}x_6 + \frac{1}{5}x_7$	$x_1 = 30 - \frac{4}{5}x_4 - \frac{3}{5}x_6 + \frac{1}{5}x_7$ $x_2 = 40 + \frac{3}{5}x_4 + \frac{1}{5}x_6 - \frac{2}{5}x_7$ $x_3 = 10 + \frac{1}{5}x_4 - \frac{1}{10}x_6 + \frac{1}{5}x_7$ $x_5 = 10 + \frac{4}{5}x_4 - \frac{2}{5}x_6 - \frac{1}{5}x_7$ <hr style="border: 0.5px solid black;"/> $z = 170 - \frac{6}{5}x_4 - \frac{2}{5}x_6 - \frac{1}{5}x_7$	Shadow prices: $\pi_1 = 0, \pi_2 = \frac{6}{5},$ $\pi_3 = 0, \pi_4 = \frac{1}{5}$  <b>optimal</b> solution to the Dual. (Why?)
--	--	--

Solution is feasible (and dually feasible)  $\rightarrow$  **optimal** solution found.

### Summary

Starting with a **dually feasible** solution:

1. Find a **basic** variable  $x_i$  of negative value.
2. If no such  $x_i$  exists  $\rightarrow$  stop, the solution is feasible  $\rightarrow$  **optimal** solution found.
3. **Ratio test**: in the dictionary, in the equation for  $x_i$ , find a non-basic variable  $x_j$  such that
  - $x_j$  appears with **positive** coefficient  $a$  in the equation for  $x_i$
  - the ratio  $\frac{c}{a}$  is smallest possible (where  $-c$  is the coefficient of  $x_j$  in  $z$ )
4. If no such  $x_j$  exists  $\rightarrow$  stop, no feasible solution exists  $\rightarrow$  report that LP is **Infeasible**.
5. **Pivot**  $x_j$  into the basis,  $x_i$  leaves the basis.  
 (the resulting dictionary is **guaranteed** to be dually feasible)
6. Repeat.

## 8.2 Upper-Bounded Simplex

Back to the toy factory problem. Subsequent market survey revealed the maximum demand for trains to be 50.

Max $3x_1 + 2x_2$ $x_1 + x_2 \leq 80$ $2x_1 + x_2 \leq 100$ $x_1 \leq 40$ $x_2 \leq 50$ $x_1, x_2 \geq 0$	Max $3x_1 + 2x_2$ $x_1 + x_2 \leq 80$ $2x_1 + x_2 \leq 100$ $0 \leq x_1 \leq 40$ $0 \leq x_2 \leq 50$	introduce new (complementary) variables $x'_1, x'_2$ : $x'_1 = 40 - x_1$ $x'_2 = 50 - x_2$ $0 \leq x'_1 \leq 40$ $0 \leq x'_2 \leq 50$
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We solve the problem as usual, but for each variable we keep track of either the variable or its complement but never both. That is, the dictionary will either contain  $x_1$  or  $x'_1$  but not both. Note that it is not necessary to keep both as one can be derived from the other by a simple substitution  $x'_1 = 40 - x_1$ . In a way, we may think of having both  $x_1$  and  $x'_1$  in the dictionary but one of them is **hidden**.

We modify the **ratio test** to account for these hidden variables. For illustration, let us solve the above.

$x_3 = 80 - x_1 - x_2$ $x_4 = 100 - 2x_1 - x_2$ <hr style="border: 0.5px solid black;"/> $z = 0 + 3x_1 + 2x_2$ $0 \leq x_3 \leq \infty$ $0 \leq x_4 \leq \infty$	$x_1$ increases to $\Delta \geq 0$ (enters) variables change as follows: $x_1 = \Delta \quad 0 \leq x_1 \leq 40$ $x_3 = 80 - \Delta \quad 0 \leq x_3 \leq \infty$ $x_4 = 100 - 2\Delta \quad 0 \leq x_4 \leq \infty$	$0 \leq \Delta \leq 40$ $0 \leq 80 - \Delta \leq \infty$ $0 \leq 100 - 2\Delta \leq \infty$ $\Delta \leq 40$ $\Delta \leq \frac{80}{1} = 80$ $\Delta \leq \frac{100}{2} = 50$
--	--	--

Most restrictive constrained imposed by the upper bound on  $x_1 \rightarrow$  we replace  $x_1$  by  $x'_1 = 40 - x_1$ .

$x_3 = 80 - x_1 - x_2$ $x_4 = 100 - 2x_1 - x_2$ <hr style="border: 0.5px solid black;"/> $z = 0 + 3x_1 + 2x_2$	$\rightarrow$ $x_3 = 80 - (40 - x'_1) - x_2$ $x_4 = 100 - 2(40 - x'_1) - x_2$ <hr style="border: 0.5px solid black;"/> $z = 0 + 3(40 - x'_1) + 2x_2$	$\rightarrow$ $x_3 = 40 + x'_1 - x_2$ $x_4 = 20 + 2x'_1 - x_2$ <hr style="border: 0.5px solid black;"/> $z = 120 - 3x'_1 + 2x_2$
--	--	--

### Ratio test

three possible types of constraints

- (1) *lower bound* on a basic variables ( $x_3, x_4$ )  $\rightarrow$  usual ratio test, var with a **negative** coeff of  $x_1 \rightarrow$  pivot
- (2) *upper bound* on the entering variable ( $x_1$ )  $\rightarrow$  replace the variable by its complementary ( $'$ ) variable
- (3) *upper bound* on a basic variable (see below)  $\rightarrow$  var with a **positive** coeff of  $x_1 \rightarrow$  replace by  $'$  var

$$\begin{array}{l}
 x_3 = 40 + x'_1 - x_2 \\
 x_4 = 20 + 2x'_1 - x_2 \\
 \hline
 z = 120 - 3x'_1 + 2x_2
 \end{array}
 \quad
 \begin{array}{l}
 x_2 \text{ increased to } \Delta, \text{ ratio test:} \\
 (1) x_3 = 40 - \Delta \geq 0 \\
 \quad x_4 = 20 - \Delta \geq 0 \\
 (2) x_2 = \Delta \leq 50 \\
 (3) \text{ no constraint}
 \end{array}
 \quad
 \begin{array}{l}
 x_3 : \Delta \leq \frac{40}{1} = 40 \\
 x_4 : \Delta \leq \frac{20}{1} = 20 \\
 x_2 : \Delta \leq 50 \\
 x_4 \text{ leaves, we pivot}
 \end{array}
 \quad
 \begin{array}{l}
 x_4 = 20 + 2x'_1 - x_2 \\
 \downarrow \\
 x_2 = 20 + 2x'_1 - x_4
 \end{array}$$

$$\begin{array}{l}
 x_2 = \quad \quad \quad (20 + 2x'_1 - x_4) \\
 x_3 = 40 + x'_1 - (20 + 2x'_1 - x_4) \\
 \hline
 z = 120 - 3x'_1 + 2(20 + 2x'_1 - x_4)
 \end{array}
 \rightarrow
 \begin{array}{l}
 x_2 = 20 + 2x'_1 - x_4 \\
 x_3 = 20 - x'_1 + x_4 \\
 \hline
 z = 160 + x'_1 - 2x_4
 \end{array}$$

$$\begin{array}{l}
 x'_1 \text{ increased, ratio test:} \\
 (1) x_3 = 20 - \Delta \geq 0 \\
 (2) x'_1 = \Delta \leq 40 \\
 (3) x_2 = 20 + 2\Delta \leq 50
 \end{array}
 \quad
 \begin{array}{l}
 x_3 : \Delta \leq \frac{20}{1} = 20 \\
 x'_1 : \Delta \leq 40 \\
 x_2 : \Delta \leq \frac{50-20}{2} = 15
 \end{array}
 \quad
 \begin{array}{l}
 \text{Most restrictive constraint imposed by } x_2 \\
 \rightarrow \text{ replace by } x'_2 = 50 - x_2
 \end{array}$$

$$\begin{array}{l}
 (50 - x'_2) = 20 + 2x'_1 - x_4 \\
 x_3 = 20 - x'_1 + x_4 \\
 \hline
 z = 160 + x'_1 - 2x_4
 \end{array}
 \rightarrow
 \begin{array}{l}
 x'_2 = 30 - 2x'_1 + x_4 \\
 x_3 = 20 - x'_1 + x_4 \\
 \hline
 z = 160 + x'_1 - 2x_4
 \end{array}$$

$$\begin{array}{l}
 x'_1 \text{ increased, ratio test:} \\
 (1) x'_2 : \frac{30}{2} = 15 \\
 \quad x_3 : \frac{20}{1} = 20 \\
 (2) x'_1 : 40 \\
 (3) \text{ no constraint}
 \end{array}
 \quad
 \begin{array}{l}
 x'_2 \text{ leaves, } x'_1 \text{ enters} \\
 x'_2 = 30 - 2x'_1 + x_4 \\
 2x'_1 = 30 - x'_2 + x_4 \\
 x'_1 = 15 - \frac{1}{2}x'_2 + \frac{1}{2}x_4
 \end{array}
 \quad
 \begin{array}{l}
 x'_1 = \quad \quad \quad (15 - \frac{1}{2}x'_2 + \frac{1}{2}x_4) \\
 x_3 = 20 - (15 - \frac{1}{2}x'_2 + \frac{1}{2}x_4) + x_4 \\
 \hline
 z = 160 + (15 - \frac{1}{2}x'_2 + \frac{1}{2}x_4) - 2x_4
 \end{array}$$

$$\begin{array}{l}
 x'_1 = 15 - \frac{1}{2}x'_2 + \frac{1}{2}x_4 \\
 x_3 = 5 + \frac{1}{2}x'_2 + \frac{1}{2}x_4 \\
 \hline
 z = 175 - \frac{1}{2}x'_2 - \frac{3}{2}x_4
 \end{array}
 \quad
 \begin{array}{l}
 \text{optimal solution found} \\
 \rightarrow \text{ basic variables: } x'_1 = 15, x_3 = 5 \\
 \rightarrow \text{ non-basic variables: } x'_2 = 0, x_4 = 0
 \end{array}
 \quad
 \begin{array}{l}
 x_1 = 40 - x'_1 = 25 \\
 x_2 = 50 - x'_2 = 50 \\
 x_3 = 5 \\
 x_4 = 0
 \end{array}$$

**Note:** Be careful when expressing the values of variables from the dictionary not to confuse the variables and their complements. In the above,  $x'_2$  is **non-basic** and so its value is **zero**. However, this does not mean that  $x_2$  is and in fact, the value of  $x_2$  is  $50 - x'_2 = 50$ . In particular, observe that both  $x_2$  and  $x'_2$  can **never** be non-basic at the same time (one of them is always positive). The fact that we don't see a variable in the final dictionary does not mean its value is zero.

### 8.3 Lower bounds

The above procedure allows us to also handle general lower bounds (other than 0).

If a variable  $x_i$  is constrained by bounds  $\ell_i \leq x_i \leq r_i$  where  $\ell_i, r_i$  are numbers, then we introduce a new variable  $x_j$  and substitute  $x_i = (x_j + \ell_i)$ . In the resulting problem, the variable  $x_j$  is non-negative with an upper bound  $x_j \leq r_i - \ell_i$ . (If  $r_i = \infty$ , we don't have an upper bound for  $x_j$ .) We then solve the problem using the new variable, and then calculate  $x_i$  from the resulting value of  $x_j$  by taking  $x_i = x_j + \ell_i$ .

Contractual obligations with a big retailer demand that at least 30 toy soldiers be produced.

$$\begin{aligned} \text{Max } & 3x_1 + 2x_2 \\ & x_1 + x_2 \leq 80 \\ & 2x_1 + x_2 \leq 100 \\ & 30 \leq x_1 \leq 40 \\ & 0 \leq x_2 \leq 50 \end{aligned}$$

introduce  $x_5$  to replace  $x_1$ :

$$\begin{aligned} x_1 &= 30 + x_5 \\ 0 &\leq x_5 \leq 10 \end{aligned}$$

$$\begin{aligned} \text{Max } & 3(30 + x_5) + 2x_2 \\ & (30 + x_5) + x_2 \leq 80 \\ & 2(30 + x_5) + x_2 \leq 100 \\ & 0 \leq x_5 \leq 10 \\ & 0 \leq x_2 \leq 50 \end{aligned}$$

$$\begin{aligned} \text{Max } & 2x_2 + 3x_5 + 90 \\ & x_2 + x_5 \leq 50 \\ & x_2 + 2x_5 \leq 40 \\ & 0 \leq x_5 \leq 10 \\ & 0 \leq x_2 \leq 50 \end{aligned}$$

introduce complementary

variables  $x'_2, x'_5$ :

$$\begin{aligned} x'_2 &= 50 - x_2 \\ x'_5 &= 10 - x_5 \\ 0 &\leq x'_2 \leq 50 \\ 0 &\leq x'_5 \leq 10 \end{aligned}$$

Initial dictionary:

$$\begin{aligned} x_3 &= 50 - x_2 - x_5 \\ x_4 &= 40 - x_2 - 2x_5 \\ z &= 90 + 2x_2 + 3x_5 \end{aligned} \rightarrow$$

 $x_5$  enters, ratio test:

$$(1) x_3 : \frac{50}{1} = 50$$

$$\rightarrow x_4 : \frac{40}{2} = 20$$

$$(2) x_5 : 10$$

$$(3) \text{ no constraint}$$

 $x_5$  replaced by  $x_5 = 10 - x'_5$ 

$$\begin{aligned} x_3 &= 40 - x_2 + x'_5 \\ x_4 &= 20 - x_2 + 2x'_5 \\ z &= 120 + 2x_2 - 3x'_5 \end{aligned}$$

 $x_2$  enters, ratio test:

$$(1) x_3 : \frac{40}{1} = 40$$

$$\rightarrow x_4 : \frac{20}{1} = 20$$

$$(2) x_2 : 50$$

$$(3) \text{ no constraint}$$

 $x_4$  leaves,  $x_2 = 20 - x_4 + 2x'_5$ 

$$\begin{aligned} \rightarrow x_2 &= 20 - x_4 + 2x'_5 \\ x_3 &= 20 + x_4 - x'_5 \\ z &= 160 - 2x_4 + x'_5 \end{aligned}$$

 $x'_5$  enters, ratio test

$$(1) x_2 : \frac{20}{1} = 20$$

$$(2) x'_5 : 10$$

$$(3) x_2 : \frac{50-20}{2} = 15$$

 $x'_5$  replaced by  $x'_5 = 10 - x_5$ 

$$\begin{aligned} \rightarrow x_2 &= 40 - x_4 - 2x_5 \\ x_3 &= 10 + x_4 + x_5 \\ z &= 170 - 2x_4 - x_5 \end{aligned}$$

**optimal solution:**

$$x_2 = 40, x_3 = 10, x_4 = 0, x_5 = 0$$

calculate  $x_1$  from  $x_5 \rightarrow x_1 = 30 + x_5 = 30$ 

$$\rightarrow \begin{aligned} x_1 &= 30 \\ x_2 &= 40 \end{aligned} \quad \text{solution of value } z = 170$$

## 8.4 Dual Simplex with Upper Bounds

Instead of selling via a retailer, we decided to ship directly to customers. This involves extra packaging step before shipping requiring 3 sheets of cardboard per a toy soldier, and 4 sheets per a toy train.

Being a green company, we wish to minimize the amount of packaging material used. However, we also want to make profit, namely at least \$150. This leads to the following minimization problem.

$$\begin{aligned} \text{Min } & 3x_1 + 4x_2 \\ & x_1 + x_2 \leq 80 \quad [\text{carving}] \\ & 2x_1 + x_2 \leq 100 \quad [\text{finishing}] \\ & 3x_1 + 2x_2 \geq 150 \quad [\text{profit}] \\ & 0 \leq x_1 \leq 40 \\ & 0 \leq x_2 \leq 50 \end{aligned}$$

introduce  $x'_1, x'_2$  andslack/excess  $x_3, x_4, x_5$ 

$$x'_1 = 40 - x_1$$

$$x'_2 = 50 - x_2$$

$$0 \leq x'_1 \leq 40$$

$$0 \leq x'_2 \leq 50$$

$$0 \leq x_3 \leq \infty$$

$$0 \leq x_4 \leq \infty$$

$$0 \leq x_5 \leq \infty$$

$$\text{Max } z = -3x_1 - 4x_2$$

$$x_3 = 80 - x_1 - x_2$$

$$x_4 = 100 - 2x_1 - x_2$$

$$x_5 = -150 + 3x_1 + 2x_2$$

$$z = 0 - 3x_1 - 4x_2$$

dictionary is **dually feasible**  $\rightarrow$  all coefficients in  $z$  are non-positive  $\rightarrow$  can use Dual Simplex methodfirst, we check **upper bounds** of basic variables:  $x_3 = 80 \leq \infty$ ,  $x_4 = 100 \leq \infty$ ,  $x_5 = -150 \leq \infty \rightarrow$  OKnext, we check **lower bounds**:  $x_3 = 80 \geq 0$ ,  $x_4 = 100 \geq 0$ ,  $x_5 = -150 \not\geq 0 \rightarrow$  not OK, must increase  $x_5$  $x_5$  leaves, **ratio test**  $\rightarrow x_1 : \frac{3}{3} = 1$ ,  $x_2 : \frac{4}{2} = 2$  (compare coefficients in  $z$  and (positive coeffs) in  $x_5$ )take smallest  $\rightarrow x_1$  enters,  $x_5 = -150 + 3x_1 + 2x_2 \rightarrow x_1 = 50 - \frac{2}{3}x_2 + \frac{1}{3}x_5$

$$\begin{array}{rcl}
 x_1 = & (50 - \frac{2}{3}x_2 + \frac{1}{3}x_5) & \\
 x_3 = & 80 - (50 - \frac{2}{3}x_2 + \frac{1}{3}x_5) - x_2 & \\
 x_4 = & 100 - 2(50 - \frac{2}{3}x_2 + \frac{1}{3}x_5) - x_2 & \\
 \hline
 z = & 0 - 3(50 - \frac{2}{3}x_2 + \frac{1}{3}x_5) - 4x_2 & \\
 \end{array}
 \quad \rightarrow \quad
 \begin{array}{rcl}
 x_1 = & 50 - \frac{2}{3}x_2 + \frac{1}{3}x_5 & \\
 x_3 = & 30 - \frac{1}{3}x_2 - \frac{1}{3}x_5 & \\
 x_4 = & \frac{1}{3}x_2 - \frac{2}{3}x_5 & \\
 \hline
 z = & -150 - 2x_2 - x_5 & 
 \end{array}$$

check **upper bounds**:  $x_1 = 50 \not\leq 40$ ,  $x_3 = 30 \leq \infty$ ,  $x_4 = 0 \leq \infty \rightarrow$  not OK, replace  $x_1$  by  $x_1 = 40 - x'_1$

$$\begin{array}{rcl}
 (40 - x'_1) = & 50 - \frac{2}{3}x_2 + \frac{1}{3}x_5 & \\
 x_3 = & 30 - \frac{1}{3}x_2 - \frac{1}{3}x_5 & \\
 x_4 = & \frac{1}{3}x_2 - \frac{2}{3}x_5 & \\
 \hline
 z = & -150 - 2x_2 - x_5 & 
 \end{array}
 \quad \rightarrow \quad
 \begin{array}{rcl}
 x'_1 = & -10 + \frac{2}{3}x_2 - \frac{1}{3}x_5 & \\
 x_3 = & 30 - \frac{1}{3}x_2 - \frac{1}{3}x_5 & \\
 x_4 = & \frac{1}{3}x_2 - \frac{2}{3}x_5 & \\
 \hline
 z = & -150 - 2x_2 - x_5 & 
 \end{array}$$

upper bounds:  $x'_1 = -10 \leq 40$ ,  $x_3 = 30 \leq \infty$ ,  $x_4 = 0 \leq \infty \rightarrow$  OK

lower bounds:  $x'_1 = -10 \not\geq 0$ ,  $x_3 = 30 \geq 0$ ,  $x_4 = 0 \geq 0 \rightarrow$  not OK, must increase  $x'_1$

$x'_1$  leaves, ratio test  $\rightarrow x_2 = \frac{2}{(2/3)} = 3$ ,  $x_5$  : *no constraint* (negative coefficient in  $x'_1$ )  $\rightarrow x_2$  enters

$$\begin{array}{rcl}
 x'_1 = -10 + \frac{2}{3}x_2 - \frac{1}{3}x_5 & x_2 = & (15 + \frac{3}{2}x'_1 + \frac{1}{2}x_5) & x_2 = & 15 + \frac{3}{2}x'_1 + \frac{1}{2}x_5 \\
 \frac{2}{3}x_2 = 10 + x'_1 + \frac{1}{3}x_5 & x_3 = & 30 - \frac{1}{3}(15 + \frac{3}{2}x'_1 + \frac{1}{2}x_5) - \frac{1}{3}x_5 & x_3 = & 25 - \frac{1}{2}x'_1 - \frac{1}{2}x_5 \\
 \rightarrow x_2 = 15 + \frac{3}{2}x'_1 + \frac{1}{2}x_5 & x_4 = & \frac{1}{3}(15 + \frac{3}{2}x'_1 + \frac{1}{2}x_5) - \frac{2}{3}x_5 & x_4 = & 5 + \frac{1}{2}x'_1 - \frac{1}{2}x_5 \\
 & z = & -150 - 2(15 + \frac{3}{2}x'_1 + \frac{1}{2}x_5) - x_5 & z = & -180 - 3x'_1 - 2x_5
 \end{array}$$

upper bounds:  $x_2 = 15 \leq 50$ ,  $x_3 = 25 \leq \infty$ ,  $x_4 = 5 \leq \infty \rightarrow$  OK

lower bounds:  $x_2 = 15 \not\geq 0$ ,  $x_3 = 25 \geq 0$ ,  $x_4 = 5 \geq 0 \rightarrow$  OK

}  $\implies$  **optimal solution** found

values of variables: basic  $x_2 = 15$ ,  $x_3 = 25$ ,  $x_4 = 5$ , non-basic  $x_5 = 0$ ,  $x'_1 = 0 \rightarrow x_1 = 40 - x'_1 = 40$

$\rightarrow x_1 = 40$ ,  $x_2 = 15$  is optimal with value  $z = -180$

$\implies$  Minimum amount of packaging required to make \$150 of profit is 180 units.

## 8.5 Goal Programming

We consider again the (usual) toy factory production problem with two products, *toy soldiers* and *toy trains*, with sale prices \$3 respectively \$2, requiring 1 hour each in the carving shop, and 2 hours respectively 1 hour in the finishing shop; the total hours being 80 and 100 respectively, as usual. Each toy soldier requires 3 units of packaging material, and each toy train requires 4.

The company became non-profit and set itself several new goals.

**Goal #1:** not to produce more than 40 toy soldiers; only up to 40 can be sold at the price \$3

**Goal #2:** make at least \$140 of profit to pay workers wages

**Goal #3:** use at most 130 units of packaging material; only this much is currently available

Each goal can be met individually, however trying to meet all three at once may not be (and it is not) possible. *Failure* to meet a goal carries a **penalty**, proportional to the amount by which the goal is not met.

**Penalty #1:** each toy soldier over 40 units will not be sold; penalty of \$3 of missing profit

**Penalty #2:** any profit below \$140 has to be obtained by borrowing at 20% interest

**Penalty #3:** any missing packaging material has to be bought at \$0.80 a unit

Our combined **goal** is to **minimize** the **total penalty** incurred. This leads to the following:

$$\begin{array}{rcll}
x_1 + x_2 \leq 80 & \text{[Carving]} & & \\
2x_1 + x_2 \leq 100 & \text{[Finishing]} & & \\
x_1 \leq 40 & \text{[Goal #1]} & \xrightarrow{\text{add}} & \\
3x_1 + 2x_2 \geq 140 & \text{[Goal #2]} & \text{slack} & \\
3x_1 + 4x_2 \leq 130 & \text{[Goal #3]} & & 
\end{array}
\qquad
\begin{array}{rcl}
x_1 + x_2 + x_3 & = & 80 \\
2x_1 + x_2 + x_4 & = & 100 \\
x_1 + x_5 & = & 40 \\
3x_1 + 2x_2 - x_6 & = & 140 \\
3x_1 + 4x_2 + x_7 & = & 130
\end{array}$$

Slack variables  $x_3$  and  $x_4$  are assumed to be always non-negative; they come from **hard** constraints – must be satisfied **unconditionally**. On the other hand, we can allow  $x_5$ ,  $x_6$ , and  $x_7$  to be unrestricted, since the corresponding constraints are **soft** – we may violate any of them, but we incur a penalty.

Depending on whether each of  $x_5, x_6, x_7$  is positive or negative, we incur different penalties. For instance, if  $x_5$  is positive, there is no penalty, the penalty is 0; if  $x_5$  is negative, the penalty is  $-3x_5$ , since we lose \$3 for each unsold toy soldier (there is “minus” in penalty because  $x_5$  is negative, not because we lose profit).

Similarly, if  $x_6$  is positive, the penalty is 0, while if  $x_6$  is negative, the penalty is  $-1.2x_6$ , since if profit drops below \$140, to pay the workers we have borrow at 20% interest. Note that we can specify different penalties for both positive and negative slack (not necessarily only zero) and the method works the same way.

The question now is: **how do we formulate this problem as a linear program?**

We express each of the variables  $x_5, x_6, x_7$  as a **difference** of two **non-negative** variables. Namely we write  $x_5$  as  $x_5 = x_5^+ - x_5^-$  where  $x_5^+$  and  $x_5^-$  are new non-negative variables. Similarly we write  $x_6 = x_6^+ - x_6^-$  and  $x_7 = x_7^+ - x_7^-$ . As we discussed at the beginning of the course, under some assumptions, the values of these new variables (in an optimal solution) will be precisely the positive/negative parts of the variables  $x_5, x_6, x_7$ .

This allows us to **construct** an objective function that captures the sum of all the penalties that we incur in different situations (depending on the signs of the variables  $x_5, x_6, x_7$ ).

$$\begin{array}{rcll}
x_5 = x_5^+ - x_5^- & \min & 3x_5^- & + 1.2x_6^- & + 0.8x_7^- \\
x_6 = x_6^+ - x_6^- & & x_1 + x_2 + x_3 & & = 80 \\
x_7 = x_7^+ - x_7^- & & 2x_1 + x_2 + x_4 & & = 100 \\
x_5^+, x_5^-, x_6^+, x_6^-, x_7^+, x_7^- \geq 0 & & x_1 & + x_5^+ - x_5^- & = 40 \\
& & 3x_1 + 2x_2 & - x_6^+ + x_6^- & = 140 \\
& & 3x_1 + 4x_2 & + x_7^+ - x_7^- & = 130
\end{array}$$

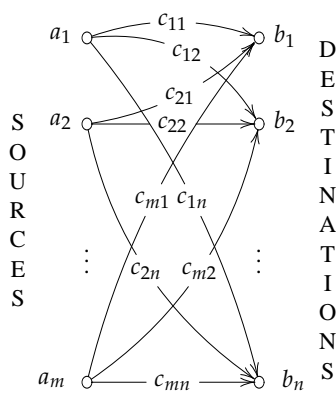
We solve this problem using standard methods (Simplex method or Dual Simplex method). Alternatively, we can solve the dual and from that construct a solution to this problem.

**Exercise.** Construct the dual LP to the above problem.

# Transportation Problem

### Historical note

The problem and its solution described first in 1941 by Hitchcock – predates the Simplex Method – independently discovered by Koopmans during World War II used to minimize shipping times for cargo ships – the method spearheaded research on linear and non-linear programs in economics. The Assignment problem (a special case of the Transportation Problem) can be solved more efficiently using the Hungarian Method, found in the work of Egerváry in 1931 and first explicitly described in 1955 by Kuhn who named it the Hungarian Method; recently it was found that the method was actually first found a century earlier by Jacobi in the context of solving systems of ordinary differential equations (work published posthumously in 1890). Since then, more efficient (but more complicated) algorithms have been found for the two problems.



In the transportation problem, we have

- $m$  sources (warehouses, factories) producing items, and
- $n$  destinations (shops, businesses) requiring these items.

The items need to be **transported** from sources to destinations which has **associated cost**.

The goal is to **minimize** the total **cost of transportation**.

- The  $i$ -th source has  $a_i$  **available** items
- The  $j$ -th destination **demands**  $b_j$  items to be delivered.
- It costs  $c_{ij}$  to deliver one item from  $i$ -th source to  $j$ -th destination.

**Assumption:** destinations do not care from which source the items come, and sources do not care to which destinations they deliver.

**Decision variables:**  $x_{ij}$  = the number of items transported from the  $i$ -th source to  $j$ -th destination

Minimize  $\sum_{i=1}^m \sum_{j=1}^n c_{ij}x_{ij}$

**Balanced Transportation Problem**

subject to  $\sum_{j=1}^n x_{ij} = a_i$  for  $i = 1, \dots, m$  [i-th sources has  $a_i$  available items]

$\sum_{i=1}^m x_{ij} = b_j$  for  $j = 1, \dots, n$  [j-th demands  $b_j$  items]

$x_{ij} \geq 0$  for  $i = 1, \dots, m$  and  $j = 1, \dots, n$  [non-negative amounts transported]

A solution exists if and only if  $\overbrace{\sum_{i=1}^m a_i}^{\text{total supply}} = \overbrace{\sum_{i=1}^n b_i}^{\text{total demand}}$  (the total supply is equal to the total demand)

**Why equalities?** If the total supply is bigger than the total demand by  $\Delta > 0$ , then we introduce a *dummy* destination with demand  $\Delta$  and with **zero cost** of transportation from all sources. If the total demand is bigger than the total supply by  $\Delta > 0$ , then **no feasible solution** exists. However, we can still model the **shortage** by introducing a *dummy* source with supply  $\Delta$  and some the costs of transportation (say zero).

**Example**

A furniture company owns three warehouses in the New York City area and needs to deliver chairs to its three shops in the city for tomorrow. The three shops demand 4, 2, and 3 units respectively. Current stock level of chairs in the three warehouses is 8, 6, and 3 respectively. Delivery costs from each warehouse to each store are different due to different distances. These are as follows (in \$ per unit).

		Shop #			Demand:	Supply:
<b>Delivery costs:</b>	Warehouse #1	\$7	\$3	\$4	Shop #1: 4 units	Warehouse #1: 8 units
	Warehouse #2	\$4	\$2	\$2	Shop #2: 2 units	Warehouse #2: 6 units
	Warehouse #3	\$2	\$1	\$5	Shop #4: 3 units	Warehouse #3: 3 units

**Find** the least expensive way to deliver the chairs to the stores.

**Total Demand:** 9 units  
**Total Supply:** 17 units

} Solution exists, since total supply is at least the total demand. The excess supply of 8 units will be assigned to a **dummy destination** at zero cost.

		Shops #				Demand:	Supply:
<b>Delivery costs:</b>	Warehouse #1	\$7	\$3	\$4	\$0	Shop #1: 4 units	Warehouse #1: 8 units
	Warehouse #2	\$4	\$2	\$2	\$0	Shop #2: 2 units	Warehouse #2: 6 units
	Warehouse #3	\$2	\$1	\$5	\$0	Shop #4: 3 units	Warehouse #3: 3 units
						Dummy: 8 units	

$$\begin{aligned} \text{Min } & 7x_{11} + 3x_{12} + 4x_{13} + 0x_{14} + 4x_{21} + 2x_{22} + 2x_{23} + 0x_{24} + 2x_{31} + 1x_{32} + 5x_{33} + 0x_{34} \\ \text{s.t. } & x_{11} + x_{12} + x_{13} + x_{14} = 8 \\ & x_{21} + x_{22} + x_{23} + x_{24} = 6 \\ & x_{31} + x_{32} + x_{33} + x_{34} = 3 \\ & x_{11} + x_{21} + x_{31} = 4 \\ & x_{12} + x_{22} + x_{32} = 2 \\ & x_{13} + x_{23} + x_{33} = 3 \\ & x_{14} + x_{24} + x_{34} = 8 \end{aligned}$$

all variables non-negative

**9.1 Transportation Simplex Method**

Tableau form:

$c_{11}$	$c_{12}$	$c_{13}$	$c_{14}$	$u_1$
$x_{11}$	$x_{12}$	$x_{13}$	$x_{14}$	$a_1$
$c_{21}$	$c_{22}$	$c_{23}$	$c_{24}$	$u_2$
$x_{21}$	$x_{22}$	$x_{23}$	$x_{24}$	$a_2$
$c_{31}$	$c_{32}$	$c_{33}$	$c_{34}$	$u_3$
$x_{31}$	$x_{32}$	$x_{33}$	$x_{34}$	$a_3$
$v_1$	$v_2$	$v_3$	$v_4$	$z = ?$
$b_1$	$b_2$	$b_3$	$b_4$	

7	3	4	0	$u_1$
				8
4	2	2	0	$u_2$
				6
2	1	5	0	$u_3$
				3
$v_1$	$v_2$	$v_3$	$v_4$	
4	2	3	8	

**Shadow prices**

- $u_1, u_2, u_3$  for the first three constraints (*supply constraints*),
- $v_1, v_2, v_3, v_4$  for the latter four constraints (*demand constraints*)



### Initialization

We need to find a starting **basic feasible solution**. One method is the **Minimum-cost rule**:

1. If the tableau has only 1 row, then for every variable  $x_{ij}$ : set  $x_{ij} = b_j$ , and add  $x_{ij}$  to the basis; then stop.
2. Otherwise find a variable  $x_{ij}$  of smallest cost  $c_{ij}$ .
3. If  $a_i \leq b_j$ , then set  $x_{ij} = a_i$ , add  $x_{ij}$  to the basis, remove the  $i$ -th row, and decrease  $b_j$  to  $b_j - a_i$ .
4. If  $a_i > b_j$ , then set  $x_{ij} = b_j$ , add  $x_{ij}$  to the basis, remove the  $j$ -th column, and decrease  $a_i$  to  $a_i - b_j$ .
5. Repeat.

<del>x</del>	7	3	4	0	<del>8</del>	$u_1$
	4	2	2	0	6	$u_2$
	2	1	5	0	3	$u_3$
4	$v_1$	2	3	<del>8</del> 0		

<del>x</del>	7	3	4	8	<del>8</del>	$u_1$
	4	2	2	0	6	$u_2$
	2	1	5	0	<del>3</del> 1	$u_3$
4	<del>x</del>	2	3	<del>8</del> 0		

<del>x</del>	7	3	4	8	0	<del>8</del>	$u_1$
	4	<del>x</del>	2	2	0	6	$u_2$
	2	1	5	0	0	<del>3</del> 1	$u_3$
4	<del>x</del>	<del>x</del>	3	<del>8</del> 0			

<del>x</del>	7	3	4	8	0	<del>8</del>	$u_1$
	4	2	2	0	0	<del>6</del> <del>3</del>	$u_2$
	3	<del>x</del>	3	0	0	<del>6</del> <del>3</del>	$u_2$
	2	1	5	0	0	<del>3</del> <del>1</del>	$u_3$
1	<del>x</del>	<del>x</del>	<del>x</del>	<del>8</del> 0			

### Basic Feasible Solutions

The problem with  $m$  sources and  $n$  destinations has  $m \times n$  variables and  $m + n$  constraint. The constraints are **not independent**, but each  $m + n - 1$  of them are. This implies the **fundamental property**:

- every **basis** contains **exactly**  $m + n - 1$  **variables**,
- each **constraint** contains **at least one basic variable**.

We **mark** all cells that correspond to the **basic variables**.

- the table will contain  $m + n - 1$  marked cells
- every unmarked cell has value zero
- each row contains at least one marked cell
- each column contains at least one marked cell
- **consequently**: there always exists a row or column with **exactly one** marked cell.

	7	3	4	0	$u_1$	
	4	2	2	0	$u_2$	
	2	1	5	0	$u_3$	
4	$v_1$	2	3	8	$v_4$	$z = 22$

### Shadow prices

**Economic interpretation:**  $(-u_i)$  = price of chair in the  $i$ -th warehouse  
 $v_j$  = sale price of chair in the  $j$ -th shop.

**Optimality criterion:** the current solution is optimal if for **every** warehouse  $i$  and **every** shop  $j$  the **cost of transportation is more than** (or equal to) the **difference in prices**, i.e., if  $c_{ij} \geq v_j + u_i$

### How to determine shadow prices?

If  $x_{ij}$  is **basic**, then we break even when transporting from  $i$ -th source to  $j$ -th destination, i.e., if  $c_{ij} = v_j + u_i$  (we have no incentive *increasing* or *decreasing* the transported amount, since that does not change our profit).

This yields  $m + n - 1$  equations over  $m + n$  variables  $u_1, \dots, u_m, v_1, \dots, v_n$ . This is an **underdetermined** system. Since we are only interested in price **differences**, we may simply assume that one of the shadow prices is zero. With that, the system will have a solution which will give us the shadow prices we are interested in. **Instead** of writing down and solving the system, we find shadow prices **directly**.

Set  $u_i = 0$  for some  $i \in \{1, \dots, m\}$ . Then repeat the following until all shadow prices have been determined.

- Find  $i$  such that  $u_i$  has been determined  $\rightarrow$  for every basic variable  $x_{ij}$ , set  $v_j = c_{ij} - u_i$ .
- Find  $j$  such that  $v_j$  has been determined  $\rightarrow$  for every basic variable  $x_{ij}$ , set  $u_i = c_{ij} - v_j$ .

We choose to start by assigning  $u_2 = 0$ .

7	3	4	0	$u_1$
			(8)	8
4	2	2	0	(0)
(3)		(3)		6
2	1	5	0	$u_3$
(1)	(2)		(0)	3
4	2	3	8	
4	$v_2$	$v_4$		

→

7	3	4	0	$u_1$
			(8)	8
4	2	2	0	0
(3)		(3)		6
2	1	5	0	-2
(1)	(2)		(0)	3
4	2	3	8	
4	$v_2$	$v_4$		

→

7	3	4	0	$u_1$
			(8)	8
4	2	2	0	0
(3)		(3)		6
2	1	5	0	(-2)
(1)	(2)		(0)	3
4	2	3	8	
4	$v_2$	$v_4$		

→

7	3	4	0	-2
			(8)	8
4	2	2	0	0
(3)		(3)		6
2	1	5	0	-2
(1)	(2)		(0)	3
4	2	3	8	
4	$v_2$	$v_4$		

We **price out** non-basic variables by taking  $u_i + v_j$  (the value in blue) for every non-basic  $x_{ij}$ . This value represents the **price difference** between  $i$ -th warehouse and  $j$ -th shop. If this value is **more** than the corresponding transportation cost  $c_{ij}$ , then it pays to make  $x_{ij}$  **basic**, i.e., transport items from  $i$ -th warehouse to  $j$ -th shop, since the net profit is positive. Such variable is  $x_{22}$ , since  $c_{22} = 2$  and  $u_2 + v_2 = 3$ .

### Pivoting to a new basis

We increase  $x_{22}$  to  $\Delta > 0$  and need to find out which variable leaves the basis. This is done by finding a **loop** (a cycle, circuit) in the tableau: starting from  $x_{22}$ , choosing basic variables in turn, either from the same row, or same column where we alternate between rows and columns, and coming back to  $x_{22}$ .

7	3	4	0	-2
			(8)	8
$-\Delta$	4	2	0	-4
(3)	$\Delta$	(3)		6
$+\Delta$	2	1	0	-2
(1)	(2)		(0)	3
4	2	3	8	

We increase  $x_{22}$  to  $\Delta$ . Since total value in column 2 must be exactly 2, we decrease  $x_{32}$  from 2 to  $2 - \Delta$ . This decreases the total value in row 3 by  $\Delta$ , so we compensate by increasing  $x_{31}$  from 1 to  $1 + \Delta$ . Then the total in column 1 increases, so we decrease  $x_{21}$  from 3 to  $3 - \Delta$ . Finally, all row and column totals are back in order.

Largest value  $\Delta$  for which all current basic variables are non-negative is  $\Delta = 2$ ; for this value,  $x_{32}$  decreases to 0, leaves the basis, and  $x_{22}$  enters the basis.

**How to find the loop?** Make the entering variable basic (circle it in the tableau). Then cross out every basic variable that belongs a row or column which has no other basic variable (or has other basic variables, but they all have been crossed out). Repeat as long as possible. What remains is the desired loop.

7	3	4	<del>8</del> 0	-2
4	2	2	0	-4
③	Δ	<del>3</del>		6
2	1	5	0	-2
①	②		④	3
4	2	3	6	2
4	2	3	8	

→

7	3	4	<del>8</del> 0	-2
4	2	2	0	-4
③	Δ	<del>3</del>		6
2	1	5	0	-2
①	②		<del>④</del>	3
4	2	3	6	2
4	2	3	8	

After pivoting, we obtain the following tableau. We calculate the shadow prices, and price out cells

7	3	4	<del>8</del> 0	$u_1$
4	2	2	0	$u_2$
①	②	③		6
2	1	5	0	$u_3$
③			④	3
$v_1$	$v_2$	$v_3$	$v_4$	$z = 20$
4	2	3	8	

→

7	3	4	<del>8</del> 0	-2
2	0	0	⑧	8
①	②	③	2	0
③	0	0	④	3
4	2	3	6	2
4	2	3	8	2
4	2	3	8	$z = 20$

→  $x_{24}$  enters the basis, increased to  $\Delta$ , largest  $\Delta = 0$  when  $x_{34}$  leaves the basis.

7	3	4	<del>8</del> 0	-2
4	2	2	0	0
①-Δ	<del>2</del>	<del>3</del>	Δ	6
2	1	5	0	-2
③+Δ			④-Δ	3
4	2	3	6	2
4	2	3	8	$z = 20$

→

7	3	4	<del>8</del> 0	0
4	2	2	0	0
①	②	③	④	6
2	1	5	0	-2
③	0	0	-2	3
4	2	3	6	2
4	2	3	8	$z = 20$

**Optimal** solution found.

$$x_{21} = 1, x_{22} = 2, x_{23} = 3, x_{31} = 3.$$

**Conclusion:** Least costly way to deliver the chairs to stores is to deliver no chairs from warehouse #1, deliver 1, 2, and 3 chairs from warehouse #2 to shops #1, #2, and #3 respectively, and in addition deliver 3 chairs from warehouse #3 to shop #1. The total transportation cost is \$20.

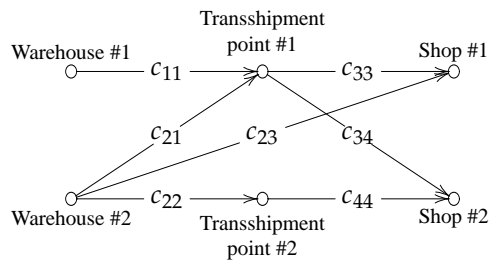
### Assignment Problem

We have  $m$  machines that we want to assigned  $n$  jobs. Machines are different and to execute job  $j$  on machine  $i$  requires  $c_{ij}$  resources (time, space, energy). We want to minimize total amount of resources used.

We can model it as a Transportation Problem with  $m$  sources–machines,  $n$  destinations–jobs, costs  $c_{ij}$ , each demand  $a_i$  equal to 1, and each available supply  $b_j$  also equal 1. By adding dummy machines and jobs, we may assume that  $n = m$ . We may solve the problem using the standard (Simplex) method for the Transportation problem. This is unfortunately not very efficient, since **every basis** of the corresponding LP contains  $n - 1$  **degenerate** basic variables. That is why specialized algorithms, such as the Hungarian Method, exist for the Assignment Problem.

### Transshipment Problem

A generalization of the Transportation Problem where we may ship some items via several intermediate (**transshipment**) locations. We can model it as an ordinary Transportation Problem by **treating** each intermediate node as **both** a source and a destination (it appears twice in the tableau); its demand is equal to its supply (equal to its storage capacity), and has zero transportation cost between itself.

**Transportation costs matrix:**

$c_{11}$	$\infty$	$\infty$	$\infty$	Warehouse #1
$c_{21}$	$c_{22}$	$c_{23}$	$\infty$	Warehouse #2
0	$\infty$	$c_{33}$	$c_{34}$	Transship #1
$\infty$	0	$\infty$	$c_{44}$	Transship #2
Transship #1	Transship #2	Shop #1	Shop #2	

## Network problems

### Historical note

The Shortest Path Problem is one of the most important efficient computational tools at our disposal. It is used everywhere, from GPS navigation and network communication to project management, layout design, robotics, computer graphics, and the list goes on. Often many problems reduce to finding shortest paths or use shortest paths as guidelines to optimal solutions. First algorithms for the Shortest Path problem were designed by Ford (1956), Bellman (1958), and Moore (1959). For non-negative weights, a more efficient algorithm was first suggested by Dijkstra (1959). Since then, many improved and specialized algorithms have been developed, with close to linear running time. For instance, in the context of navigation, current record holder is the Hub labelling algorithm (2011), which uses precomputed distances from carefully chosen nodes–hubs to speed up path finding on maps from hours to tenths of microseconds. For All-pairs Shortest Path problem the first algorithms were found by Shimbel (1953), and by Roy (1959), Floyd (1962), and Warshall (1962). A more efficient approach is a combination of Dijkstra’s and Bellman-Ford’s algorithms known as Johnson’s algorithm (1977).

The Minimum Spanning Tree problem also has a rich history. The first known algorithm was developed by Borůvka (1926) for efficient distribution of electricity. Later independently discovered by many researchers over the years: Jarník (1930), Kruskal (1956), Prim (1957), and Dijkstra (1959). All are based on similar ideas and have very efficient, almost linear-time implementations. There also exist efficient parallel and distributed implementations. The latter is notably used in minimum spanning tree protocols found frequently in routing broadcast/multicast communication in computer and wireless networking.

$G$  = **graph** or **network** consists of

- a set  $V$  of **vertices** (nodes, points) and
- a set  $E$  of **edges** (arcs, lines) which are connections between vertices.

write  $G = (V, E)$ ; write  $V(G)$  for vertices of  $G$ , and  $E(G)$  for edges of  $G$ .

(vertices are usually denoted  $u$  or  $v$  with subscripts; edges we usually denote  $e$ )

edges may have **direction**: an edge  $e$  between  $u$  and  $v$  may go from  $u$  to  $v$ , we write  $e = (u, v)$ ,

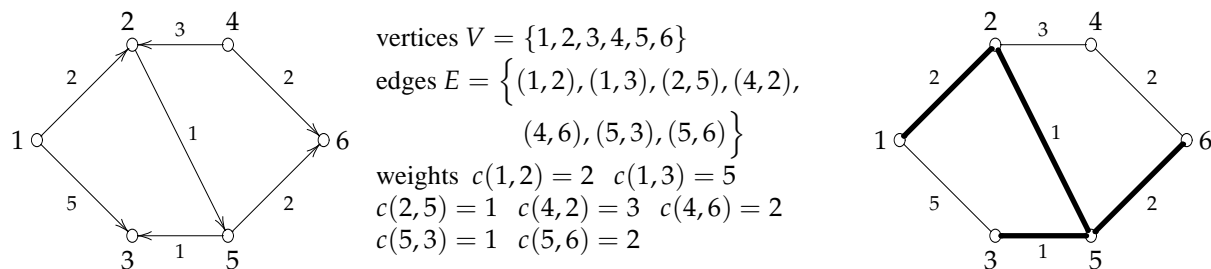


Figure 10.1: network (left), undirected network (right), edges  $(1, 2)$ ,  $(2, 5)$ ,  $(3, 5)$ ,  $(5, 6)$  form a tree of weight 5

or from  $v$  to  $u$ , we write  $e = (v, u)$

(if an edge  $e$  does not have a direction, we treat it the same way as having both directions)

if all edges do not have a direction (are undirected), we say that the network is **undirected**

edges may have **weight**: a weight of edge  $e = (u, v)$  is a real number denoted  $c(e)$  or  $c(u, v)$ ,  $c_e$ ,  $c_{uv}$

a sequence of nodes and edges  $v_1, e_1, v_2, e_2, \dots, v_{k-1}, e_{k-1}, v_k$  is

- a **path** (directed path) if each  $e_i$  goes from  $v_i$  to  $v_{i+1}$
- a **chain** (undirected path) if each  $e_i$  connects  $v_i$  and  $v_{i+1}$  (in some direction)

(often we write:  $e_1, e_2, \dots, e_k$  is a path (we omit vertices) or write:  $v_1, v_2, \dots, v_k$  is a path (we omit edges))

a network is **connected** if for every two nodes there is a path connecting them; otherwise it is **disconnected**

a **cycle** (loop, circuit) is a path starting and ending in the same node, never repeating any node or edge

a **forest** (acyclic graph) is an undirected graph that contains no cycles

a **tree** is a connected forest

**Claim:** A tree with  $n$  nodes contains exactly  $n - 1$  edges. Adding any edge to a tree creates a cycle.  
Removing any edge from a tree creates a disconnected forest.

### 10.1 Shortest Path Problem

Given a network  $G = (V, E)$  with two distinguished vertices  $s, t \in V$ , find a shortest path from  $s$  to  $t$

**Example:** In Figure 1 (left), a shortest path from  $s = 1$  to  $t = 6$  is 1, 2, 5, 6 of total length 5, while for  $t = 3$  a shortest path is 1, 2, 5, 3 of length 4. We say that **distance** from node 1 to node 6 is 5. Note that there is no path from  $s$  to  $t = 4$ ; we indicate this by defining the distance to 4 as  $\infty$ .

**LP formulation:** decision variables  $x_{ij}$  for each  $(i, j) \in E$

$$\begin{aligned} \text{Min} \quad & \sum_{(i,j) \in E} w_{ij} x_{ij} \\ \sum_{j:(i,j) \in E} x_{ij} - \sum_{j:(j,i) \in E} x_{ji} = & \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases} & \text{for each } i \in V \\ x_{ij} \in & \{0, 1\} & \text{for each } (i, j) \in E \end{aligned}$$

### Modeling

A new car costs \$12,000. Annual maintenance costs are as follows:  $m_1 = \$2,000$  first year,  $m_2 = \$4,000$  second year,  $m_3 = \$5,000$  third year,  $m_4 = \$9,000$  fourth year, and  $m_5 = \$12,000$  fifth year and on. The car can be sold for  $s_1 = \$7,000$  in the first year, for  $s_2 = \$6,000$  in the second year, for  $s_3 = \$2,000$  in the third year, and for  $s_4 = \$1,000$  in the fourth year of ownership.

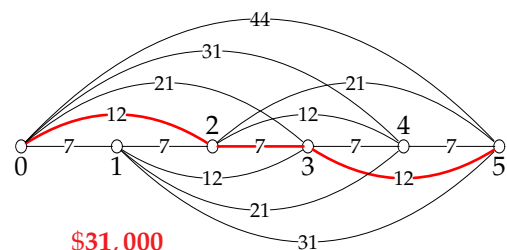
An existing car can be sold at any time and another new car purchased at \$12,000. What buying/selling strategy for the next 5 years minimizes the total cost of ownership?

Nodes =  $\{0, 1, 2, 3, 4, 5\}$

Edge  $(i, j)$  represents the act of buying a car in year  $i$  and selling in year  $j$ . The weight is the price difference plus the maintenance cost, i.e., the weight is

$$c(i, j) = \$12,000 - s_{(i-j)} + m_1 + m_2 + \dots + m_{(i-j)}$$

**Answer:** the length of a shortest path from node 0 to node 5.



A company wants to introduce a new model of its ever-popular *cell phone* to the market. To gauge the demand, a limited batch of product is to be brought to the market. The product is assembled from two parts, a circuit board, and housing. In addition, workers need to be trained and raw material procured.

After the initial assesment, the project manager put together a list of all tasks along with estimated duration of each task. The tasks are interdependent, some need to finish before others can start. First, (A) workers need to be trained and (B) raw materials purchased which takes 6 and 9 days, respectively. After these tasks are finished, both parts of the product, (C) the circuit board and (D) the housing, are manufactured taking 7 and 8 days, respectively. Each circuit board, once manufactured, needs to undergo additional testing (E) to comply with FCC regulation which takes 10 days. Afterwards, the cell phone is assembled, packaged, and shipped (F) which takes 12 days.

What is the minimum number of days before the product can reach the market? What is the **critical path** of the production, i.e., the sequence of tasks delaying any of which delays the whole production ?

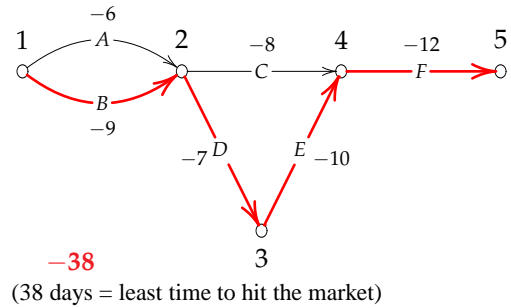
We identify main check-points:

- (1) start, (2) completion of tasks A and B,
- (3) completion of task D, (4) completion of tasks C,E,
- and (5) finish.

Nodes {1,2,3,4,5}

Edges correspond to tasks that connect checkpoints, weights are durations with negative sign.

**Answer:** A shortest path from 1 to 5 is a critical path.



### Dijkstra's algorithm

Algorithm finds the length of a shortest path from  $s$  to every vertex of  $G$  (not only  $t$ )

Weights of edges are assumed to be **non-negative**, else the algorithm may output incorrect answer.

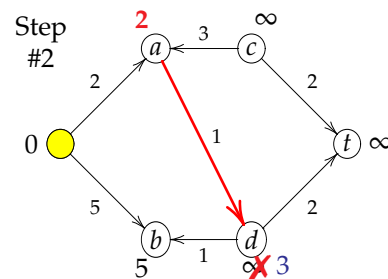
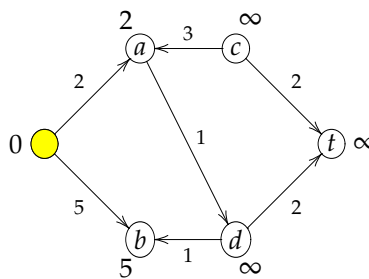
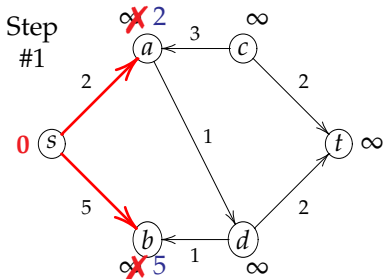
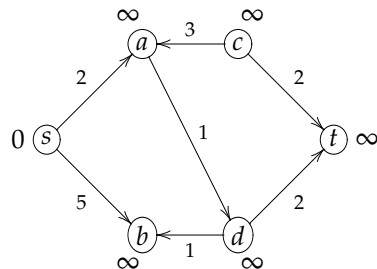
**variables:**  $d_u$  for each  $u \in V$ , an **estimate** on the distance from  $s$  to  $u$

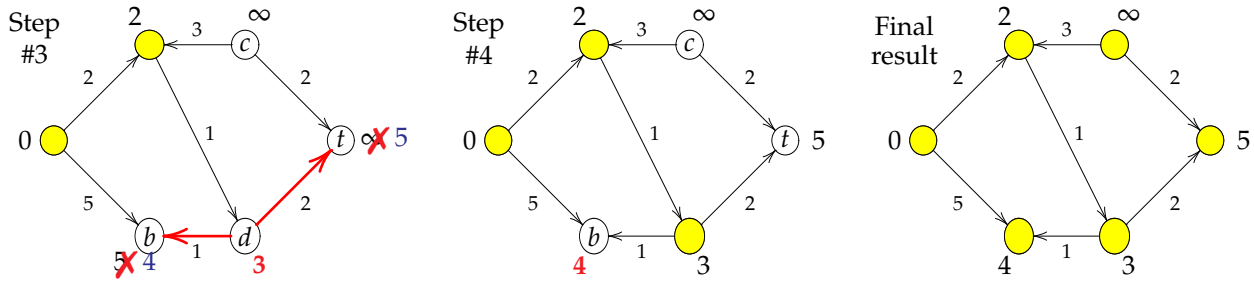
**initialize:**  $d_u = \begin{cases} 0 & \text{if } u = s \\ \infty & \text{otherwise} \end{cases}$  all vertices are initially *unprocessed*

1. Find an *unprocessed* vertex  $u$  with smallest  $d_u$
2. For each  $(u, v) \in E$ , update  $d_v = \min\{d_v, d_u + c_{uv}\}$
3. Mark  $u$  as processed; repeat until all vertices are processed.
4. Report  $d_t$  as distance from  $s$  to  $t$

Step#	$s$	$a$	$b$	$c$	$d$	$t$
1.	<u>0</u>	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$
2.	$0^*$	<u>2</u>	5	$\infty$	$\infty$	$\infty$
3.	$0^*$	$2^*$	5	$\infty$	<u>3</u>	$\infty$
			4			5
4.	$0^*$	$2^*$	<u>4</u>	$\infty$	$3^*$	5
5.	$0^*$	$2^*$	$4^*$	$\infty$	$3^*$	<u>5</u>
6.	$0^*$	$2^*$	$4^*$	<u><math>\infty</math></u>	$3^*$	$5^*$
final	$0^*$	$2^*$	$4^*$	$\infty^*$	$3^*$	$5^*$

**Example:**





Can be augmented to actually find a shortest path.

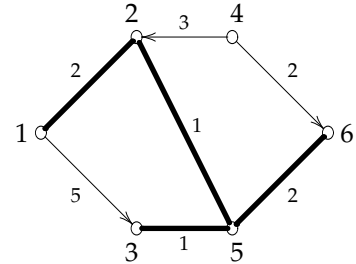
2' For each  $(u, v) \in E$ , if  $d_v > d_u + w_{uv}$ , then  $d_v = d_u + w_{uv}$  and set  $p_v = u$ .

For the above example, we have:

$p_1 = \text{undefined}, p_2 = 1, p_3 = 5, p_4 = \text{undefined}, p_5 = 2, p_6 = 5$ .

We can depict this by marking the edge from  $p_u$  to  $u$  for each  $u \in V$ . (the picture on right). Note that the edges we marked form a tree.

A shortest path from 1 to 6 is found backwards by taking 6,  $p_6, p_{p_6}, \dots$ . The path this yields is 1, 2, 5, 6, since  $5 = p_6, 2 = p_5$ , and  $1 = p_2$ .



## 10.2 Minimum Spanning Tree

A power company delivers electricity from its power plant to neighbouring cities. The cities are interconnected by power lines operated by various operators. The power company wants to rent power lines in the grid of least total cost that will allow it to send electricity from its power plant to all cities.

Given an undirected network  $G = (V, E)$  find a collection  $F \subseteq E$  of minimum weight so that  $(V, F)$  is a tree.

(we say that  $(V, F)$  is a **spanning tree** because it spans all vertices)

**Example:** The tree in Figure 1 (right) is not spanning because it does not contain the vertex 4. Adding the edge  $(2, 4)$  yields a spanning tree of weight 8, while adding the edge  $(4, 6)$  yields a spanning tree of weight 7. Note that adding the edge  $(1, 3)$  is not possible, for it creates a cycle 1, 2, 5, 3 which is not allowed in a tree.

### Kruskal's (Prim's) algorithm

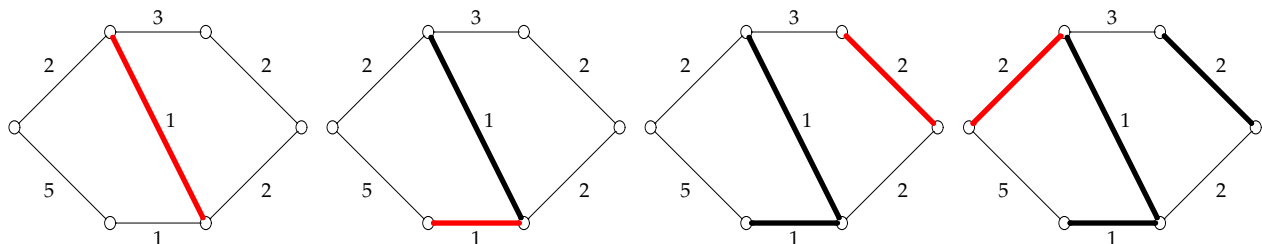
**initialize:**  $F$  to be empty; all edges are initially *unprocessed*

**Kruskal's algorithm:**

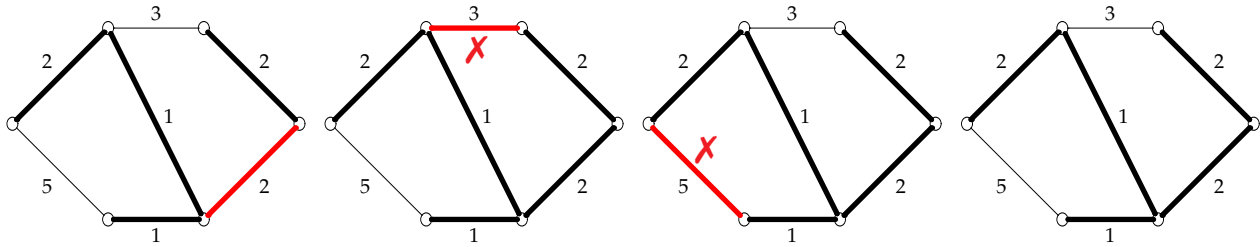
1. Find an unprocessed edge  $e$  of smallest weight  $w_e$ .
2. If  $(V, F \cup \{e\})$  is a forest, then add  $e$  to  $F$ .
3. Mark  $e$  as processed and repeat until all edges have been processed.
4. Report  $(V, F)$  as a minimum-weight spanning tree.

**Prim's algorithm:** replace 1 by 1'

- 1' Find an unprocessed edge  $e$  of smallest weight that shares an endpoint with some edge in  $F$

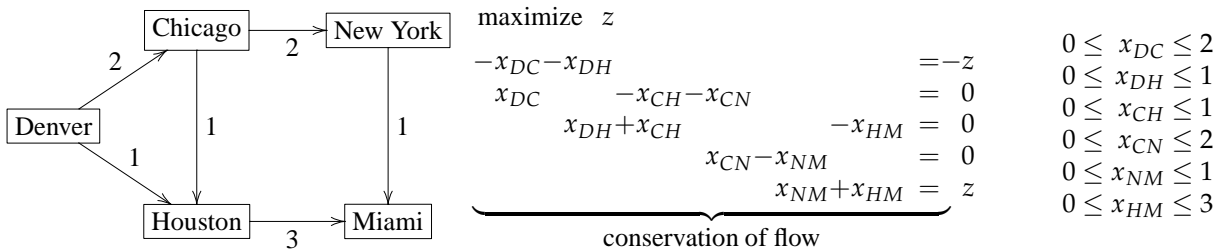






### 10.3 Maximum Flow problem

A delivery company runs a delivery network between major US cities. Selected cities are connected by routes as shown below. On each route a number of delivery trucks is dispatched daily (indicated by labels on the corresponding edges). A customer is interested in hiring the company to deliver his products daily from Denver to Miami, and needs to know how much product can be delivered on a daily basis.



In general, network  $G = (V, E)$ :

$s$  = source (Denver)

$t$  = sink (Miami)

$u_{ij}$  = capacity of an edge  $ij$  (# trucks dispatched daily between  $i$  and  $j$ )

$x_{ij}$  = flow on an edge  $ij$  (# trucks delivering the customer's products)

$$\max z \quad \underbrace{\sum_{\substack{j \in V \\ ji \in E}} x_{ji}}_{\text{flow into } i} - \underbrace{\sum_{\substack{j \in V \\ ij \in E}} x_{ij}}_{\text{flow out of } i} = \begin{cases} -z & i = s \\ z & i = t \\ 0 & \text{otherwise} \end{cases}$$

$$0 \leq x_{ij} \leq u_{ij} \quad \text{for all } ij \in E$$

#### Ford-Fulkerson algorithm

Initial feasible flow  $x_{ij} = 0$  for all  $ij \in E$ .

A sequence of nodes  $v_1, v_2, \dots, v_n$  is a *chain* if  $v_i v_{i+1} \in E$  (forward edge) or  $v_{i+1} v_i \in E$  (backward edge) for all  $i = 1, \dots, n - 1$ . If  $v_1 = s$  and  $v_n = t$ , then we call it an  $(s, t)$ -chain. Consider an  $(s, t)$ -chain  $P$ .

The *residual capacity* of a forward edge  $ij$  on  $P$  is defined as  $u_{ij} - x_{ij}$  (the remaining capacity on the edge  $ij$ ). The *residual capacity* of a backward edge  $ij$  on  $P$  is defined as  $x_{ij}$  (the used capacity of the edge  $ij$ ).

The *residual capacity* of  $P$  is the **minimum** taken over residual capacities of edges on  $P$ .

If the *residual capacity* of  $P$  is positive  $\epsilon > 0$ , then  $P$  is an **augmenting chain**. If this happens, we can increase the flow by increasing the flow on all forward edges by  $\epsilon$ , and decreasing the flow on all backward edges by  $\epsilon$ . This yields a feasible flow of larger value  $z + \epsilon$ . (Notice the similarity with the Transportation Problem and the ratio test in the Simplex Method – same thing in disguise.)

**Optimality criterion:** The flow  $x_{ij}$  is optimal if and only if there is no augmenting chain.

### Residual network

Finding an augmenting chain efficiently  $\rightarrow$  residual network  $G_x$  constructed as follows:

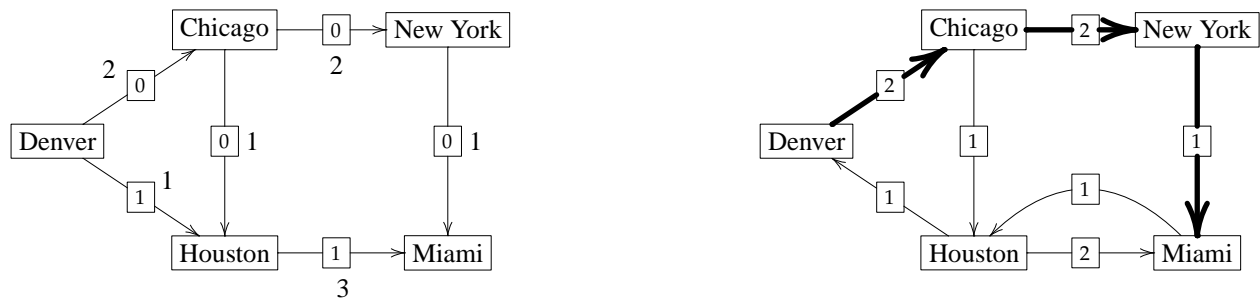
- start by making  $V$  the nodes of  $G_x$  (no edges yet)
- then for every edge  $ij \in E$ ,
  - (a) add edge  $ij$  to  $G_x$  if  $x_{ij} < u_{ij}$
  - (b) add edge  $ji$  to  $G_x$  if  $x_{ij} > 0$
 (if both happen then the residual network contains both the edge  $ij$  and  $ji$ )

Any path from  $s$  to  $t$  in the residual network is an augmenting chain.

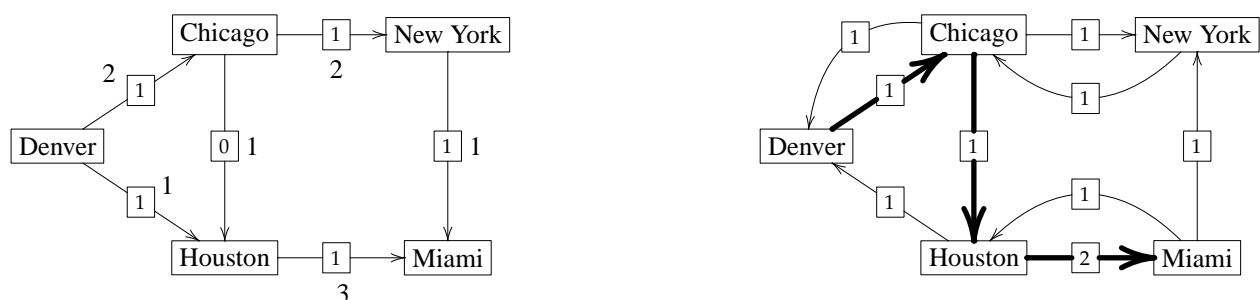
Starting feasible flow  $x_{ij} = 0$  (indicated in boxes)  $\rightarrow$  residual network (residual capacity shown on edges)



augmenting chain of residual capacity 1  $\rightarrow$  increase flow by 1



augmenting chain of residual capacity 1  $\rightarrow$  increase flow by 1



augmenting chain of residual capacity 1  $\rightarrow$  increase flow by 1



no path from Denver to Miami in the residual network  $\rightarrow$  no augmenting chain  $\rightarrow$  **optimal solution** found  
 $\rightarrow$  maximum flow has value 3

### Minimum Cut

For a subset of vertices  $A \subseteq V$ , the edges going between the nodes in  $A$  and the rest of the graph is called a **cut**. We write  $(A, \bar{A})$  to denote this cut. The edges going out of  $A$  are called **forward edges**, the edges coming into  $A$  are **backward edges**. If  $A$  contains  $s$  but not  $t$ , then it is an  $(s, t)$ -cut.

The **capacity** of a cut  $(A, \bar{A})$  is the sum of the capacities of its forward edges.

For example, let  $A = \{\text{Denver, Chicago}\}$ . Then  $(A, \bar{A})$  is an  $(s, t)$ -cut of capacity 4. Similarly, let  $A_* = \{\text{Denver, Chicago, New York}\}$ . Then  $(A_*, \bar{A}_*)$  is an  $(s, t)$ -cut of capacity 3.

**Theorem 5.** The maximum value of an  $(s, t)$ -flow is equal to the minimum capacity of an  $(s, t)$ -cut.

This is known as the Max-Flow-Min-Cut theorem – a consequence of strong duality of linear programming.

$$\begin{array}{rcl}
 \text{maximize } z & & 0 \leq x_{DC} \leq 2 \\
 -x_{DC} - x_{DH} & = -z & 0 \leq x_{DH} \leq 1 \\
 x_{DC} & & 0 \leq x_{CH} \leq 1 \\
 x_{DH} + x_{CH} & - x_{HM} = 0 & 0 \leq x_{CN} \leq 2 \\
 x_{CN} - x_{NM} & = 0 & 0 \leq x_{NM} \leq 1 \\
 x_{NM} + x_{HM} & = z & 0 \leq x_{HM} \leq 3
 \end{array}$$

Dual:

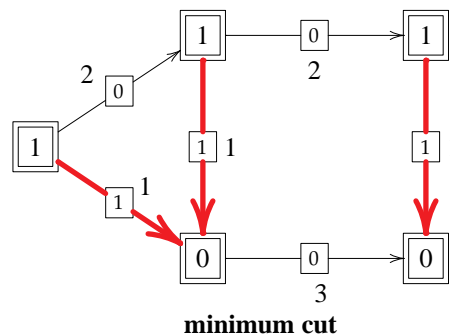
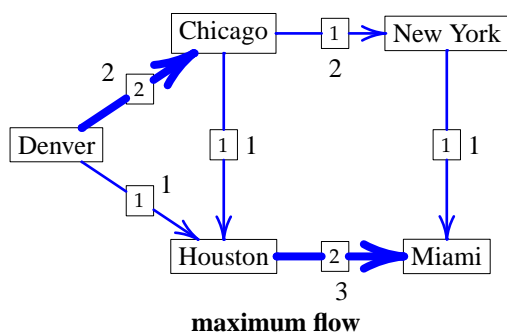
$$\text{minimize } 2v_{DC} + v_{DH} + v_{CH} + 2v_{CN} + v_{NM} + 3v_{HM}$$

$$\begin{array}{rcl}
 y_D - y_C & \leq & v_{DC} \\
 y_D - y_H & \leq & v_{DH} \\
 y_C - y_H & \leq & v_{CH} \\
 y_C - y_N & \leq & v_{CN} \\
 y_N - y_M & \leq & v_{NM} \\
 y_H - y_M & \leq & v_{HM} \\
 y_D & & -y_M \geq 1 \\
 v_{DC}, v_{DH}, v_{CH}, v_{CN}, v_{NM}, v_{HM} & \geq & 0 \\
 y_D, y_C, y_H, y_N, y_M & \text{unrestricted} &
 \end{array}$$

Optimal solution ( of value 3)

$$\begin{array}{l}
 y_D = y_C = y_N = 1 \quad \rightarrow \quad A = \{D, C, N\} \\
 y_H = y_M = 0 \quad \quad \quad \text{min-cut} \\
 v_{DH} = v_{CH} = v_{NM} = 1 \\
 v_{DC} = v_{CN} = v_{HM} = 0
 \end{array}$$

$\rightarrow$  given an optimal solution, let  $A$  be the nodes whose  $y$  value is the same as that of source  
 $\rightarrow (A, \bar{A})$  **minimum cut**

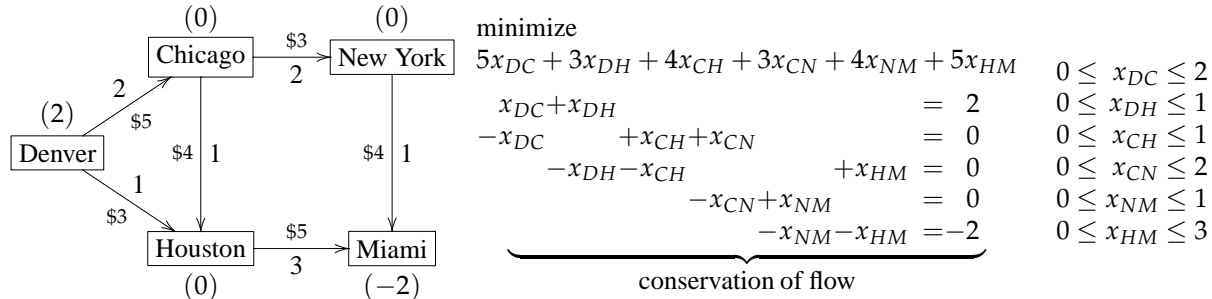


$$\begin{array}{l}
 \max z \\
 \sum_{j \in V} x_{ji} - \sum_{j \in V} x_{ij} = \begin{cases} -z & i = s \\ z & i = t \\ 0 & \text{otherwise} \end{cases} \\
 0 \leq x_{ij} \leq u_{ij} \quad \text{for all } ij \in E
 \end{array}$$

$$\begin{array}{l}
 \min \sum_{ij \in E} u_{ij} v_{ij} \\
 y_i - y_j \leq v_{ij} \quad \text{for all } ij \in E \\
 y_s - y_t \geq 1 \\
 v_{ij} \geq 0 \quad \text{for all } ij \in E \\
 y_i \text{ unrestricted} \quad \text{for all } i \in V
 \end{array}$$

### 10.4 Minimum-cost Flow problem

A delivery company runs a delivery network between major US cities. Selected cities are connected by routes as shown below. On each route a number of delivery trucks is dispatched daily (indicated by labels on the corresponding edges). Delivering along each route incurs a certain cost (indicated by the \$ figure (in thousands) on each edge). A customer hired the company to deliver two trucks worth of products from Denver to Miami. What is the least cost of delivering the products?



### Minimum-cost Network Flow problem

Network  $G = (V, E)$ :

$u_{ij}$  = capacity of an edge  $(i, j) \in E$  (# trucks dispatched daily between  $i$  and  $j$ )

$x_{ij}$  = flow on an edge  $(i, j) \in E$  (# trucks delivering the customer's products)

$c_{ij}$  = cost on an edge  $(i, j) \in E$  (cost of transportation per each truck)

$b_i$  = net supply of a vertex  $i \in V$  (amount of products produced/consumed at node  $i$ )

$$\min \sum_{(i,j) \in E} c_{ij}x_{ij}$$

$$\underbrace{\sum_{\substack{j \in V \\ ij \in E}} x_{ij}}_{\text{flow out of } i} - \underbrace{\sum_{\substack{j \in V \\ ji \in E}} x_{ji}}_{\text{flow into } i} = \underbrace{b_i}_{\text{net supply}}$$

$$0 \leq x_{ij} \leq u_{ij} \quad \text{for all } ij \in E$$

Necessary condition:  $\sum_i b_i = 0$ .

If there are no capacity constraints, the problem is called the **Transshipment problem**.

### 10.5 Network Simplex Algorithm

**Primal**

$$\min \sum_{(i,j) \in E} c_{ij}x_{ij}$$

$$\sum_{\substack{j \in V \\ (i,j) \in E}} x_{ij} - \sum_{\substack{j \in V \\ (j,i) \in E}} x_{ji} = \underbrace{b_i}_{\text{net supply}} \quad \text{for all } i \in V$$

$$x_{ij} \geq 0 \quad \text{for all } (i, j) \in E$$

**Dual**

$$\max \sum_{i \in V} b_i y_i$$

$$y_i - y_j \leq c_{ij} \quad \text{for all } (i, j) \in E$$

$$y_i \text{ unrestricted} \quad \text{for all } i \in V$$

The Network Simplex method is a specialized form of the Simplex method for solving the Minimum-cost network flow problem. That is, starting from a basic feasible solution, we seek to improve the solution by increasing values

of non-basic variables. If this is not possible, we arrive at an optimal solution. If it is possible for some variable, we increase its value as much as possible until some other variable is reduced to 0 at which point we swap the two variables in our basis and repeat. To this end, we need to discuss the following issues:

1. what is basis like for the Transshipment problem?
2. how do we calculate shadow prices and reduced costs for a given basis?
3. how do we determine optimality?
4. how do we pivot – how do we find which variable leaves the basis?
5. how do we find starting feasible basis?

### Basis

Note that in the Transshipment problem there are  $n$  equations, one for each vertex, but they are not independent (much like in the Transportation problem and the Maximum flow problem). However, any  $n - 1$  of them are independent (assuming the network is connected). So we can summarize this as follows.

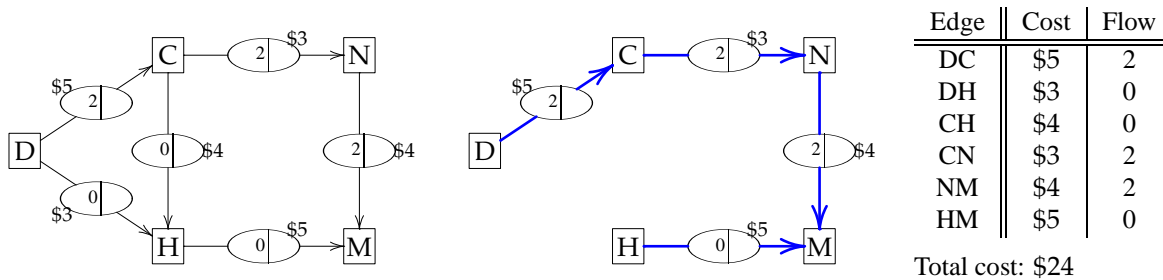
**Fact 1.** Every basis consist of  $n - 1$  basic variables.

Since variables correspond to the edges of the network, this in turn implies that

**Fact 2.** Every basis yields a **spanning tree** of the network  $G$ .

Namely, every basis has the form  $x_B = \{x_{ij} \mid (i, j) \in F\}$  where  $F \subseteq E$  is set of edges of  $G$  forming a spanning tree  $(V, F)$  of  $G$ . (We ignore here the directions of edges in  $F$ .) To see this, recall that  $F$  contains exactly  $n - 1$  elements (since every basis has  $n - 1$  variables), and it cannot contain a cycle (ignoring the directions of edges), since the columns of the basis matrix  $B$  corresponding to the edges of such a cycle would not be independent (we can combine them to produce the zero vector), which is impossible, since  $B$  is invertible.

For example, consider the delivery company example without the capacities. Recall that we seek to deliver 2 units of flow (truck-loads) from Denver to Miami. An example of such a flow (**feasible** flow) is shown below. It constitutes a **basic** (feasible) solution because at most  $n - 1$  edges carry non-zero flow (justify this fact for yourself; from this, remember that not every feasible flow is basic). The corresponding spanning tree (**basis**) is produced by simply taking all edges with non-zero flow. If there are less than  $n - 1$  edges with non-zero flow (the solution is degenerate), additional edges with 0 flow might be needed to be added to form a basis – spanning tree. The choice can be arbitrary as long as there are no cycles.



### Shadow prices

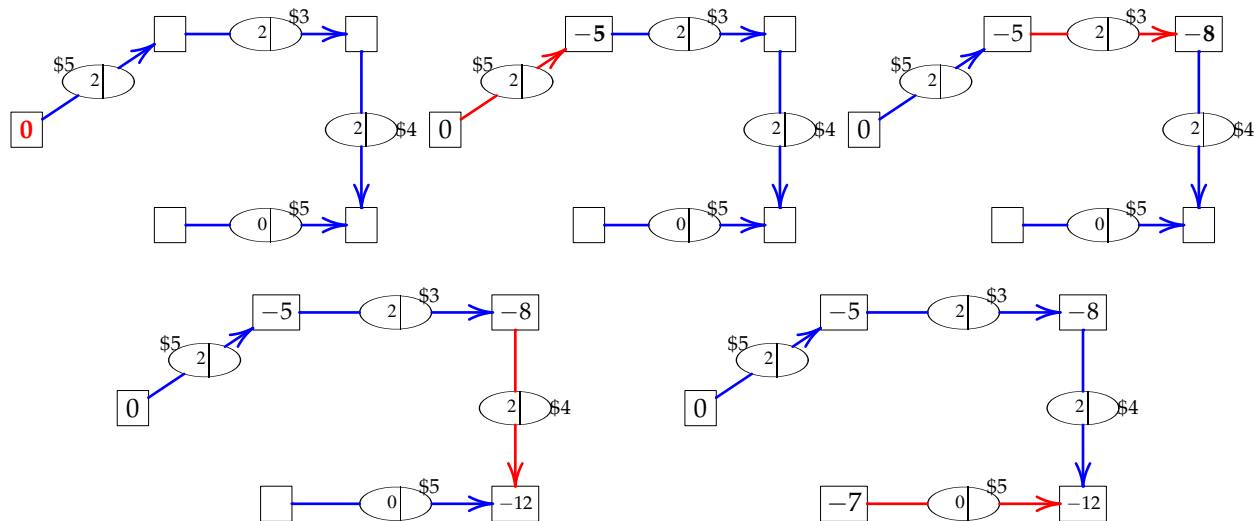
Note that shadow prices, one for each equation, correspond to nodes of  $G$ . Namely, we have a shadow price  $y_i$  for each  $i \in V$  where this shadow price corresponds to the equation for conservation of flow at the node  $i$ . To find shadow prices, we use complementarity – the dual constraints corresponding to basic variables must be tight. By inspecting the dual, we see that the dual constraints are  $y_i - y_j \leq c_{ij}$  for each  $(i, j) \in E$ . Thus for edges  $(i, j)$  forming our basis, we must satisfy this with equality,  $y_i - y_j = c_{ij}$ . This yields  $n - 1$  equations in  $n$  variables, an **underdetermined system**. To determine this system we arbitrarily set  $y_1 = 0$ . Then we can find all the other shadow prices as follows:

- for every basic  $(i, j) \in E$ :
  - if  $y_j$  has been determined but not  $y_i$ , then set  $y_i = y_j + c_{ij}$ ;
  - if  $y_i$  has been determined but not  $y_j$ , then set  $y_j = y_i - c_{ij}$ .

- repeat until all shadow prices are determined.

(This should remind you of a similar process that we used in the Transportation Simplex algorithm.)

Set  $y_D = \$0$ . Then since the edge  $DC$  is basic, we must have  $y_C = y_D - c_{DC} = y_D - \$5$ . Since  $y_D = \$0$ , we have  $y_C = -\$5$ . Likewise for all other basic edges, we have  $y_N = y_C - \$3$  and  $y_M = y_N - \$4$  and  $y_M = y_H - \$5$ . Therefore  $y_N = -\$8$ ,  $y_M = -\$12$ , and  $y_H = -\$7$ .

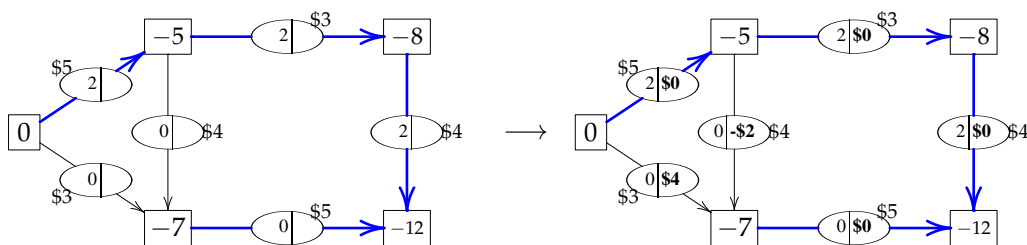


## Reduced costs

Once shadow prices are found, we can determine reduced costs of all non-basic variables. We “price out” all non-basic edges. From the dual constraints, we see (since we are minimizing) that

$$\text{the reduced cost of edge } (i, j) \in E \text{ is } y_i - y_j - c_{ij}$$

We calculate reduced costs of every **non-basic edge** using the shadow prices we calculated above. For instance, the reduced cost of the edge  $DH$  is  $y_D - y_H - c_{DH} = \$0 - (-\$7) - \$3 = \$4$ . Similarly, the reduced cost of the edge  $CH$  is  $y_C - y_H - c_{CH} = -\$5 - (-\$7) - \$4 = -\$2$ . Note that, by definition, the reduced cost of every **basic edge** is  $\$0$  (check this for yourself).



## Optimality criterion

The solution is optimal if and only if each non-basic edge  $(i, j)$  has **non-positive** reduced cost, i.e.,

$$y_i - y_j - c_{ij} \leq 0$$

If not, then there is an edge  $(i, j) \in E$  such that  $x_{ij} = 0$  and  $y_i - y_j > c_{ij}$ . The solution can be improved by entering  $x_{ij}$  into the basis and increasing it appropriately. To do this, we need to determine the leaving variable. We do this in the following section.

In our scenario above, the edge DH has positive reduced cost \$4 while the edge CH has negative reduced cost  $-\$2$ . Thus the solution is not optimal and we can get a better solution by increasing the flow on the edge DH (has positive reduced cost).

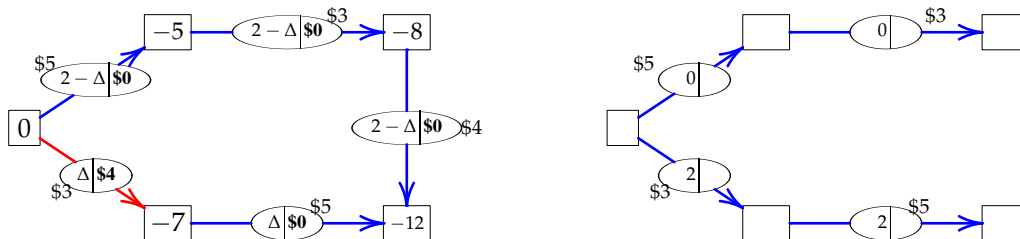
### Finding a leaving variable

We change  $x_{ij}$  by value  $\Delta \geq 0$ . This violates conservation of flow constraints and we need to fix it. We must increase/decrease some other edges to balance the decrease/increase in  $x_{ij}$  which in turn forces some other edges to increase/decrease as well and so on. To make this work, we observe that adding  $(i, j)$  to our set of basic edges (which form a spanning tree) yields a subgraph with **exactly one** loop (cycle); this follows from the defining properties of trees (being minimally acyclic – adding any one edge creates a cycle). In particular, the cycle contains the edge  $(i, j)$ . We go around this cycle and increase/decrease the flow on the edges of the cycle by  $\Delta$ . Namely, if the edge is in the same direction as  $(i, j)$ , we change the flow on this edge by  $\Delta$ ; if it is in the opposite direction, then we change it by  $-\Delta$ . Observe that for any value of  $\Delta$  this modified flow satisfies the conservation of flow equations. It remains to find the smallest value of  $\Delta$  for which the modified flow remains feasible (non-negative). Namely, we have:

- (1) each edge  $(i', j')$  in the direction opposite to  $(i, j)$  on the cycle:  $x_{i'j'}$  decreases to  $x_{i'j'} - \Delta \geq 0$
- (2) each edge  $(i', j')$  in the same direction as  $(i, j)$  on the cycle:  $x_{i'j'}$  increases to  $x_{i'j'} + \Delta \geq 0$

The most strict constraint of all the above determines the leaving variable. Namely, the variable giving the smallest upper bound on  $\Delta$  is leaving. Then we modify the flow on the edges of the cycle by adding  $\Delta$  to the flow on edges in the same direction as  $(i, j)$  and subtracting  $\Delta$  to the rest.

We decided to increase the flow to  $\Delta$  on the edge DH. Adding DH into the basis creates a loop D-H-M-N-C-D. We increase/decrease flow along this loop. Namely, the flow on the edge HM is increased by  $\Delta$ , since this edge has the same direction (along the loop) as DH. The flow on DC, CN, NM is decreased by  $\Delta$ , since the direction of these edges (with respect to the loop) is opposite to that of DH.



The largest value for  $\Delta$  (guaranteeing that the flow remains non-negative on every edge) is  $\Delta = 2$ . With that, the flow on edges DH, HM is increased to 2, while the flow on DC, CN, NM is reduced to 0. One of these three edges is a candidate for leaving the basis. We can choose any one of DC, CN, NM (since the flow on all these three edges reduces to zero simultaneously). We choose NM. Thus we get a new basis shown above (on the right).

### Economic interpretation

The value  $(-y_i)$  can be interpreted as the price of the commodity (we are transporting) in node  $i$ . For instance, in the above we have  $y_D = \$0$ ,  $y_C = -\$5$  and  $y_H = -\$7$ . Thus the commodity costs \$0 in Denver, \$5 in Chicago, and \$7 in Houston.

It pays to transport the commodity (along an unused non-basic edge) if the **cost of transportation** is less than the **difference in price**. Namely if  $y_i - y_j$  is more than  $c_{ij}$ .

For instance, the price difference between Denver and Houston is \$7 dollars and it costs \$3 to pay transportation. Thus transportation is cheaper than the price difference and consequently it pays to send along the edge DH. Comparing Chicago and Houston, we see that the price difference is \$2 and transportation costs \$4. Therefore it does not pay to send along the edge CH.

## Unbounded LP

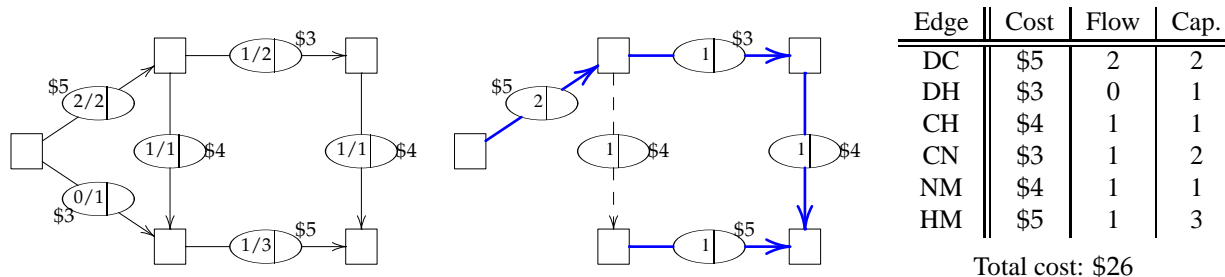
Note that it is possible that the above test finds no bound on  $\Delta$ . In this case  $\Delta$  can be made arbitrarily large which in turn makes the objective arbitrarily negative. In other words, the problem is unbounded. This happens precisely when  $x_{ij} = 0$  and each edge on the loop is in the direction of  $(i, j)$ .

**Fact 3.** The problem is **unbounded** if and only if there exists a **negative cost cycle**.

## 10.6 Network Simplex Algorithm with capacities

To solve the more general problem with capacities, we need to slightly modify our algorithm. In particular, the algorithm will be a variant of the Upper-Bounded Simplex method.

In addition to basic edges, we will need to keep track of **saturated edge** (edges that carry flow equal to their capacity). They will not necessarily be part of the spanning tree. We will indicate them using **dashed** lines. In addition to increasing flow on a non-basic edge (with zero flow), we will also consider **decreasing** flow on a saturated edge. This will require modifying our optimality criterion and pivoting steps.



### Optimality criterion

The solution is optimal if and only if

- each non-basic edge with **zero flow** has **non-positive** reduced costs
- each **saturated edge** has **non-negative** reduced costs.

If not, then there is either an edge  $(i, j) \in E$  such that  $x_{ij} = 0$  and  $y_i - y_j > c_{ij}$ , or there is an edge  $(i, j) \in E$  such that  $x_{ij} = u_{ij}$  and  $y_i - y_j < c_{ij}$ . The solution can be improved by entering  $x_{ij}$  into the basis and increasing/decreasing it appropriately. To do this, we need to determine the leaving variable.

### Finding a leaving variable

We change  $x_{ij}$  by value  $\Delta$ . Now  $\Delta$  can be both positive and negative. We determine the value of  $\Delta$  by considering the following inequalities.

- (1) edge  $(i', j')$  in the direction opposite to  $(i, j)$  on the cycle:  $0 \leq x_{i'j'} - \Delta \leq u_{i'j'}$
- (2) edge  $(i, j)$ :  $0 \leq x_{ij} + \Delta \leq u_{ij}$
- (3) edge  $(i', j')$  in the same direction as  $(i, j)$  on the cycle:  $0 \leq x_{i'j'} + \Delta \leq u_{i'j'}$

The most strict constraint of all the above determines the leaving variable. Namely, if the chosen edge  $(i, j)$  carried no flow, i.e.  $x_{ij} = 0$ , then the variable giving the smallest upper bound on  $\Delta$  is leaving (unless it is  $x_{ij}$  who gives this bound). If the chosen edge  $(i, j)$  was saturated, i.e.,  $x_{ij} = u_{ij}$ , then the variable giving the largest lower bound leaves (unless it is  $x_{ij}$  who gives this bound). Then we modify the flow on the edges of the cycle by adding  $\Delta$  to the flow on edges in the same direction as  $(i, j)$  and subtracting  $\Delta$  to the rest. (Note that  $\Delta$  can be negative.) If an edge becomes saturated by this, we indicate it by a dashed line in the diagrams. This is to help keep track of the current objective value.



### Finding a starting feasible basis

Unlike in the Transportation problem, here we cannot use a greedy heuristic to find a starting feasible solution, since we have to, in addition, satisfy the capacity constraints (without the capacity constraints, the minimum-cost method will still work – picking edges in turn and exhausting supply on one end or saturating demand on the other).

For small problems, we can get away with trial and error. However, in general this may not be very efficient. In fact, there may not exist any feasible solution to the problem. We need a more robust method. This is provided to us in the form of the two-phase method (just like with the regular Simplex algorithm).

In Phase I, we solve an auxiliary problem whose solution will give us a starting feasible solution to our problem; then we go on from there in Phase II optimizing this solution as described earlier.

The problem in Phase I is obtained as follows. We add a new **slack** node  $s$ . For every other node  $i \in V$ , we

- add the edge  $(s, i)$  of capacity  $-b_i$  if  $b_i < 0$ , or
- add the edge  $(i, s)$  of capacity  $b_i$  if  $b_i \geq 0$ .

The edges we added in this process we call **artificial** while the original edges are called **real** edges.

Now we assign new costs: each real edge will cost \$0, while each artificial edge will cost \$1. We call this the Phase I problem. Notice that this problem has a feasible basis, namely the basis formed by taking all artificial edges and saturating them; assigning them flow equal to their capacity.

**Fact 4.** The original problem has a feasible basis if and only if the Phase I problem has optimum value 0.

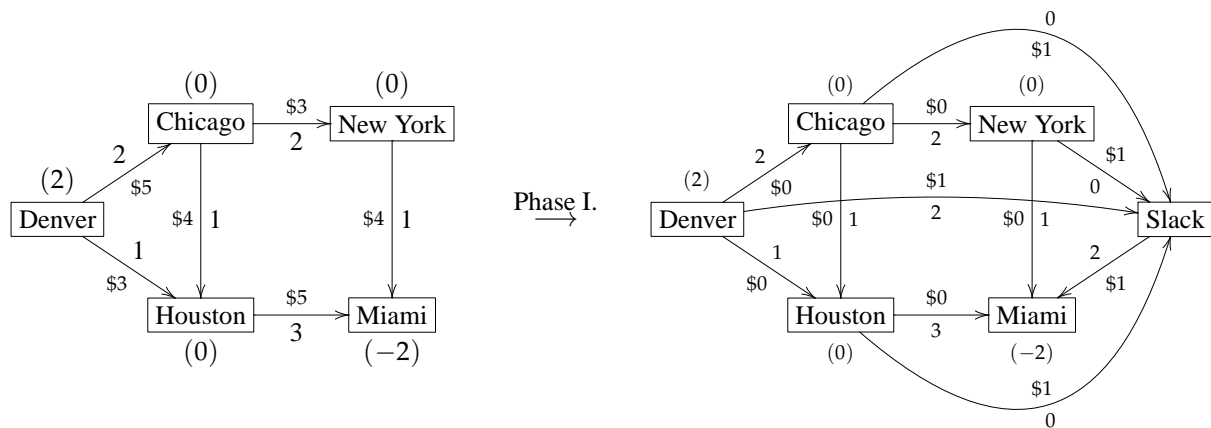
If we discover that the Phase I problem does not have optimum value 0, then this yields a set  $S$  such that

$$\sum_{i \in S} b_i > \sum_{\substack{i \in S, j \notin S \\ (i,j) \in E}} u_{ij}$$

Namely, the set  $S$  consists of those nodes whose shadow price is less than or equal to the shadow price of the slack node. (If we set the shadow price of the slack node to 0, then the set  $S$  simply consists of all nodes with non-positive shadow price.) This certifies that no feasible solution can exist.

## 10.7 Complete example

Phase I: Slack node, artificial edges to all nodes; capacities and directions of artificial edges are dictated by the net supply. If the net supply  $b$  of a node is positive (e.g. Denver), then the edge goes from this node to the slack node and capacity of this edge is  $b$ ; if the net supply  $b$  is non-positive (e.g. Miami), then the edge goes from the slack node to this node and capacity is  $-b$ . (To remember this just note that capacity musn't be negative.) All artificial edges have cost \$1 and all other (real) edges have cost \$0.

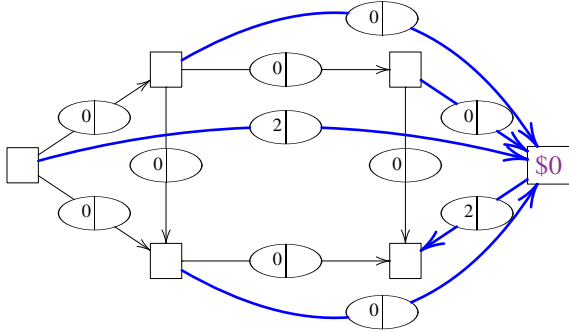


Starting basic feasible solution for Phase I: assign artificial edges flow equal to their capacity.

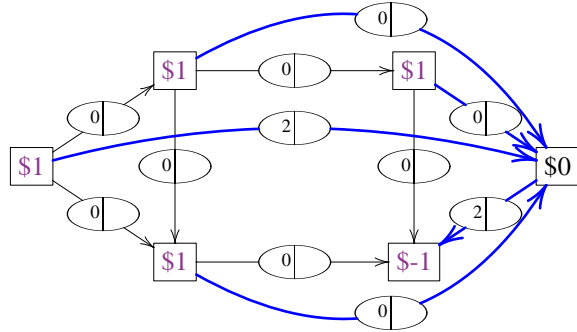
Starting basis: **all artificial edges**.

Basis shown in blue (labels on edges denote the flow and reduced costs). Everything else will not change (costs, capacities). See table below.

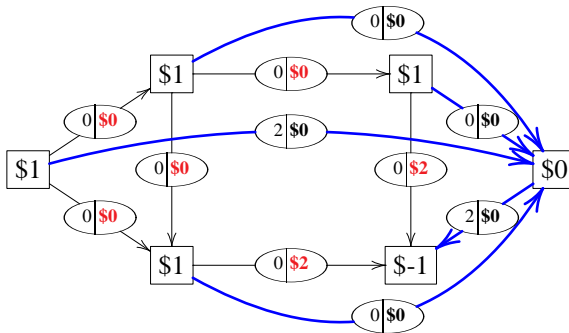
Calculate shadow prices: set  $y_S = \$0$  for slack.



Knowing shadow price of Slack, we calculate shadow prices of nodes connected to Slack by basic edges; we use the fact that shadow prices satisfy  $y_i - y_j = c_{ij}$  for every basic edge  $(i, j)$ ; if  $j = S$ , then the value  $y_j$  is known and so we plug it in to get  $y_i$ .



Calculate reduced costs  $y_i - y_j - c_{ij}$  for all edges

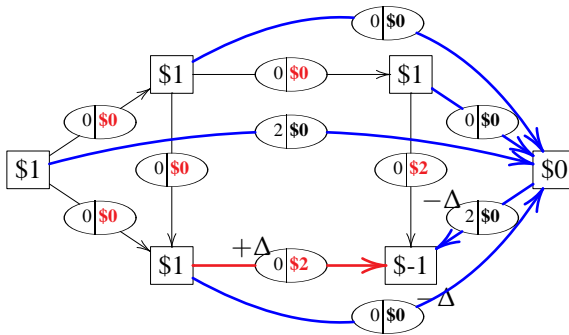


Edge	DC	DH	CH	CN	NM	HM
Cost	\$0	\$0	\$0	\$0	\$0	\$0
Flow	0	0	0	0	0	0
Capacity	2	1	1	2	1	3
Reduced cost	\$0	\$0	\$0	\$0	\$2	\$2

Edge	DS	CS	HS	NS	SM
Cost	\$1	\$1	\$1	\$1	\$1
Flow	2	0	0	0	2
Capacity	2	0	0	0	2
Reduced cost	\$0	\$0	\$0	\$0	\$0

Edge  $(H, M)$  has zero flow and positive reduced cost  $\rightarrow$  adding it into basis can make the objective smaller.

Find the loop and maximum  $\Delta$ .

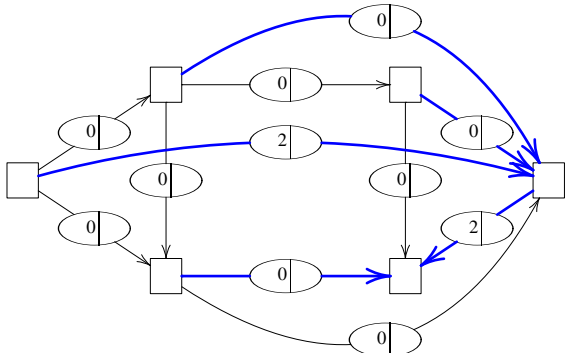


Three edges are modified:

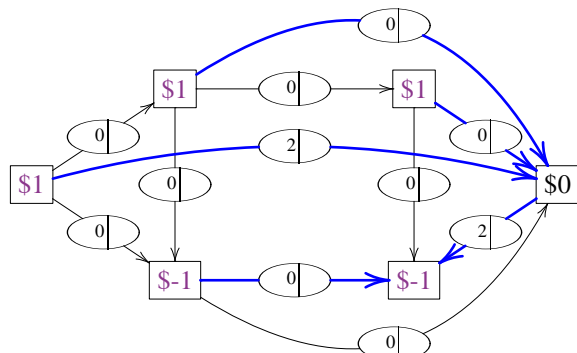
- flow on  $(H, M)$  is increased from 0 to  $\Delta$ ,
- flow on  $(S, M)$  is decreased from 2 to  $2 - \Delta$
- flow on  $(H, S)$  is decreased from 0 to  $-\Delta$  (both edges have **opposite** direction than  $(H, M)$  on the loop)

Maximum  $\Delta$  is 0 because the artificial edge  $(H, S)$  has no flow on it and thus the flow on this edge cannot decrease. The edge  $(H, S)$  leaves the basis while  $(H, M)$  enters. Flow does not change.

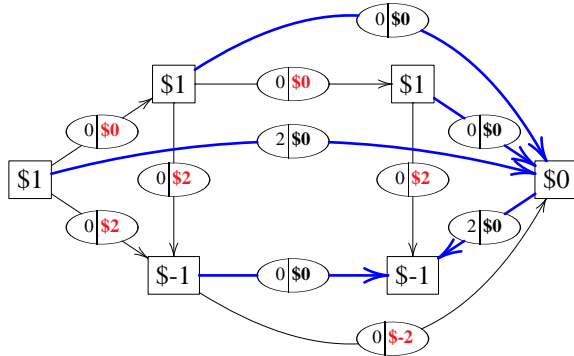
New basis:



Calculate shadow prices by setting  $y_S = \$0$ :

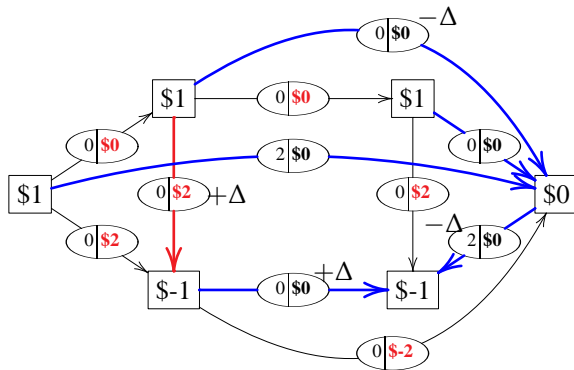


Calculate reduced costs  $y_i - y_j - c_{ij}$ :



Edge	DC	DH	CH	CN	NM	HM
Cost	\$0	\$0	\$0	\$0	\$0	\$0
Flow	0	0	0	0	0	0
Capacity	2	1	1	2	1	3
Reduced cost	\$0	\$2	\$2	\$0	\$2	\$0

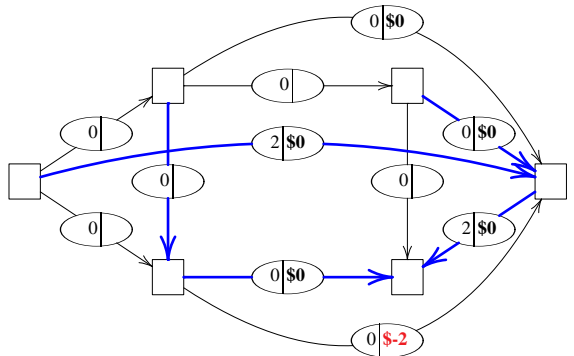
Edge	DS	CS	HS	NS	SM
Cost	\$1	\$1	\$1	\$1	\$1
Flow	2	0	0	0	2
Capacity	2	0	0	0	2
Reduced cost	\$0	\$0	\$-2	\$0	\$0



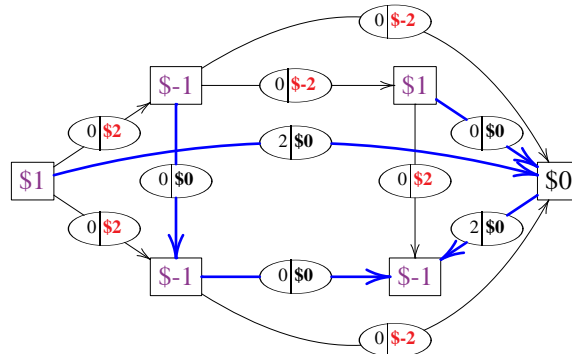
The edge  $(C, H)$  has zero flow and positive reduced cost. We increase the flow on  $(C, H)$  to  $\Delta$ .

Flow on the edge  $(H, M)$  is increased to  $\Delta$  (since  $(H, M)$  has same direction as  $(C, H)$  on the loop), while on  $(M, S)$  is decreased to  $2 - \Delta$  and on  $(C, S)$  is decreased to  $-\Delta$  (both have opposite directions). Thus  $\Delta = 0$  and  $(C, S)$  leaves.

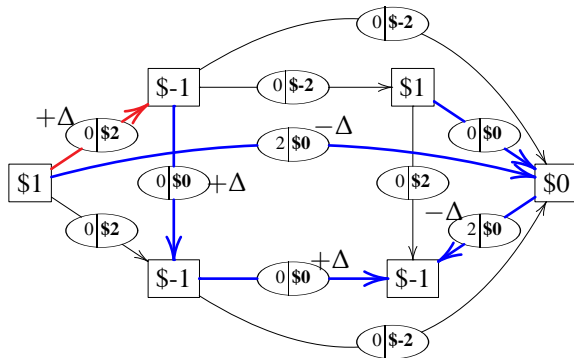
New basis:



Calculate shadow prices by setting  $y_S = \$0$ ; then determine reduced costs:



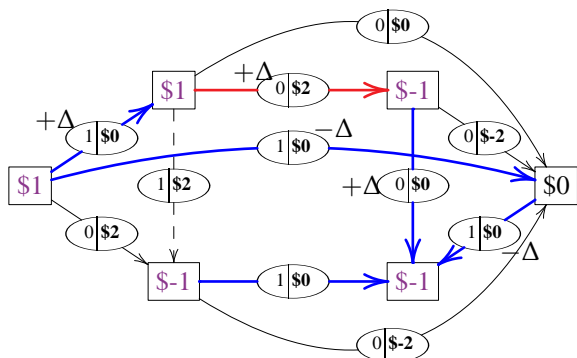
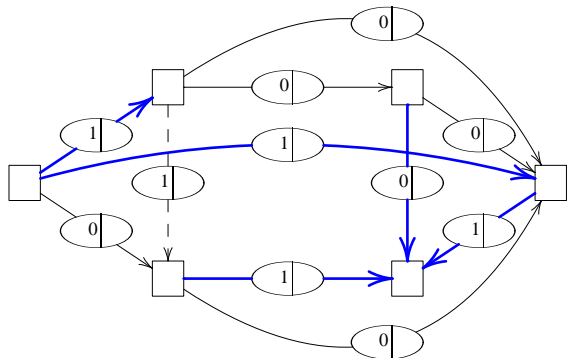
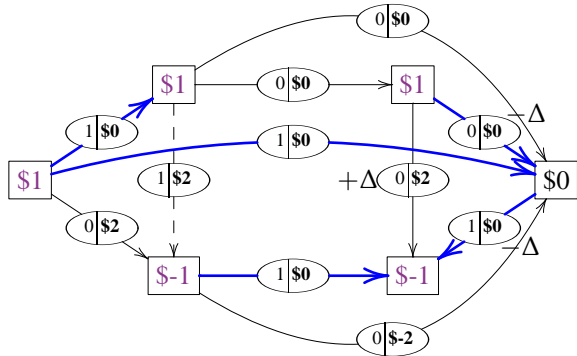
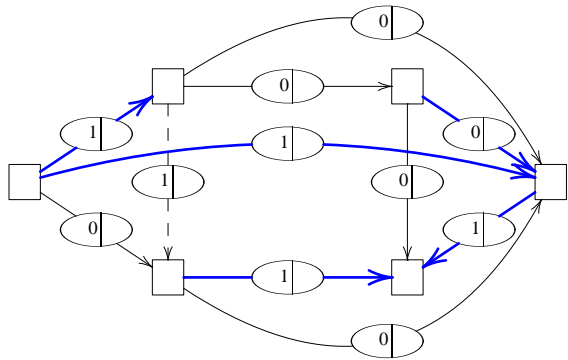
Edge  $(D, C)$  enters the basis; flow increases to  $\Delta$ :



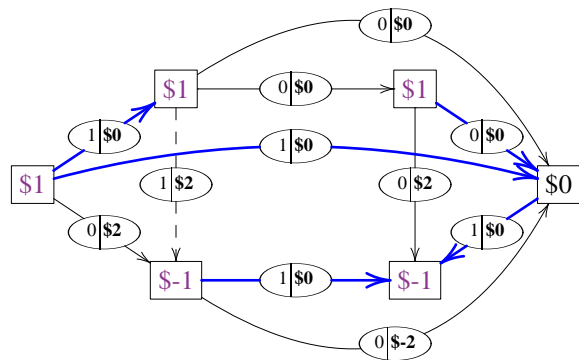
Flow increases to  $\Delta$  on edges  $(C, H)$ ,  $(H, M)$  and decreases to  $2 - \Delta$  on edges  $(D, S)$  and  $(S, M)$ . The capacity of  $(D, C)$  is 2, the capacity of  $(C, H)$  is 1, and of  $(H, M)$  is 3. Thus  $\Delta = 1$  and  $(C, H)$  leaves the basis. This time the flow finally changes.

**Important Note:** the edge  $(C, H)$  is not in the basis. Nonetheless it carries flow. The flow on  $(C, H)$  is equal to its capacity – we say it is **saturated**. We mark this by making the edge  $(C, H)$  dashed.

New basis:



Calculate shadow prices and reduced costs:

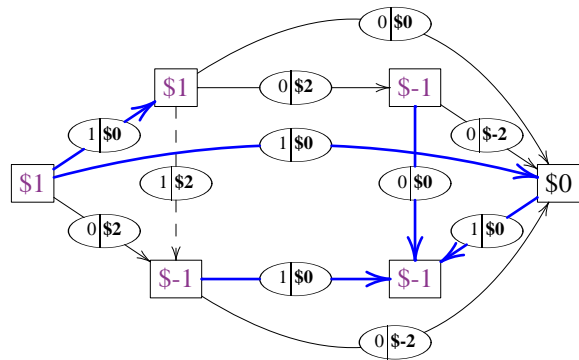


$(N, M)$  has zero flow and positive reduced cost  
 $\rightarrow$  increase flow to  $\Delta$

(note that  $(C, H)$  has also positive reduced cost; however, it is saturated – saturated edges are entered if they have **negative** reduced cost)

$\Delta = 0$  and the edge  $(N, S)$  leaves the basis

Calculate shadow prices and reduced costs:

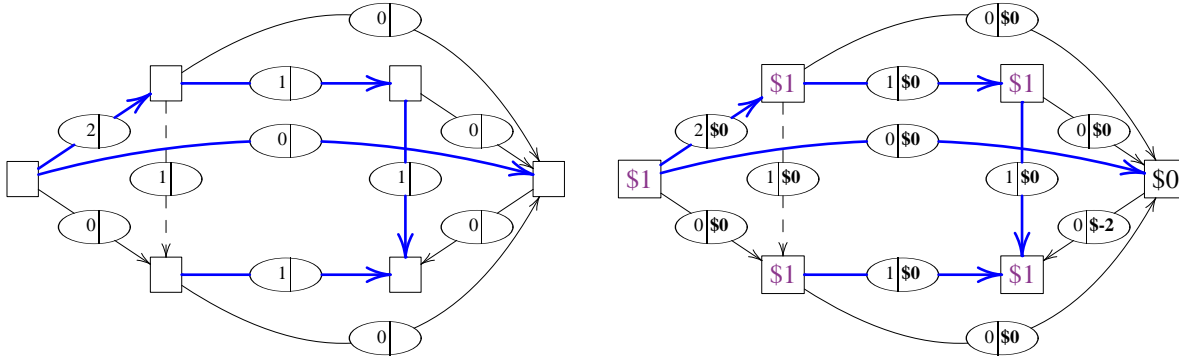


$(C, N)$  has no flow and  $> 0$  reduced cost  $\rightarrow$  enters

Flow decreases to  $1 - \Delta$  on edges  $(D, S)$  and  $(S, M)$ .

Flow increases to  $1 + \Delta$  on edge  $(D, C)$  of capacity 2 and increases to  $\Delta$  on edges  $(C, N)$  and  $(N, M)$  of capacities 2 and 1, respectively.

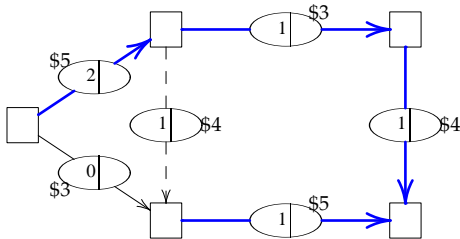
$\Delta = 1$  and we have a tie for the leaving variable. We arbitrarily choose  $(S, M)$  to leave. Note that flow changes this time (since  $\Delta > 0$ ).



Optimal solution (of Phase I) reached:

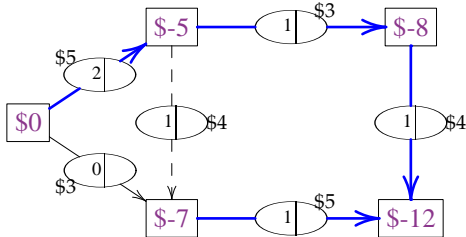
- each non-basic variable with zero flow has non-positive reduced cost.
- each saturated non-basic variable has non-negative reduced cost.

We convert this to a feasible solution to the original problem by removing the Slack node and artificial edges. Then we start Phase II. Note that from now on we use the original edge costs (unlike in Phase I where we had cost \$0 for real edges and \$1 for artificial edges).

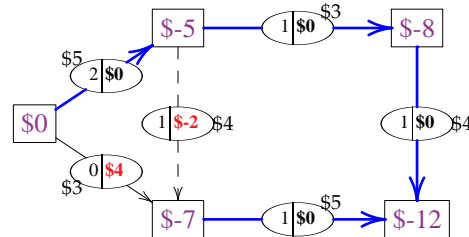


Edge	DC	DH	CH	CN	NM	HM
Cost	\$5	\$3	\$4	\$3	\$4	\$5
Flow	2	0	1	1	1	1
Capacity	2	1	1	2	1	3

Calculate shadow prices by setting  $y_D = \$0$ :



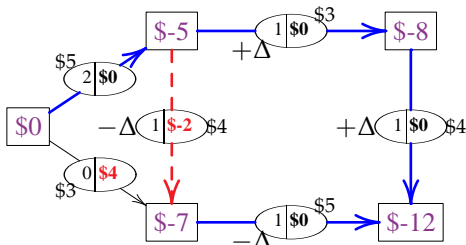
Calculate reduced costs  $y_i - y_j - c_{ij}$ :



Now we have two options:

- the edge  $(D, H)$  has **no flow** and **positive** reduced cost  $\rightarrow$  we can increase the flow on  $(D, H)$
- the edge  $(C, H)$  is **saturated** and has **negative** reduced cost  $\rightarrow$  we can decrease the flow on  $(C, H)$

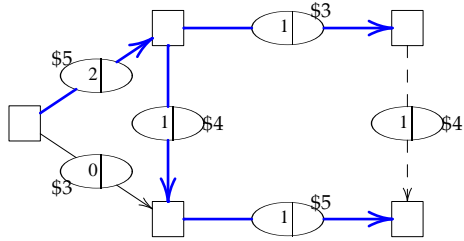
We choose (arbitrarily) do decrease the flow on  $(C, H)$  by  $\Delta$ .



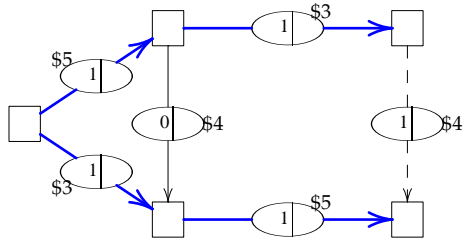
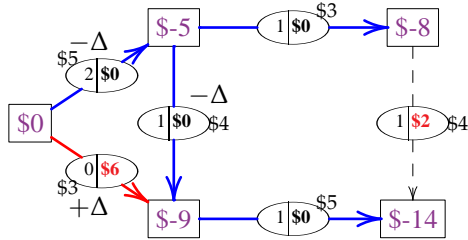
Flow on  $(C, H)$  and  $(H, M)$  changes to  $1 - \Delta$ .  
 Flow on  $(C, N)$  and  $(N, M)$  changes to  $1 + \Delta$ .  
 The capacity of  $(C, N)$  is 2, and the capacity of  $(N, M)$  is 1.

This implies that largest  $\Delta$  is  $\Delta = 0$ , since we cannot increase the flow on  $(N, M)$  as it is already saturated

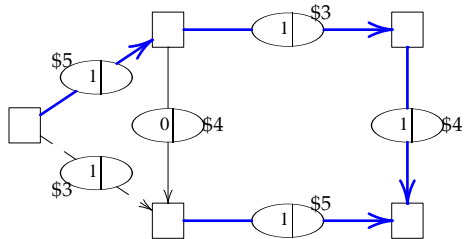
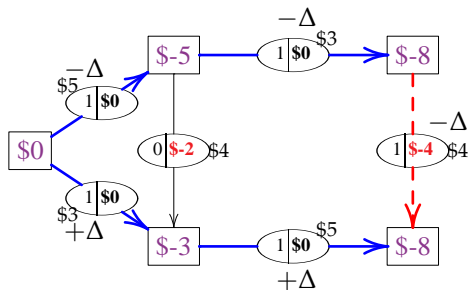
$\rightarrow (N, M)$  leaves the basis (but it stays saturated)



$(D, H)$  has no flow, positive reduce cost, enters



$(N, M)$  is saturated and has negative reduced cost  
 → decrease flow on  $(N, M)$

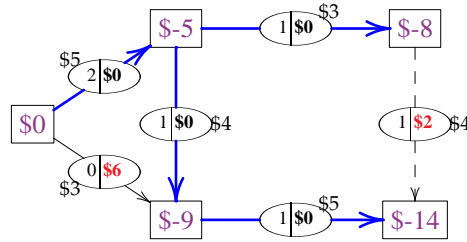


**Optimal solution found**

- all edges with **no flow** have **non-positive** reduced cost (e.g.  $(C, H)$  has no flow and reduced cost  $-\$2$ )
- all **saturated** edges have **non-negative** reduced cost (e.g.  $(D, H)$  is saturated with reduced cost  $\$4$ )

Optimal flow consists of one unit across all edges but  $(C, H)$ . The cost is  $\$5+\$3+\$3+\$5+\$4=\$20$ .

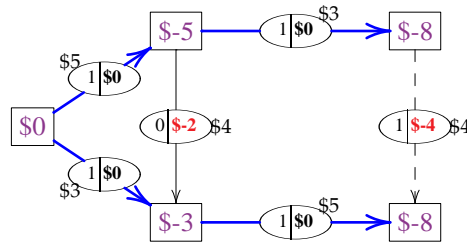
Calculate shadow prices and reduced costs:



Flow on  $(D, C)$  decreases to  $2 - \Delta$ . Flow on  $(C, H)$  decreases to  $1 - \Delta$ . Flow on  $(D, H)$  of capacity 1 increases to  $\Delta$ .

→  $\Delta = 1$  and  $(C, H)$  leaves

Calculate shadow prices and reduced costs:

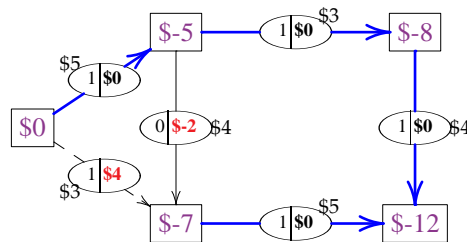


Flow on  $(D, C)$ ,  $(C, N)$ , and  $(N, M)$  decreases to  $1 - \Delta$ .  
 Flow on  $(D, H)$  and  $(H, M)$  increases to  $1 + \Delta$ .  
 The capacity of  $(D, H)$  is 1 and the capacity of  $(H, M)$  is 3.

$\Delta = 0$  since  $(D, H)$  is already saturated

→  $(D, H)$  leaves (but stays saturated)

Calculate shadow prices and reduced costs:



## 10.8 Summary

Network  $G = (V, E)$  has **nodes**  $V$  and **edges**  $E$ .

- Each edge  $(i, j) \in E$  has a **capacity**  $u_{ij}$  and **cost**  $c_{ij}$ .
- Each vertex  $i \in V$  provides **net supply**  $b_i$ .

For a set  $S \subseteq V$ , write  $\bar{S}$  for  $V \setminus S$  and write  $E(S, \bar{S})$  for the set of edges  $(i, j) \in E$  with  $i \in S$  and  $j \in \bar{S}$ . The pair  $(S, \bar{S})$  is called a **cut**. (Where applicable) there are two distinguished nodes:  $s = \text{source}$  and  $t = \text{sink}$ .

### Minimum spanning tree

**Primal**

$$\begin{aligned} \min \quad & \sum_{(i,j) \in E} c_{ij} x_{ij} \\ & \underbrace{\sum_{(i,j) \in E(S, \bar{S})} x_{ij}}_{\text{edges from } S \text{ to } \bar{S}} > 0 && \text{for all } S \subseteq V \\ & && \text{where } \emptyset \neq S \neq V \\ & x_{ij} \geq 0 && \text{for all } (i, j) \in E \end{aligned}$$

**Obstruction (to feasibility):**

$$\text{set } S \subseteq V \text{ with } \emptyset \neq S \neq V \text{ such that } E(S, \bar{S}) = \emptyset$$

### Shortest path problem

**Primal**

$$\begin{aligned} \min \quad & \sum_{(i,j) \in E} c_{ij} x_{ij} \\ & \underbrace{\sum_{\substack{j \in V \\ (i,j) \in E}} x_{ij}}_{\text{flow out of } i} - \underbrace{\sum_{\substack{j \in V \\ (j,i) \in E}} x_{ji}}_{\text{flow into } i} = \begin{cases} 1 & i = s \\ -1 & i = t \\ 0 & \text{else} \end{cases} && \text{for all } i \in V \\ & x_{ij} \geq 0 && \text{for all } (i, j) \in E \end{aligned}$$

**Dual**

$$\begin{aligned} \max \quad & y_s - y_t \\ & y_i - y_j \leq c_{ij} && \text{for all } (i, j) \in E \\ & y_i \text{ unrestricted} && \text{for all } i \in V \end{aligned}$$

**Obstruction (to feasibility):** set  $S \subseteq V$  with  $s \in S$  and  $t \in \bar{S}$  such that  $E(S, \bar{S}) = \emptyset$

### Maximum-flow problem

**Primal**

$$\begin{aligned} \max \quad & z \\ & \sum_{\substack{j \in V \\ (i,j) \in E}} x_{ij} - \sum_{\substack{j \in V \\ (j,i) \in E}} x_{ji} = \begin{cases} z & i = s \\ -z & i = t \\ 0 & \text{else} \end{cases} && \text{for all } i \in V \\ & 0 \leq x_{ij} \leq u_{ij} && \text{for all } (i, j) \in E \\ & z \text{ unrestricted} \end{aligned}$$

**Dual**

$$\begin{aligned} \min \quad & \sum_{(i,j) \in E} u_{ij} v_{ij} \\ & y_i - y_j + v_{ij} \geq 0 && \text{for all } (i, j) \in E \\ & y_t - y_s = 1 \\ & v_{ij} \geq 0 && \text{for all } (i, j) \in E \\ & y_i \text{ unrestricted} && \text{for all } i \in V \end{aligned}$$

**Obstruction (to feasibility):** set  $S \subseteq V$  with  $s \in S$  and  $t \in \bar{S}$  such that  $z > \underbrace{\sum_{(i,j) \in E(S, \bar{S})} u_{ij}}_{\text{capacity of the cut } (S, \bar{S})}$

(no flow bigger than the capacity of a cut)

capacity of the cut  $(S, \bar{S})$

**Minimum-cost  $(s, t)$ -flow problem****Primal**

$$\min \sum_{(i,j) \in E} c_{ij} x_{ij}$$

$$\sum_{\substack{j \in V \\ (i,j) \in E}} x_{ij} - \sum_{\substack{j \in V \\ (j,i) \in E}} x_{ji} = \begin{cases} f & i = s \\ -f & i = t \\ 0 & \text{else} \end{cases} \quad \text{for all } i \in V$$

$$0 \leq x_{ij} \leq u_{ij} \quad \text{for all } (i, j) \in E$$

**Dual**

$$\max f y_s - f y_t - \sum_{(i,j) \in E} u_{ij} v_{ij}$$

$$y_i - y_j - v_{ij} \leq c_{ij} \quad \text{for all } (i, j) \in E$$

$$v_{ij} \geq 0 \quad \text{for all } (i, j) \in E$$

$$y_i \text{ unrestricted} \quad \text{for all } i \in V$$

**Obstruction (to feasibility):** set  $S \subseteq V$  with  $s \in S$  and  $t \in \bar{S}$  such that  $f > \sum_{(i,j) \in E(S, \bar{S})} u_{ij}$

**Transshipment problem****Primal**

$$\min \sum_{(i,j) \in E} c_{ij} x_{ij}$$

$$\sum_{\substack{j \in V \\ (i,j) \in E}} x_{ij} - \sum_{\substack{j \in V \\ (j,i) \in E}} x_{ji} = \underbrace{b_i}_{\text{net supply}} \quad \text{for all } i \in V$$

$$x_{ij} \geq 0 \quad \text{for all } (i, j) \in E$$

**Dual**

$$\max \sum_{i \in V} b_i y_i$$

$$y_i - y_j \leq c_{ij} \quad \text{for all } (i, j) \in E$$

$$y_i \text{ unrestricted} \quad \text{for all } i \in V$$

**Obstruction (to feasibility):** set  $S \subseteq V$  such that  $\sum_{i \in S} b_i > 0$  and  $E(S, \bar{S}) = \emptyset$

**Minimum-cost network flow problem****Primal**

$$\min \sum_{(i,j) \in E} c_{ij} x_{ij}$$

$$\sum_{\substack{j \in V \\ (i,j) \in E}} x_{ij} - \sum_{\substack{j \in V \\ (j,i) \in E}} x_{ji} = b_i \quad \text{for all } i \in V$$

$$0 \leq x_{ij} \leq u_{ij} \quad \text{for all } (i, j) \in E$$

**Dual**

$$\max \sum_{i \in V} b_i y_i - \sum_{(i,j) \in E} u_{ij} v_{ij}$$

$$y_i - y_j - v_{ij} \leq c_{ij} \quad \text{for all } (i, j) \in E$$

$$v_{ij} \geq 0 \quad \text{for all } (i, j) \in E$$

$$y_i \text{ unrestricted} \quad \text{for all } i \in V$$

**Obstruction (to feasibility):** set  $S \subseteq V$  such that  $\sum_{i \in S} b_i > \sum_{(i,j) \in E(S, \bar{S})} u_{ij}$



## Game Theory

During the 8PM to 9PM time slot, two TV networks compete for an audience of 100 million viewers. The networks announce their schedule ahead of time and do not know of each other's decision until the show time. Based on that a certain number of people will tune to Network 1 while the rest will watch Network 2. The market research revealed the following expected number of viewers of Network 1.

Network 1	Network 2		
	Western	Soap Opera	Comedy
Western	35	15	60
Soap Opera	45	58	50
Comedy	38	14	70

For instance, if Network 1 shows a Western while Network 2 shows a Comedy, then 60 million viewers will watch Network 1, and  $100 - 60 = 40$  million watch Network 2.

**Question:** What strategy should the two networks use to maximize their viewership?

Terminology:

- Network 1 is a **row player**.
- Network 2 is a **column player**.
- The above matrix is called a **payoff matrix**.
- This is a **constant-sum** game (the outcome for both players always sums up to a constant 100 million).

How to solve this game? Let us look at the structure of outcomes. For instance, if Network 1 chooses to show Western, then it can get as many as 60 million viewers (if Network 2 chooses to show a Comedy) but also as little as 15 million (if Network 2 shows a Soap Opera). Thus this choice cannot guarantee more than 15 million viewers for Network 1 (in the worst case). If the network instead chooses to show a Comedy, the situation is even worse, since then we can guarantee only 14 million viewers (the minimum in the 3rd row). The best therefore for Network 1 is to choose to show a Soap Opera in which case 45 million or more viewers will tune to Network 1 regardless of what Network 2 does. Note that in this strategy Network 1 (being the row player) simply calculates the **row minimum** of each row and then chooses the row with **largest** row minimum.

By the same argument, Network 2 (being the column player) can maximize its viewership by calculating each **column maximum** and choosing column with the smallest column maximum.

It is easy to see that the two outcomes will satisfy the following inequality

$$\max_{\text{all rows}} (\text{row minimum}) \leq \min_{\text{all columns}} (\text{column maximum})$$

In this example, Network 1 chooses Soap Opera and Network 2 chooses Western whereby 45 million viewers will watch Network 1 and 55 million will watch Network 2. Note that this choice is simultaneously best for both Network

1 and Network 2 (we have equality  $\max(\text{row minimum}) = \min(\text{col maximum})$ ). This is called a **saddle point** and the common value of both sides of the equation is called the **value** of the game.

- An **equilibrium point** of the game: choice of strategies for both players such that neither player can improve their outcome by changing his strategy.
- A **saddle point** of a game is an example of an equilibrium.

## 11.1 Pure and Mixed strategies

Note that in the above example each player's strategy was deterministic; they each examined possible outcomes and made a specific single choice to follow. This is called a **pure strategy**.

Unfortunately, there are games with no saddle points and following a pure strategy may not always give the players the best outcome. (Consider for instance Poker – see next section.) We need to generalize our definition of strategy. Instead of choosing one fixed move, the player considers all moves and chooses randomly according to a distribution. The outcome of the game will depend on the chosen distribution.

- a **mixed (randomized) strategy**  $(x_1, x_2, \dots, x_n)$  is a probability distribution over  $n$  possible moves of the player (i.e. satisfies  $x_1 + x_2 + \dots + x_n = 1$ )
- **optimal strategy** is a strategy maximizing the **expected** gain for the player

Note that since we are now using chance we can sometimes gain more, sometimes less; thus we will settle for maximizing the average outcome (if, say, we play the game repeatedly for many rounds).

### Betting Game

Player 1 draws a card from a card deck (and hides it from the other player). Then he decides to either **pass** in which case he discards the card and pays \$1 to Player 2, or he will **bet** in which case it is 2nd player's turn. Player 2 can either **fold** in which case she pays \$1 to Player 1, or she will **call** and the card is revealed.

If the revealed card is **high** (10, Jack, Queen, King, Ace), then Player 2 pays \$2 to Player 1. Otherwise, the card is **low** (2 through 9) and Player 1 pays \$2 to Player 2.

Let us analyze the possible strategies for Player 1. Given the card, the card can be either high or low. Based on that the player can either pass or bet. So there are 4 possible strategies: pass on both high and low (**PP**), pass on high and bet on low (**PB**), bet on high and pass on low (**BP**), and bet on both high and low (**BB**). Player 2 can either **Call** or **Fold**. The possible expected outcomes of the game are then as follows.

Player 1	Player 2		Row Minimum
	Call	Fold	
<b>PP</b>	-1	-1	-1
<b>PB</b>	$-1\frac{8}{13}$	$\frac{3}{13}$	$-1\frac{8}{13}$
<b>BP</b>	$\frac{2}{13}$	$-\frac{3}{13}$	$-\frac{3}{13}$
<b>BB</b>	$-\frac{6}{13}$	1	$-\frac{6}{13}$
Column Maximum	$\frac{2}{13}$	1	

Thus, for instance, suppose that Player 1 plays strategy **BP** (bet on high, pass on low) and Player 2 **calls**. Then either the card is high, Player 1 bets and wins \$2, or the card is low and Player 1 folds and loses \$1. The chance of getting a high card is  $5/13$  and thus getting a low card has probability of  $8/13$ . So on average Player 1 gains

$$\frac{5}{13}\$2 + \frac{8}{13}(-\$1) = \frac{2}{13} \approx \$0.15$$

On the other hand, if Player 1 plays **BP** but Player 2 **folds**, then the expected gain of Player 1 is only  $-\frac{3}{13} \approx -\$0.23$ . So clearly, the intuitive strategy **BP** (bet on high, pass on low) is not necessarily best against a worst-case opponent.

Note that this is a **zero-sum** game since either Player 1 pays Player 2 or vice-versa (the sum of the players' gains is zero). Looking at the table above, we see that largest row minimum is  $-\frac{3}{13}$  while smallest column maximum is  $\frac{2}{13}$ . So this game does not have a saddle point (unlike the first game we discussed).

Notice that some strategies are better than others in all possible cases. For instance, playing **BP** instead of **PP** always gives better outcome for Player 1. We say that the strategy **PP** is **dominated** by another strategy (in this case **BP**). Clearly, if a strategy is dominated, the player can always do better by playing another strategy. Thus we can safely remove any dominated strategy from consideration without changing the problem (its optimal solution). In our case, this eliminates the strategy **PP** as well as the strategy **PB** which is dominated by the strategy **BB**. This leaves us with only two strategies and the pay-off matrix shown on the right.

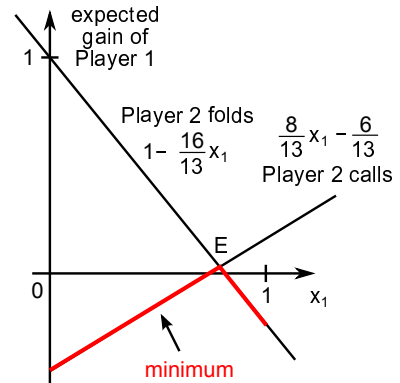
	Player 2	
	Call	Fold
Player 1		
<b>BP</b>	$\frac{2}{13}$	$-\frac{3}{13}$
<b>BB</b>	$-\frac{6}{13}$	1

Now we are ready to determine the **best mixed strategy** for Player 1. The player chooses to play **BP** with probability  $x_1$  or plays **BB** with probability  $x_2$ , where  $x_1 + x_2 = 1$ . We denote this **mixed strategy** as  $(x_1, x_2)$ . An **expected outcome** of this strategy is  $\frac{2}{13}x_1 - \frac{6}{13}x_2$  if Player 2 **calls** and is  $-\frac{3}{13}x_1 + x_2$  if Player 2 **folds**. So the worst-case outcome is simply the minimum of the two

$$\min_{(x_1, x_2)} \left\{ \frac{2}{13}x_1 - \frac{6}{13}x_2, -\frac{3}{13}x_1 + x_2 \right\}$$

Since  $x_1 + x_2 = 1$ , we can simplify this to

$$\text{outcome} = \min_{x_1} \left\{ \frac{8}{13}x_1 - \frac{6}{13}, 1 - \frac{16}{13}x_1 \right\}$$



We can plot the possible outcomes (based on the choices of  $x_1$ ) as shown on the right. From this we determine that the best mixed strategy for Player 1 is the point  $E$  corresponding to strategy  $(x_1, x_2)$  where  $x_1 = 19/24$  and  $x_2 = 5/24$ . This guarantees the player expected gain of  $1/39 \approx \$0.025$ .

**Answer:** The best strategy for Player 1 is to randomly choose between betting on high, passing on low with probability  $79\frac{1}{2}\%$  or always bet with probability  $20\frac{5}{6}\%$ . This gives him expected gain of  $\$2.5$ .

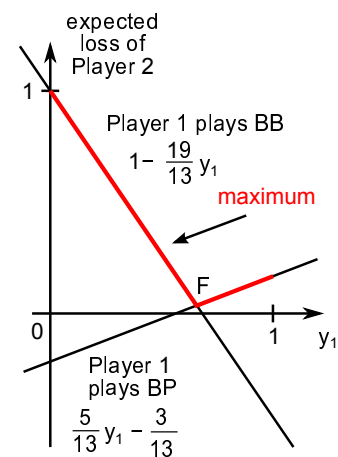
Similarly, we can determine the best mixed strategy for Player 2. Let  $(y_1, y_2)$  be the probabilities of Player 2 calling ( $y_1$ ) or folding ( $y_2$ ) where  $y_1 + y_2 = 1$ . If Player 1 plays **BP**, then the expected outcome (loss) for Player 2 is  $\frac{2}{13}y_1 - \frac{3}{13}y_2$ . If Player 1 play **BB**, then the outcome is  $-\frac{6}{13}y_1 + y_2$ . Thus the worst-case loss for Player 2 is the maximum of the two.

$$\max_{(y_1, y_2)} \left\{ \frac{2}{13}y_1 - \frac{3}{13}y_2, -\frac{6}{13}y_1 + y_2 \right\}$$

Using the fact that  $y_1 + y_2 = 1$ , this simplifies to

$$\max_{y_1} \left\{ \frac{5}{13}y_1 - \frac{3}{13}, 1 - \frac{19}{13}y_1 \right\}$$

We can plot the possible outcomes as shown on the right. The best strategy for Player 2 is the point  $F$  corresponding to strategy  $(y_1, y_2)$  where  $y_1 = \frac{2}{3}$  and  $y_2 = \frac{1}{3}$  with expected loss of  $\frac{1}{39} \approx \$0.025$ . Thus the best strategy for Player 2 is to call with  $66\frac{2}{3}\%$  probability and to fold with  $33\frac{1}{3}\%$  probability.



## Linear programming

We can now see what are the best strategies for the two players. Player 1 tries to choose probabilities  $x_1, x_2$  so as to maximize the  $\min\{\frac{2}{13}x_1 - \frac{6}{13}x_2, -\frac{3}{13}x_1 + x_2\}$  where probabilities  $x_1, x_2$  sum up to 1. Similarly, Player 2 chooses his probabilities  $y_1, y_2$  so as to minimize  $\max\{\frac{2}{13}y_1 - \frac{3}{13}y_2, -\frac{6}{13}y_1 + y_2\}$  where  $y_1 + y_2 = 1$ . Using the tricks we learnt, we see that both these problem can be transformed into linear programs as follows.

$$\begin{aligned} & \max z \\ \text{subject to} & \quad \frac{2}{13}x_1 - \frac{6}{13}x_2 \geq z \\ & \quad -\frac{3}{13}x_1 + x_2 \geq z \\ & \quad x_1 + x_2 = 1 \\ & \quad x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} & \min w \\ \text{subject to} & \quad \frac{2}{13}y_1 - \frac{3}{13}y_2 \leq w \\ & \quad -\frac{6}{13}y_1 + y_2 \leq w \\ & \quad y_1 + y_2 = 1 \\ & \quad y_1, y_2 \geq 0 \end{aligned}$$

which we can rewrite as follows:

$$\begin{aligned} & \max z \\ \text{subject to} & \quad -\frac{2}{13}x_1 + \frac{6}{13}x_2 + z \leq 0 \\ & \quad \frac{3}{13}x_1 - x_2 + z \leq 0 \\ & \quad x_1 + x_2 = 1 \\ & \quad x_1, x_2 \geq 0 \\ & \quad z \text{ unrestricted} \end{aligned}$$

$$\begin{aligned} & \min w \\ \text{subject to} & \quad -\frac{2}{13}y_1 + \frac{3}{13}y_2 + w \geq 0 \\ & \quad \frac{6}{13}y_1 - y_2 + w \geq 0 \\ & \quad y_1 + y_2 = 1 \\ & \quad y_1, y_2 \geq 0 \\ & \quad w \text{ unrestricted} \end{aligned}$$

Notice that the two programs are **duals** of each other. This is not a coincidence. In fact, it is always the case in zero-sum (constant-sum) games. This tells us that the optimal solutions to the two programs have the **same value** by Strong Duality, which defines the **value** of the game. Moreover, the optimal strategies for the two players satisfy Complementary Slackness (verify this for the solutions we found above). In other words, the solutions to the two problems form an **equilibrium** point (neither player can do better by changing his/hers strategy). In the literature, this is often called **Nash equilibrium**.

## 11.2 Nonconstant-sum Games

In the real world, more often we find situations where the gains/losses of the players are not necessarily constant. This happens for instance in cases where cooperating players can gain more together than by competing alone.

A prototypical example of this is the famous **Prisoners' dilemma**. In this problem, we have two criminals who committed a crime but there is not enough evidence against both of them. The district attorney gives each a chance to confess. If both confess, then they both go to jail for 5 years. If only one of them confesses, he is let free and the other criminal gets 20 years in jail. If neither confesses, they are each sentenced for 1 year for a misdemeanor. The two criminals are not allowed to communicate. **What is the best strategy** for each of them?

	Criminal 2	
	Confess	Don't confess
Confess	(-5, -5)	(0, -20)
Don't confess	(-20, 0)	(-1, -1)

The pay-off matrix now has different pay-offs for the two players as shown on the right. Clearly the pay-offs do not sum up to the same value each time. If they both confess they each go to jail for 5 years (sum is 10), while if they both do not confess, they each go to jail only for 1 year (sum is 2).

Note that both confessing is an **equilibrium** point, since if only one of them changes his mind (does not confess), he goes to jail for 20 year (which is more than 5 he gets by confessing). However, if they both change their mind (do not confess), they both improve their situation (each only gets 1 year in jail). This does not change our conclusion that confessing is an equilibrium. To be an equilibrium we only check that each player by himself cannot get a better outcome if he changes his mind (and others play the same way).

On the other hand, both not confessing, is not an equilibrium, since either player can change his mind and confess and thus not go to jail (while the other player gets 20 years – again, we only consider a single player changing his mind).

To solve this game, we see that the strategy **Don't confess** is **dominated** by strategy **Confess** in both players (the player can always do better by confessing when playing against a worst-case opponent). By eliminating these strategies, we are left with the unique equilibrium where both players confess.

Recall that this is a **pure** (deterministic) strategy. In other cases, eliminating dominated strategies will leave more than one choice and we formulate the maximization problem just like we did before. Solving it for both players gives us their optimal **mixed strategy**.

## Integer programming

## Integer Linear Program

$\max \quad \mathbf{c}\mathbf{x}$   
 $\mathbf{A}\mathbf{x} = \mathbf{b}$   
 $\mathbf{x} \geq 0$   
 $\mathbf{x}$  integer

- **pure** IP = all variables integer
- **mixed** IP = some variables integer
- **LP relaxation** of an ILP = dropping the integer restriction

## Example:

(PURE) INTEGER PROGRAM	MIXED INTEGER PROGRAM	LP RELAXATION
$\max 4x_1 + 2x_2 + 3x_3$ $5x_1 + 3x_2 + 4x_3 \leq 7$ $x_1, x_2, x_3 \in \{0, 1\}$	$\max 4x_1 + 2x_2 + 3x_3$ $5x_1 + 3x_2 + 4x_3 \leq 7$ $0 \leq x_1, x_2 \leq 1$ $x_3 \in \{0, 1\}$	$\max 4x_1 + 2x_2 + 3x_3$ $5x_1 + 3x_2 + 4x_3 \leq 7$ $0 \leq x_1, x_2, x_3 \leq 1$
optimal solution: $x_1 = 0, x_2 = x_3 = 1, z = 5$	optimal solution: $x_1 = 0.6, x_2 = 0, x_3 = 1, z = 5.4$	optimal solution: $x_1 = 1, x_2 = 0, x_3 = 0.5, z = 5.5$

## 12.1 Problem Formulation

## Review LP formulations

Standard LP problems:

- diet problem
- product mix
- blending
- inventory
- scheduling
- budgeting

Tricks:

1. maximum/minimum:

$$\max\{f, g\} \leq h \iff \begin{array}{l} 2 \text{ constraints } f \leq h \\ g \leq h \end{array}$$

$$\min\{f, g\} \geq h \iff \begin{array}{l} 2 \text{ constraints } f \geq h \\ g \geq h \end{array}$$

2. absolute value:  $|f - g| \leq h \iff \begin{array}{l} 2 \text{ constraints } f - g \leq h \\ g - f \leq h \end{array}$

(to see this, note that  $|f - g| \leq h$  means that  $g$  lies between  $f - h$  and  $f + h$ ; that is,  $f - h \leq g \leq f + h$ )

3. positive/negative values  $f = x^+ - x^-$  where  $x^+, x^-$  are new non-negative variables

$\Rightarrow$  now  $x^+$  is the positive part of  $f$  (or zero) and  $x^-$  the negative part of  $f$  (or zero)

**Important:** must make sure that  $x^+$  and  $x^-$  are never both basic in the optimum, else it doesn't work

alternatively: 2 constraints  $f \leq x^+$  and  $-f \leq x^-$  (with similar caveats)



$x_i \in \{0, 1\}$  = indicates if a facility is opened in city  $i$

City	1	2	3	4	5	6
Neighbouring cities	1,2	1,2,6	3,4	3,4,5	4,5,6	2,5,6

$$\min z = 500x_1 + 340x_2 + 450x_3 + 250x_4 + 200x_5 + 300x_6$$

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 + x_2 + x_6 &\geq 1 \\ x_3 + x_4 &\geq 1 \\ x_3 + x_4 + x_5 &\geq 1 \\ x_4 + x_5 + x_6 &\geq 1 \\ x_2 + x_5 + x_6 &\geq 1 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\in \{0, 1\} \end{aligned}$$

**Non-linear objectives** – price discounts (piece-wise linear functions)

- first 500 gallons – 25 cents
- next 500 gallons – 20 cents
- any gallon above that – 15 cents

$z$  = cost of  $x$  gallons

write  $x = x_1 + x_2 + x_3$   
 where  $0 \leq x_1, x_2 \leq 500$   
 and  $x_3 \geq 0$

then  $z = 25x_1 + 20x_2 + 15x_3$

$x_1 \geq 500y_1 \geq x_2 \geq 500y_2$   
 $My_2 \geq x_3 \quad y_1, y_2 \in \{0, 1\}$

**Network problems** - Travelling salesman

- find a shortest cycle through all nodes of  $G = (V, E)$  where  $V = \{1, \dots, N\}$
- $x_{ij} = 1$  if edge  $(i, j)$  on the route
- $u_i = k$  if  $i$  is  $k$ -th node on the route

$$\begin{aligned} \min \sum_{(i,j) \in E} c_{ij}x_{ij} \\ \sum_j x_{ij} = 1 \quad \text{and} \quad \sum_j x_{ji} = 1 \quad \text{for all } i \in V \\ u_i - u_j + Nx_{ij} \leq N - 1 \quad \text{for all } (i, j) \in E, i, j \neq 1 \\ u_i \geq 0 \quad \text{and} \quad x_{ij} \in \{0, 1\} \end{aligned}$$

**Why this works?** The first two constraints make sure that exactly one edge is coming in and one edge going out of each node. So the chosen edges form a collection of cycles. If edge  $(i, j)$  is used, i.e. if  $x_{ij} = 1$ , then we have  $u_i - u_j + N \leq N - 1$  and so  $u_i < u_j$ . This excludes any cycle on nodes  $\{2, \dots, N\}$ . So there can only be one cycle, through all nodes. The objective function selects such a cycle of minimum cost.

**Scheduling problems** - non pre-emptive job scheduling

- schedule jobs on a machine(s), jobs have arrival time and processing time, minimize waiting time
- $x_{ijk} = 1$  if job  $j$  is  $i$ -th to be processed on machine  $k$
- once job is started, it cannot be interrupted (no pre-emption)

**Tricks**

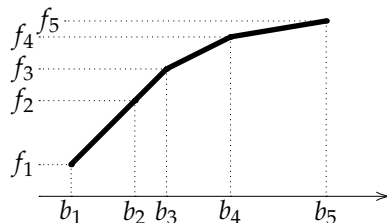
**Or constraint:**  $f \leq 0$  or  $g \leq 0 \iff 2$  constraints  $f \leq My$  and  $g \leq M(1 - y)$   $y \in \{0, 1\}$  and  $M$  is a large number

**If-then:** if  $f > 0$ , then  $g \geq 0 \iff 2$  constraints  $-g \leq My$  and  $f \leq M(1 - y)$   $y \in \{0, 1\}$  and  $M$  is a large number

**Piece-wise linear function**  $z$

- breakpoints  $b_1, b_2, \dots, b_t$
- with values  $f_1, f_2, \dots, f_t$

$x = b_1x_1 + b_2x_2 + \dots + b_tx_t$   
 $z = f_1x_1 + f_2x_2 + \dots + f_tx_t$   
 $x_1 + x_2 + \dots + x_t = 1$



$$\begin{aligned} y_1 + y_2 + \dots + y_{t-1} &= 1 \\ y_1, \dots, y_t &\in \{0, 1\} \\ x_1 &\leq y_1 \\ x_2 &\leq y_1 + y_2 \\ x_3 &\leq y_2 + y_3 \\ &\vdots \\ x_{t-1} &\leq y_{t-2} + y_{t-1} \\ x_t &\leq y_{t-1} \end{aligned}$$

**Why this works?** The variable  $y_i$  indicates that  $x$  is between  $b_i$  and  $b_{i+1}$ ; if  $y_i = 1$ , then only  $x_i$  and  $x_{i+1}$  can be positive; so we have  $x_i + x_{i+1} = 1$  and thus  $x = b_ix_i + b_{i+1}(1 - x_i)$  and  $z = f_ix_i + f_{i+1}(1 - x_i)$  where  $0 \leq x_i \leq 1$ ; this exactly corresponds to the segment between  $(b_i, f_i)$  and  $(b_{i+1}, f_{i+1})$

## 12.2 Cutting Planes

Let us go back to our toy factory problem. The toys are sold to a wholesale distributor who demands shipments in packs (boxes) of 30. There's no limit on how much we can produce.

$$\begin{array}{l}
 \text{Max } 3x_1 + 2x_2 \\
 x_1 + x_2 \leq 80 \\
 2x_1 + x_2 \leq 100 \\
 x_1, x_2 \geq 0
 \end{array}
 \rightarrow
 \begin{array}{l}
 30w_1 = x_1 \\
 30w_2 = x_2 \\
 w_1, w_2 \geq 0 \\
 \text{integer}
 \end{array}
 \rightarrow
 \begin{array}{l}
 \text{Max } 90w_1 + 60w_2 \\
 30w_1 + 30w_2 \leq 80 \\
 60w_1 + 30w_2 \leq 100 \\
 w_1, w_2 \geq 0 \\
 \text{integer}
 \end{array}$$

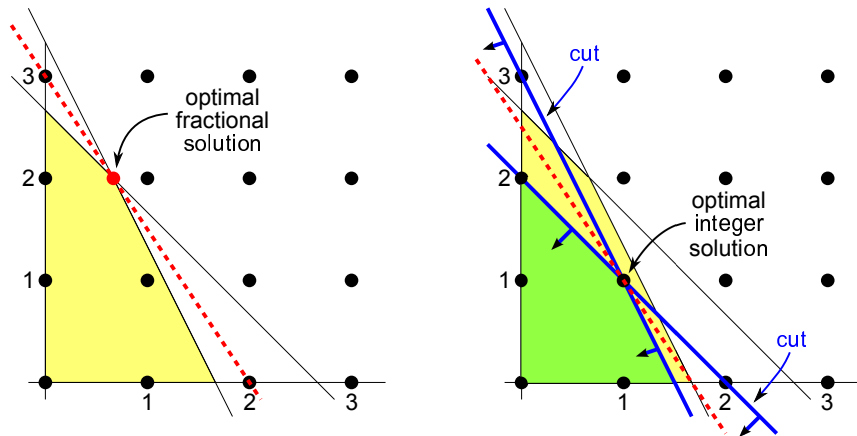
Consider the first constraint  $30w_1 + 30w_2 \leq 80$ . Let's divide both sides by 30. We obtain  $w_1 + w_2 \leq \frac{8}{3}$ . Notice that the left-hand side of the constraint is an integer, while the right-hand side is a fraction. Therefore

$$\underbrace{w_1 + w_2}_{\text{integer}} \leq \frac{8}{3} = 2 + \frac{2}{3} \rightarrow \underbrace{w_1 + w_2 - 2}_{\text{integer}} \leq \underbrace{\frac{2}{3}}_{<1} \rightarrow \underbrace{w_1 + w_2 - 2}_{\text{cut}} \leq 0$$

**What happens?** We rewrite the constraint so that the left-hand side is integer and the right-hand side is a positive fraction less than one. Then, since the left-hand side is integer and is less than a fraction of one, it must be zero or less. This gives us a *new constraint* that we call a **cut** or a **cutting plane**. We can add this cut as our new constraint **without** removing any **integer** feasible solution.

Doing the same for the second constraint  $60w_1 + 30w_2 \leq 100$ :

$$60w_1 + 30w_2 \leq 100 \rightarrow \underbrace{2w_1 + w_2}_{\text{integer}} \leq \frac{10}{3} = 3 + \frac{1}{3} \rightarrow \underbrace{2w_1 + w_2 - 3}_{\text{integer}} \leq \underbrace{\frac{1}{3}}_{<1} \rightarrow \underbrace{2w_1 + w_2 - 3}_{\text{cut}} \leq 0$$



### Integer Linear Program

$$\begin{array}{l}
 \text{Max } 90w_1 + 60w_2 \\
 w_1 + w_2 \leq 2 \\
 2w_1 + w_2 \leq 3 \\
 w_1, w_2 \geq 0 \\
 \text{integer}
 \end{array}$$

### LP relaxation

$$\begin{array}{l}
 \text{Max } 90w_1 + 60w_2 \\
 w_1 + w_2 \leq 2 \\
 2w_1 + w_2 \leq 3 \\
 w_1, w_2 \geq 0
 \end{array}$$

**Note:** Every feasible solution to an integer linear program is also a feasible solution to its LP relaxation. Thus if the **LP relaxation has integer optimal solution**, then this is an **optimal solution** to the **ILP**.

The LP relaxation has an optimal solution  $w_1 = w_2 = 1$  of value  $z = 150$ . Since this is an integer solution, it is also an optimal solution to the original integer linear program.

**Conclusion:** for optimal production, we should produce 1 box of each of the two toys.

Now, suppose that a different distributor demands that toy soldiers are delivered in boxes of 20, while toy trains in boxes of 25. We simplify the constraints by dividing both sides by 5 (and the objective by 10).

$$\begin{array}{l}
 \text{Max } 3x_1 + 2x_2 \\
 x_1 + x_2 \leq 80 \\
 2x_1 + x_2 \leq 100 \\
 x_1, x_2 \geq 0
 \end{array}
 \quad
 \begin{array}{l}
 20w_1 = x_1 \\
 25w_2 = x_2 \\
 w_1, w_2 \geq 0 \\
 \text{integer}
 \end{array}
 \quad
 \begin{array}{l}
 \text{Max } 60w_1 + 50w_2 \\
 20w_1 + 25w_2 \leq 80 \\
 40w_1 + 25w_2 \leq 100 \\
 w_1, w_2 \geq 0 \\
 \text{integer}
 \end{array}
 \quad
 \begin{array}{l}
 \text{Max } 6w_1 + 5w_2 \\
 4w_1 + 5w_2 \leq 16 \\
 8w_1 + 5w_2 \leq 20 \\
 w_1, w_2 \geq 0 \\
 \text{integer}
 \end{array}$$

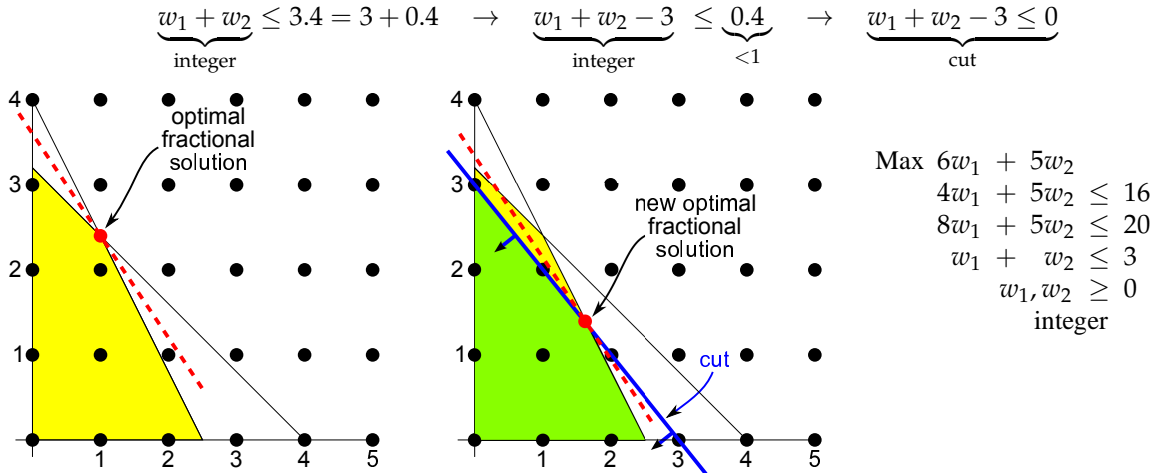
Notice that this time we cannot directly get fractions on the right-hand side while keeping the left-hand side integer.



However, let us multiply the first constraint by 3 and add it to the second constraint.

$$20w_1 + 20w_2 = 3(\underbrace{4w_1 + 5w_2}_{\leq 16}) + (\underbrace{8w_1 + 5w_2}_{\leq 20}) \leq 3 \times 16 + 20 = 68$$

We can now divide both sides by 20 to obtain  $w_1 + w_2 \leq 3.4$ . So again we can derive a cut.



We need a more systematic approach to finding cuts.

### Gomory cuts

We can derive cuts directly from solving the LP relaxation. If the relaxation has an integer optimal solution, then we are done since this is also an optimal solution to the integer problem. If the optimal solution is **fractional**, we will use **rounding** to obtain a cut (as explained below). This cut will **cut off** the fractional solution and thus make the feasible region **smaller**. This makes sure that we **improve** in each step.

Consider the optimal dictionary to our problem.

<p>Max <math>6w_1 + 5w_2</math>  <math>4w_1 + 5w_2 \leq 16</math>  <math>8w_1 + 5w_2 \leq 20</math>    add slack variables  <math>w_1 + w_2 \leq 3</math>        (non-negative integers)  <math>w_1, w_2 \geq 0</math>          integer</p> <p>Max <math>6w_1 + 5w_2</math>  <math>4w_1 + 5w_2 + x_3 = 16</math>  <math>8w_1 + 5w_2 + x_4 = 20</math>  <math>w_1 + w_2 + x_5 = 3</math>  <math>w_1, w_2, x_3, x_4, x_5 \geq 0</math>          integer</p>	$w_1 = \frac{5}{3} - \frac{1}{3}x_4 + \frac{5}{3}x_5$ $w_2 = \frac{4}{3} + \frac{1}{3}x_4 - \frac{8}{3}x_5$ $x_3 = \frac{8}{3} - \frac{1}{3}x_4 + \frac{20}{3}x_5$ <hr style="width: 50%; margin-left: 0;"/> $z = \frac{50}{3} - \frac{1}{3}x_4 - \frac{10}{3}x_5$
---	--

**Choose** any line of the dictionary with a **fractional** constant (in this case every line qualifies, including  $z$ ). This choice is **arbitrary** but good practice suggest to choose the line where the fraction is closest to  $1/2$ .

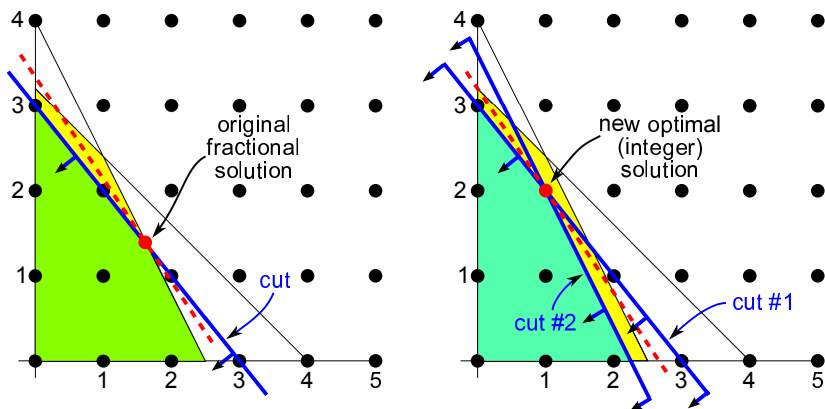
We pick the expression for  $z$  and move whole parts to the left, keeping only fractions on the right. This time we also have fractional coefficient with variables. We rearrange the expression so that on the right each variable has **negative** fractional coefficient, while the absolute constant is a **positive** fraction.

$$z = \underbrace{\frac{50}{3}}_{16 + \frac{2}{3}} - \frac{1}{3}x_4 - \underbrace{\frac{10}{3}}_{-3 - \frac{1}{3}}x_5 \rightarrow z + 3x_5 - 16 = \frac{2}{3} - \underbrace{\frac{1}{3}x_4 - \frac{1}{3}x_5}_{\text{negative fractions}} \rightarrow \underbrace{z + 3x_5 - 16}_{\text{cut}} \leq 0$$

**What happens?** Because the coefficients of variables on the right-hand side are negative, the value of the right-hand side is  $2/3$  or less (cannot be more, since the variables are non-negative). So the value of the left-hand side is also at

most  $2/3$ , but since the left-hand side is integer, the value is actually at most 0. We add the cut to our dictionary by introducing a **slack** (non-negative integer) and express it using the fractions.

$$z + 3x_5 - 16 + x_6 = 0 \rightarrow x_6 = -(z + 3x_5 - 16) = -\left(\frac{2}{3} - \frac{1}{3}x_4 - \frac{1}{3}x_5\right) \rightarrow x_6 = -\frac{2}{3} + \frac{1}{3}x_4 + \frac{1}{3}x_5$$



$$\begin{array}{r} w_1 = \frac{5}{3} - \frac{1}{3}x_4 + \frac{5}{3}x_5 \\ w_2 = \frac{4}{3} + \frac{1}{3}x_4 - \frac{8}{3}x_5 \\ x_3 = \frac{8}{3} - \frac{1}{3}x_4 + \frac{20}{3}x_5 \\ x_6 = -\frac{2}{3} + \frac{1}{3}x_4 + \frac{1}{3}x_5 \\ \hline z = \frac{50}{3} - \frac{1}{3}x_4 - \frac{10}{3}x_5 \end{array}$$

Dually feasible dictionary  
 $\rightarrow$  use Dual Simplex  
 $x_6$  leaves (value  $< 0$ )  
 ratio test:  
 $x_4 : (1/3)/(1/3) = 1$   
 $x_5 : (10/3)/(1/3) = 10$   
 $\rightarrow x_4$  enters

$$\begin{array}{r} w_1 = 1 + 2x_5 - x_6 \\ w_2 = 2 - 3x_5 + x_6 \\ x_3 = 2 + 7x_5 - x_6 \\ x_4 = 2 - x_5 + 3x_6 \\ \hline z = 16 - 3x_5 - x_6 \end{array}$$

Optimal integer solution found  $w_1 = 1, w_2 = 2$  with value  $z = 16$  (corresponds to \$160 of profit).

**Conclusion:** optimal production consists of producing one pack of toy soldiers, and two packs of toy trains.

### Cutting planes algorithm

This works **only** for **pure** integer linear programs.

1. Solve the LP relaxation.
2. If the LP is **infeasible**, then **report** that the problem is **infeasible**, and **stop**.
3. If all variables are integers, then **report** the solution and **stop**.
4. Else in the optimal dictionary, pick any line with fractional constant.
  - (a) Rewrite the line by moving whole parts to the left so that
    - the absolute constant on the right is a positive fraction (less than 1),
    - the coefficients of variables on the right are negative fractions (less than 1 in absolute value).
  - (b) Make the **right-hand side**  $\leq 0$  to form a new constraint (cut).
  - (c) Introduce a slack variable to the cut and add the resulting equation to the dictionary.
  - (d) Solve the resulting LP using the Dual Simplex method and then go back to 2.

**Example.** suppose that the dictionary contains

$$x_1 = \frac{5}{2} - \frac{3}{2}x_2 + \frac{7}{2}x_3 + 2x_4$$

Rewrite by moving whole parts to the left so that the coefficient of variables on the right are **negative**.

$$x_1 = \underbrace{\frac{5}{2}}_{2+\frac{1}{2}} - \underbrace{\frac{3}{2}}_{-1-\frac{3}{2}}x_2 + \underbrace{\frac{7}{2}}_{4-\frac{1}{2}}x_3 + 2x_4$$

$$\boxed{x_1 + x_2 - 4x_3 - 2x_4 - 2 = \frac{1}{2} - \frac{1}{2}x_2 - \frac{1}{2}x_3}$$

(Notice:  $x_3$  had positive coefficient  $+\frac{7}{2}$ , and we had to take  $\frac{7}{2} = 4 - \frac{1}{2}$  to get a negative coeff on the right.)  
 The right-hand side is  $< 1$  but must be integer (since lhs is). We introduce the cut that makes it at most 0.

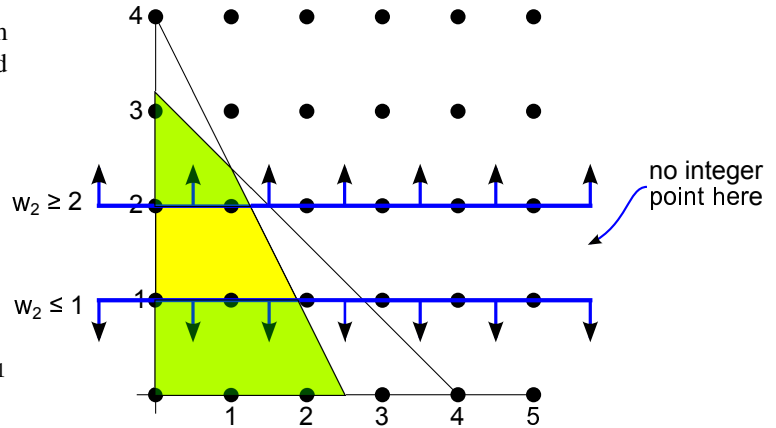
$$\frac{1}{2} - \frac{1}{2}x_2 - \frac{1}{2}x_3 \leq 0 \quad \text{add slack} \rightarrow \text{new constraint} \quad \boxed{x_5 = -\frac{1}{2} + \frac{1}{2}x_2 + \frac{1}{2}x_3}$$

### 12.3 Branch and Bound

For illustration, consider the toy factory problem where toy soldiers are delivered in boxes of 20 and toy trains in boxes of 25.

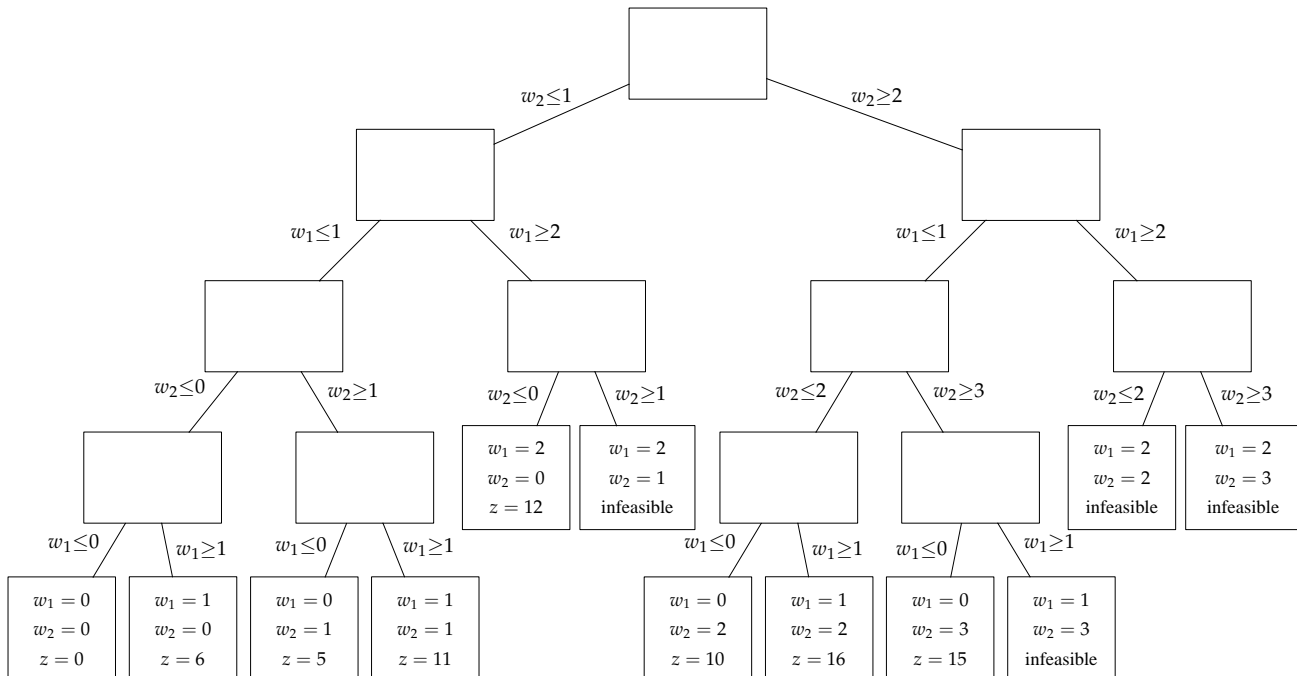
$$\begin{aligned} \text{Max } & 6w_1 + 5w_2 \\ & 4w_1 + 5w_2 \leq 16 \\ & 8w_1 + 5w_2 \leq 20 \\ & 0 \leq w_1 \leq 2 \\ & 0 \leq w_2 \leq 3 \\ & w_1, w_2 \text{ integer} \end{aligned}$$

Notice that we also assume that the variables  $w_1$  and  $w_2$  are upper-bounded (by 2 resp. 3).



#### Enumerating solutions by Branching

Consider the variable  $w_2$ . Since we are only interested solutions where this variable is an integer, we can also assume that  $w_2$  does not attain any value strictly between 1 and 2. This splits the feasible region onto two parts, one where  $w_2 \leq 1$  and one where  $w_2 \geq 2$ . We solve the problem on each of the two parts independently and pick the better of the two solutions. In other words, we **branch** into two **subcases**. In each of the subcases, we can again consider an integer variable, say  $w_1$ , and exclude all fractional values between, say zero and one. We branch into two further subcases and continue this process until the values of all integer variables have been determined. We can visualize this using a **branching tree**.



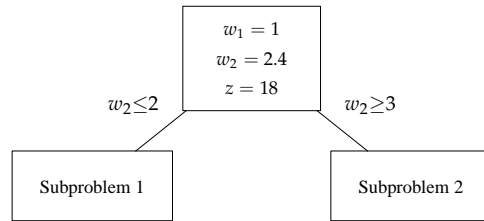


we can use the Simplex algorithm and **reuse** dictionaries in subsequent steps.

<b>Integer program</b>	<b>Max</b> $6w_1 + 5w_2$	
$4w_1 + 5w_2 \leq 16$	$4w_1 + 5w_2 + x_3 = 16$	$w_1 = 1 + 0.25x_3 - 0.25x_4$
$8w_1 + 5w_2 \leq 20$	$8w_1 + 5w_2 + x_4 = 20$	$w_2 = 2.4 - 0.4x_3 + 0.2x_4$
$w_1, w_2 \geq 0$	$w_1, w_2 \geq 0$	$z = 18 - 0.5x_3 - 0.5x_4$
integer	integer	

**Optimal fractional solution**

The optimal solution is fractional, because  $w_2 = 2.4$  is not an integer. We branch into  $w_2 \leq 2$  and  $w_2 \geq 3$ .



**Subproblem 1:**

$w_1 = 1 + 0.25x_3 - 0.25x_4$	<b>Bounds:</b>
$w_2 = 2.4 - 0.4x_3 + 0.2x_4$	$0 \leq w_1 \leq \infty$
$z = 18 - 0.5x_3 - 0.5x_4$	$0 \leq w_2 \leq 2$

**Subproblem 2:**

$w_1 = 1 + 0.25x_3 - 0.25x_4$	<b>Bounds:</b>
$w_2 = 2.4 - 0.4x_3 + 0.2x_4$	$0 \leq w_1 \leq \infty$
$z = 18 - 0.5x_3 - 0.5x_4$	$3 \leq w_2 \leq \infty$

In both subproblems the dictionary is not feasible. To fix it, we use the Upper-bounded Dual Simplex method. Let us first solve **Subproblem 1**.

$w_1 = 1 + 0.25x_3 - 0.25x_4$	<b>Bounds:</b>
$w_2 = 2.4 - 0.4x_3 + 0.2x_4$	$0 \leq w_1 \leq \infty$
$z = 18 - 0.5x_3 - 0.5x_4$	$0 \leq w_2 \leq 2$

check lower bounds:  $0 \leq 1 = w_1, 0 \leq 2.4 = w_2$   
 check upper bounds:  $w_1 = 1 \leq \infty, w_2 = 2.4 \not\leq 2$   
 $\rightarrow$  replace  $w_2$  by  $w_2 = 2 - w'_2$

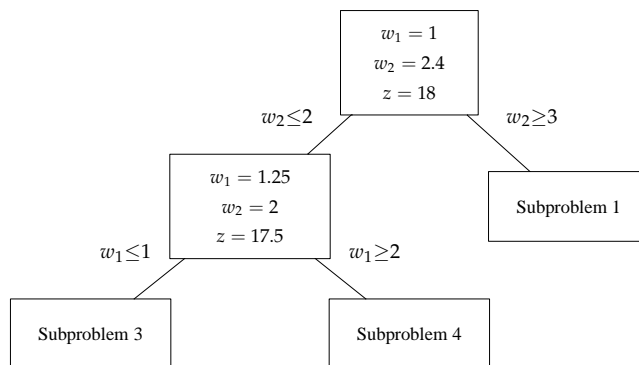
$w_1 = 1 + 0.25x_3 - 0.25x_4$	<b>Bounds:</b>
$w'_2 = -0.4 + 0.4x_3 - 0.2x_4$	$0 \leq w_1 \leq \infty$
$z = 18 - 0.5x_3 - 0.5x_4$	$0 \leq w_2 \leq 2$

check lower bounds:  $0 \leq 1 = w_1, 0 \not\leq -0.4 = w'_2$   
 $\rightarrow w'_2$  leaves, ratio test  $x_3 : 0.5/0.4 = 1.25$   
 $x_4 : \text{no constraint}$   
 $\rightarrow x_3$  enters

$w_1 = 1.25 - 0.125x_4 + 0.625w'_2$	<b>Bounds:</b>
$x_3 = 1 + 0.5x_4 + 2.5w'_2$	$0 \leq w_1 \leq \infty$
$z = 17.5 - 0.75x_4 - 1.25w'_2$	$0 \leq w_2 \leq 2$

optimal solution found:  
 $w_1 = 1.25, w'_2 = 0 \rightarrow w_2 = 2$   
 $z = 17.5$

The optimal solution is fractional, since  $w_1 = 1.25$ . We branch on  $w_1 \leq 1$  and  $w_1 \geq 2$ .



**Subproblem 3:**

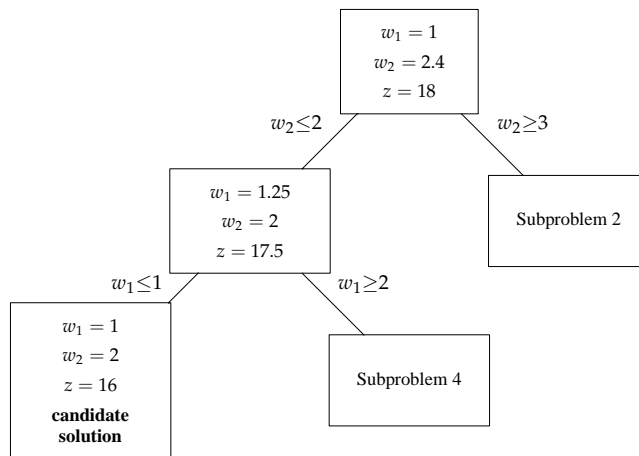
$w_1 = 1.25 - 0.125x_4 + 0.625w'_2$	<b>Bounds:</b>
$x_3 = 1 + 0.5x_4 + 2.5w'_2$	$0 \leq w_1 \leq 1$
$z = 17.5 - 0.75x_4 - 1.25w'_2$	$0 \leq w_2 \leq 2$

**Subproblem 4:**

$w_1 = 1.25 - 0.125x_4 + 0.625w'_2$	<b>Bounds:</b>
$x_3 = 1 + 0.5x_4 + 2.5w'_2$	$2 \leq w_1 \leq \infty$
$z = 17.5 - 0.75x_4 - 1.25w'_2$	$0 \leq w_2 \leq 2$

We again solve **Subproblem 3** using the Upper-bounded Dual Simplex method.

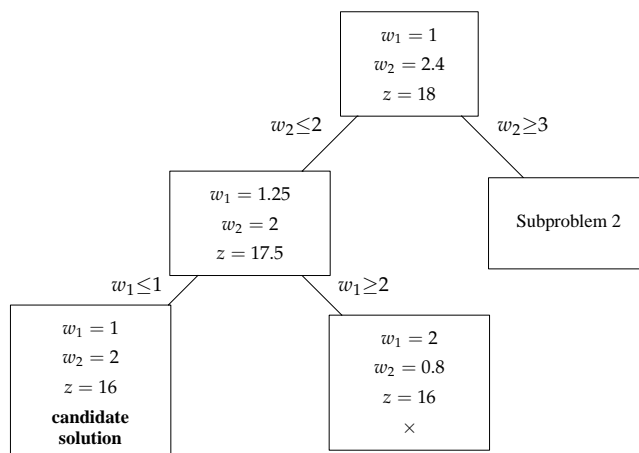
$\begin{array}{r} w_1 = 1.25 - 0.125x_4 + 0.625w'_2 \\ x_3 = 1 + 0.5x_4 + 2.5w'_2 \\ \hline z = 17.5 - 0.75x_4 - 1.25w'_2 \end{array}$	<b>Bounds:</b> $0 \leq w_1 \leq 1$ $0 \leq w_2 \leq 2$	check lower bounds: $0 \leq 1.25 = w_1, 0 \leq 1 = x_3$ check upper bounds: $w_1 = 1.25 \not\leq 1, x_3 = 1 \leq \infty$ $\rightarrow$ replace $w_1$ by $w_1 = 1 - w'_1$
$\begin{array}{r} w'_1 = -0.25 + 0.125x_4 - 0.625w'_2 \\ x_3 = 1 + 0.5x_4 + 2.5w'_2 \\ \hline z = 17.5 - 0.75x_4 - 1.25w'_2 \end{array}$	<b>Bounds:</b> $0 \leq w_1 \leq 1$ $0 \leq w_2 \leq 2$	check lower bounds: $0 \not\leq -0.25 = w'_1, 0 \leq 1 = x_3$ $\rightarrow w'_1$ leaves, ratio test $x_4 : 0.75/0.125 = 6$ $w'_2 : \text{no constraint}$ $\rightarrow x_4$ enters
$\begin{array}{r} x_3 = 2 + 4w'_1 + 5w'_2 \\ x_4 = 2 + 8w'_1 + 5w'_2 \\ \hline z = 16 - 6w'_1 - 5w'_2 \end{array}$	<b>Bounds:</b> $0 \leq w_1 \leq \infty$ $0 \leq w_2 \leq 2$	optimal (integer) solution found: $w'_1 = 0 \rightarrow w_1 = 1, w'_2 = 0 \rightarrow w_2 = 2$ $z = 16 \rightarrow$ <b>candidate</b> integer solution



Now we go back and solve **Subproblem 4**. Since  $w_1 \geq 2$  we substitute:  $w_1 = 2 + w_3$  where  $w_3 \geq 0$  is integer.

$\begin{array}{r} w_3 = -0.75 - 0.125x_4 + 0.625w'_2 \\ x_3 = 1 + 0.5x_4 + 2.5w'_2 \\ \hline z = 17.5 - 0.75x_4 - 1.25w'_2 \end{array}$	<b>Bounds:</b> $0 \leq w_3 \leq \infty$ $0 \leq w_2 \leq 2$	check lower bounds: $0 \not\leq -0.75 = w_3, 0 \leq 1 = x_3$ $\rightarrow w_3$ leaves, ratio test $x_4 : \text{no constraint}$ $w'_2 : 1.25/0.625 = 2$ $\rightarrow w'_2$ enters
$\begin{array}{r} w'_2 = 1.2 + 0.2x_4 + 1.6w_3 \\ x_3 = 4 + x_4 + 4w_3 \\ \hline z = 16 - x_4 - 2w_3 \end{array}$	<b>Bounds:</b> $0 \leq w_1 \leq \infty$ $0 \leq w_2 \leq 2$	optimal solution found: $w_3 = 0 \rightarrow w_1 = 2, w'_2 = 1.2 \rightarrow w_2 = 0.8$ $z = 16$

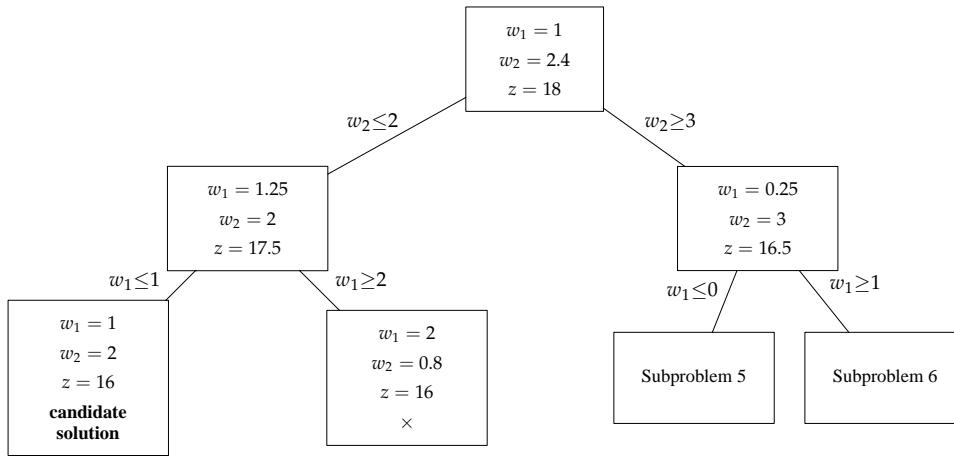
Optimal solution is fractional, but its value is  $z = 16$ . Thus we **do not branch**, since we already have a **candidate integer solution** of value  $z = 16$ .



Now we go back and solve **Subproblem 2**. Since  $w_2 \geq 3$ , we substitute  $w_2 = 3 + w_4$  where  $w_4 \geq 0$  is integer.

$w_1 = 1 + 0.25x_3 - 0.25x_4$	Bounds:	check lower bounds: $0 \leq 1 = w_1, 0 \not\leq -0.6 = w_4$
$w_4 = -0.6 - 0.4x_3 + 0.2x_4$	$0 \leq w_1 \leq \infty$	$\rightarrow w_4$ leaves, ratio test $x_3 : \text{no constraint}$
$z = 18 - 0.5x_3 - 0.5x_4$	$0 \leq w_4 \leq \infty$	$x_4 : 0.5/0.2 = 2.5$
		$\rightarrow x_4$ enters
$w_1 = 0.25 - 0.25x_3 - 1.25w_4$	Bounds:	optimal solution found:
$x_4 = 3 + 2x_3 + 5w_4$	$0 \leq w_1 \leq \infty$	$w_1 = 0.25, w_4 = 0 \rightarrow w_2 = 3$
$z = 16.5 - 1.5x_3 - 2.5w_4$	$0 \leq w_4 \leq \infty$	$z = 16.5$

Optimal solution is fractional, but its value  $z = 16.5$  is more than  $z = 16$  of the candidate solution. It is still possible that a better solution can be found by branching (here we ignore the fact that optimal  $z$  should also be an integer, otherwise we would not need to branch). Since  $w_1 = 0.25$ , we branch on  $w_1 \leq 0$  and  $w_1 \geq 1$ .



**Subproblem 5:**

$w_1 = 0.25 - 0.25x_3 - 1.25w_4$	Bounds:	
$x_4 = 3 + 2x_3 + 5w_4$	$0 \leq w_1 \leq 0$	
$z = 16.5 - 1.5x_3 - 2.5w_4$	$0 \leq w_4 \leq \infty$	

Let us now solve **Subproblem 5**.

$w_1 = 0.25 - 0.25x_3 - 1.25w_4$	Bounds:	
$x_4 = 3 + 2x_3 + 5w_4$	$0 \leq w_1 \leq 0$	
$z = 16.5 - 1.5x_3 - 2.5w_4$	$0 \leq w_4 \leq \infty$	

$w'_1 = -0.25 + 0.25x_3 + 1.25w_4$	Bounds:	
$x_4 = 3 + 2x_3 + 5w_4$	$0 \leq w_1 \leq 0$	
$z = 16.5 - 1.5x_3 - 2.5w_4$	$0 \leq w_4 \leq \infty$	

$w_4 = 0.2 - 0.2x_3 + 0.8w'_1$	Bounds:	
$x_4 = 4 - x_3 + 4w'_1$	$0 \leq w_1 \leq 0$	
$z = 16 - x_3 - 2w'_1$	$0 \leq w_4 \leq \infty$	

The solution has value  $z = 16$  which is no better than our candidate solution. We do not branch.

We now solve **Subproblem 6**. Since  $w_1 \geq 1$ , we substitute  $w_1 = 1 + w_5$ , where  $w_5 \geq 0$  and integer.

$w_5 = -0.75 - 0.25x_3 - 1.25w_4$	Bounds:	
$x_4 = 3 + 2x_3 + 5w_4$	$0 \leq w_5 \leq \infty$	
$z = 16.5 - 1.5x_3 - 2.5w_4$	$0 \leq w_4 \leq \infty$	

**Subproblem 6:**

$w_1 = 0.25 - 0.25x_3 - 1.25w_4$	Bounds:	
$x_4 = 3 + 2x_3 + 5w_4$	$1 \leq w_1 \leq \infty$	
$z = 16.5 - 1.5x_3 - 2.5w_4$	$0 \leq w_4 \leq \infty$	

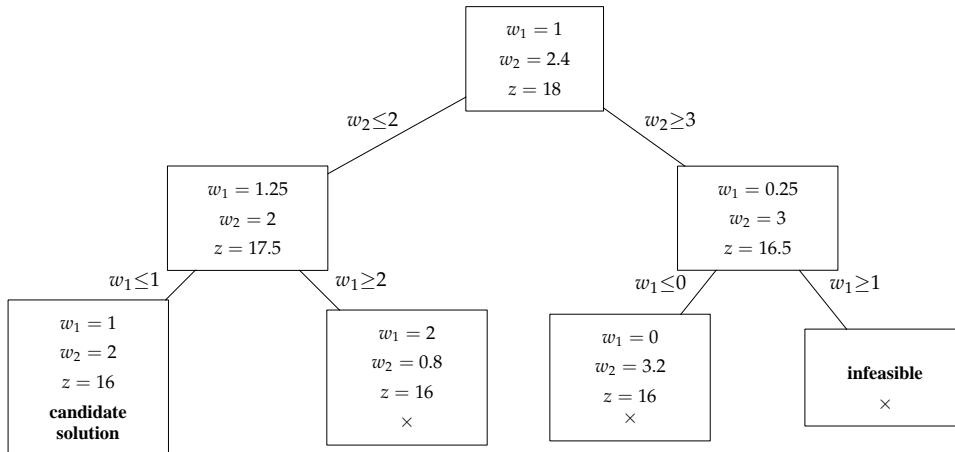
check lower bounds:  $0 \leq 0.25 = w_1, 0 \leq 3 = x_4$   
 check upper bounds:  $w_1 = 0.25 \not\geq 1, x_4 = 3 \leq \infty$   
 $\rightarrow$  replace  $w_1$  by  $w_1 = 0 - w'_1$

check lower bounds:  $0 \not\leq -0.25 = w'_1, 0 \leq 3 = x_4$   
 $\rightarrow w'_1$  leaves, ratio test  $x_3 : 1.5/0.25 = 6$   
 $w_4 : 2.5/1.25 = 2$   
 $\rightarrow w_4$  enters

optimal solution found:  
 $w'_1 = 0 \rightarrow w_1 = 0, w_4 = 0.2 \rightarrow w_2 = 3.2$   
 $z = 16$

Since the problem is infeasible, we do not need to branch anymore (no further restriction can make it feasible).

We now have solved all subproblems. We can thus summarize.



We conclude that there exist an integer feasible solution (our candidate solution), and it is also an optimal integer solution:  $w_1 = 1, w_2 = 2$  of value  $z = 16$ . This concludes the algorithm.

**Branch and bound for Knapsack**

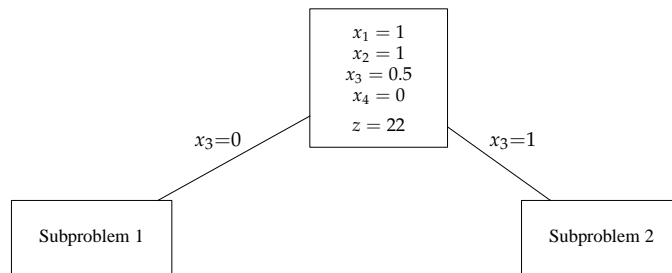
$$\begin{aligned} \text{Max } z &= 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 &\leq 14 \\ x_1, x_2, x_3, x_4 &\in \{0, 1\} \\ z &\text{ is integer} \end{aligned}$$

LP relaxation is called Fractional Knapsack. Can be solved by calculating prices of items per unit of weight and choosing items of highest unit price first.

Item	1	2	3	4
Unit price	$\frac{8}{5} = 1.6$	$\frac{11}{7} = 1.57$	$\frac{6}{4} = 1.5$	$\frac{4}{3} = 1.33$

We pick  $x_1 = 1$ , since item 1 has highest unit cost. The remaining budget is  $14 - 5 = 9$ . Then we pick  $x_2 = 1$ , since item 2 has next highest unit cost. The budget reduces to  $9 - 7 = 2$ . Then we pick item 3 which has next highest unit cost, but we can only pick  $x_3 = 0.5$  and use up all the budget.

The total price is  $z = 8 + 11 + 0.5 \times 6 = 22$ . Since  $x_3 = 0.5$ , we branch on  $x_3 = 0$  and  $x_3 = 1$ .

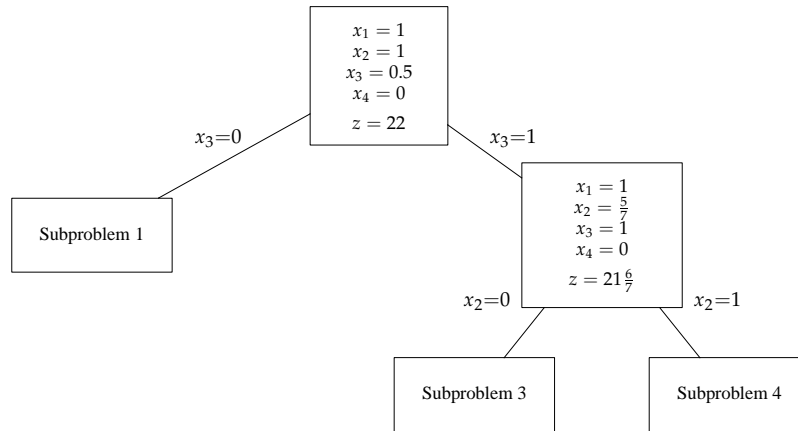


**Subproblem 2:  $x_3 = 1$**

$$\begin{aligned} \text{Max } 8x_1 + 11x_2 + 6 + 4x_4 \\ 5x_1 + 7x_2 + 4 + 3x_4 \leq 14 \end{aligned} \quad \rightarrow \quad \begin{aligned} \text{Max } 6 + 8x_1 + 11x_2 + 4x_4 \\ 5x_1 + 7x_2 + 3x_4 \leq 10 \end{aligned}$$

Again we pick  $x_1 = 1$  and budget reduces to  $10 - 5 = 5$  so we can pick only  $x_2 = 5/7$  from item 2. The total price is  $z = 6 + 8 + 11 \times \frac{5}{7} = 21\frac{6}{7}$ . We branch on  $x_2 = 0$  and  $x_2 = 1$ .





**Subproblem 3:**  $x_2 = 0$

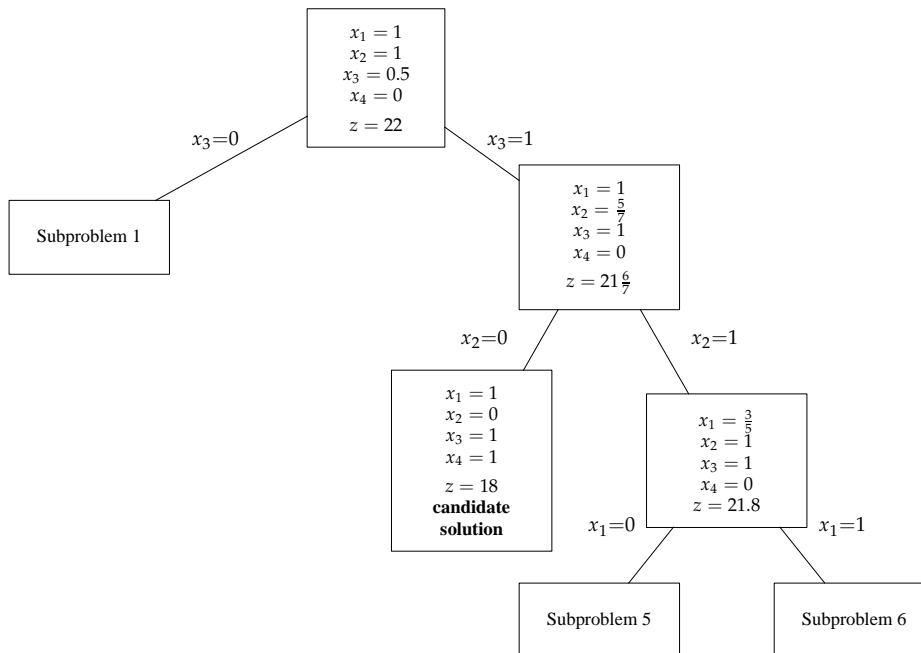
$$\begin{aligned} \text{Max } & 6 + 8x_1 + 4x_4 \\ & 5x_1 + 3x_4 \leq 10 \end{aligned}$$

pick  $x_1 = 1$  and  $x_4 = 1$  (budget is not used up completely)  
the total price is  $z = 6 + 8 + 4 = 18 \rightarrow$  **candidate solution**

**Subproblem 4:**  $x_2 = 1$

$$\begin{aligned} \text{Max } & 17 + 8x_1 + 4x_4 \\ & 5x_1 + 3x_4 \leq 3 \end{aligned}$$

pick  $x_1 = 3/5$  and use up the budget  
the total price is  $z = 17 + 8 \times \frac{3}{5} = 21.8$   
 $\rightarrow$  branch on  $x_1 = 0$  and  $x_1 = 1$



**Subproblem 5:**  $x_1 = 0$

$$\begin{aligned} \text{Max } & 17 + 4x_4 \\ & 3x_4 \leq 3 \end{aligned}$$

pick  $x_4 = 1$  and use up the budget  
the total price is  $z = 17 + 4 = 21 \rightarrow$  **candidate solution**

**Subproblem 6:**  $x_1 = 1$

$$\begin{aligned} \text{Max } & 25 + 4x_4 \\ & 3x_4 \leq -2 \end{aligned}$$

problem **infeasible**

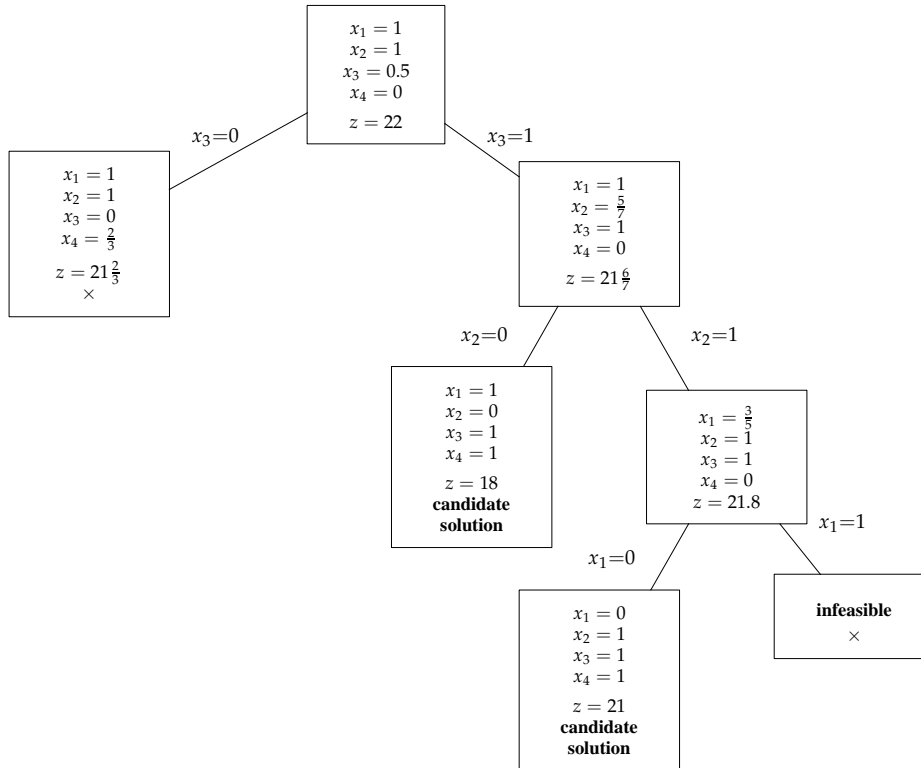
**Subproblem 1:**  $x_3 = 0$ 

$$\begin{aligned} \text{Max } & 8x_1 + 11x_2 + 4x_4 \\ & 5x_1 + 7x_2 + 3x_4 \leq 14 \end{aligned}$$

pick  $x_1 = 1$  then  $x_2 = 1$  which reduces the budget to 2;  
then we pick  $x_4 = 2/3$  to use up the budget.

The total price is  $z = 8 + 11 + 4 \times \frac{2}{3} = 21\frac{2}{3}$

→ we **do not branch**, since  $z$  must be integer and so best integer subproblem of Subproblem 1 has value  $z \leq 21$  but we already have a candidate solution of value  $z = 21$

**Short comparison of Integer Programming Methods**

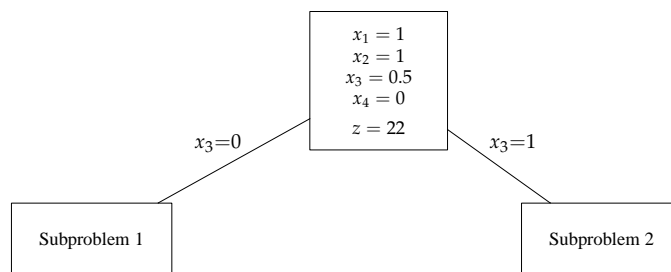
1. Cutting planes – adds a constraint at each step, does not branch, once an integer solution found we stop
2. Branch-and-bound – branches into (independent) subproblems, does not add constraints, only changes bounds on variables, if an integer solution is found, we cannot stop; all subproblems must be explored before we can declare that we found an optimal solution
3. Dynamic programming – building up a solution by reusing (dependent) subproblems; more efficient than Branch-and-Bound but only works for some problems (for instance, it works for the Knapsack Problem but not so much for the Traveling Salesman Problem)

## Dynamic Programming

### Knapsack

Consider the Knapsack problem from the previous lecture.

$$\begin{aligned} \text{Max } z &= 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ &5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ &x_1, x_2, x_3, x_4 \in \{0, 1\} \\ &z \text{ is integer} \end{aligned}$$



**Subproblem 1:**  $x_3 = 0$

$$\begin{aligned} \text{Max } &8x_1 + 11x_2 + 4x_4 \\ &5x_1 + 7x_2 + 3x_4 \leq 14 \end{aligned}$$

**Subproblem 2:**  $x_3 = 1$

$$\begin{aligned} \text{Max } &6 + 8x_1 + 11x_2 + 4x_4 \\ &5x_1 + 7x_2 + 3x_4 \leq 10 \end{aligned}$$

Note that every subproblem is characterized by the set of remaining objects (variables) and the total **budget** (the right-hand side). Observe that the above two subproblems only differ in the value of budget (as we can ignore the absolute constant in the objective function). We should exploit this symmetry.

Instead of solving just for 10 and 14, we solve the subproblem for **all** meaningful values of the right-hand side (here for values  $1, 2, \dots, 14$ ). Having done that, we pick the best solution. This may seem wasteful but it can actually be faster. To make this work efficiently, we branch systematically on variables  $x_1, x_2, \dots$

Let us describe it first more generally. Consider the problem (all coefficients are integers)

$$\begin{aligned} \text{max } &c_1x_1 + c_2x_2 + \dots + c_nx_n \\ &d_1x_1 + d_2x_2 + \dots + d_nx_n \leq B \\ &x_1, \dots, x_n \in \{0, 1\} \end{aligned}$$

For every  $i \in \{1, \dots, n\}$  and  $j \in \{1, \dots, B\}$ , we solve a subproblem with variables  $x_1, \dots, x_i$  and budget  $j$ .

$$\begin{aligned} \max \quad & c_1x_1 + c_2x_2 + \dots + c_ix_i \\ & d_1x_1 + d_2x_2 + \dots + d_ix_i \leq j \\ & x_1, \dots, x_i \in \{0, 1\} \end{aligned}$$

Let  $f_i(j)$  denote the value of this solution. We want the value of  $f_n(B)$ .

**How can we calculate  $f_i(j)$ ?** We observe that the optimal solution consisting of first  $i$  items either contains the  $i$ -th item or it does not. If it does not contain the  $i$ -th item, then the value of  $f_i(j)$  is the same as that of  $f_{i-1}(j)$ ; the best solution using just the first  $i - 1$  items. If, on the other hand, an optimal solution using the first  $i$  items contains the  $i$ -th item, then removing this item from the solution gives an **optimal solution** for **first  $i - 1$  items with budget  $j - d_i$** . (Convince yourself of this fact.) So the value of  $f_i(j)$  is obtained by taking  $f_{i-1}(j - d_i)$  and adding the value  $c_i$  of item  $i$ . We don't know which of the two situations happens, but by taking the better of the two, we always choose correctly.

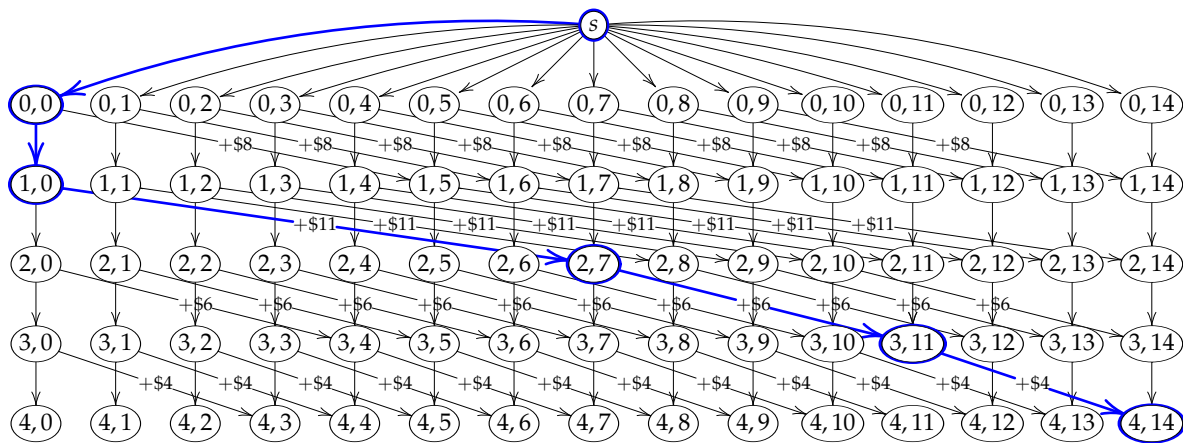
What this shows is that from optimal solutions to smaller subproblems we can build an optimal solution to a larger subproblem. We say that the problem has **optimal substructure**.

As we just described, the function  $f_i(j)$  satisfies the following **recursion**:

$$f_i(j) = \begin{cases} 0 & i = 0 \\ \max \{ f_{i-1}(j), c_i + f_{i-1}(j - d_i) \} & i \geq 1 \end{cases}$$

We can calculate it by filling the table of all possible values.

$f_i(j)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	8	8	8	8	8	8	8	8	8	8
2	0	0	0	0	0	8	8	11	11	11	11	11	19	19	19
3	0	0	0	0	6	8	8	11	11	14	14	17	19	19	19
4	0	0	0	4	6	8	8	11	12	14	15	17	19	19	21



(edges with no cost indicated have \$0 cost)

The longest path (in **bold**) from  $s$  to  $t = (4, 14)$  gives the optimal solution.

**Dynamic programming characteristics**

- Problem can be divided into stages
- Each stage has a (finite) number of possible **states**, each associated a value.
- Next stage only depends on the values of states of the previous stage

### Resource allocation

Knapsack with fixed costs.

**Investment 1:** investing  $x_1 > 0$  dollars yields  $c_1(x_1) = 7x_1 + 2$  dollars, where  $c_1(0) = 0$

**Investment 2:** investing  $x_2 > 0$  dollars yields  $c_2(x_2) = 3x_2 + 7$  dollars, where  $c_2(0) = 0$

**Investment 3:** investing  $x_3 > 0$  dollars yields  $c_3(x_3) = 4x_3 + 5$  dollars, where  $c_3(0) = 0$

(Note that if no money is invested, then there is no yield; the yield is non-linear.)

Suppose that we can invest \$6,000 and each investment must be a multiple of \$1,000. We identify stages

**stages:** in stage  $i$ , we consider only investments  $1, \dots, i$ .

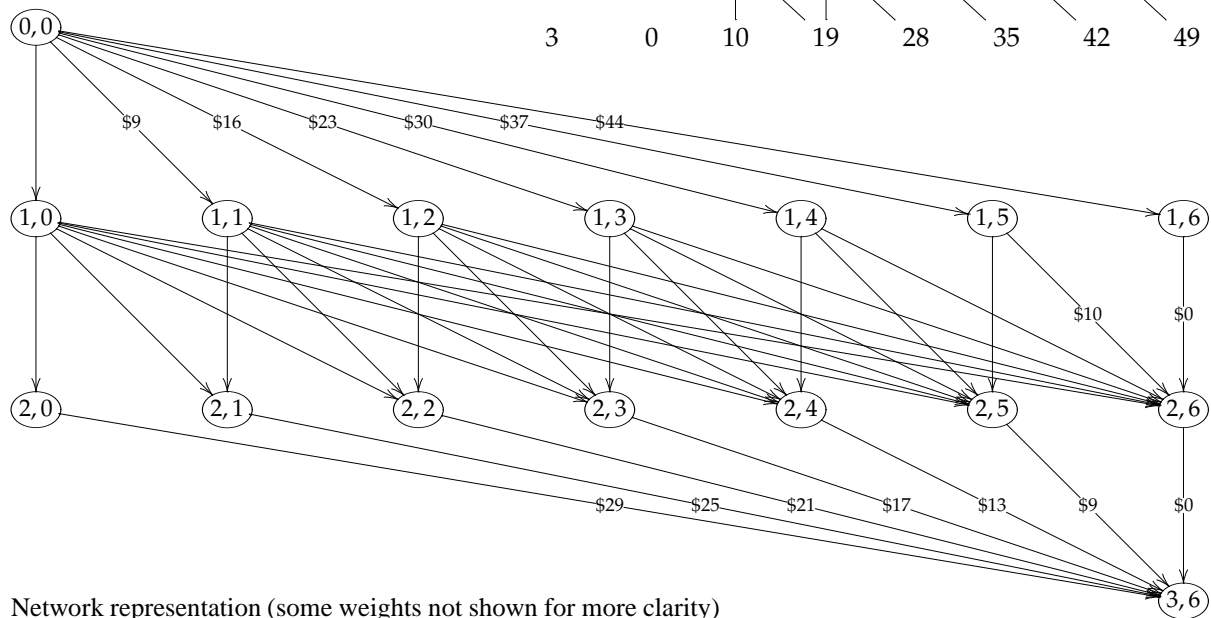
**states:** the available budget as a multiple of \$1,000, up to \$6,000.

**values:**  $f_i(j)$  = maximum yield we can get by investing  $j$  thousands of \$ into investments #1, ..., #i.

$$\text{recursion: } f_i(j) = \begin{cases} 0 & i = 0 \\ \max_{k \in \{0,1,\dots,j\}} \{c_i(k) + f_{i-1}(j-k)\} & i \geq 1 \end{cases}$$

**Note:** the arrow indicates from which subproblem was an optimal solution obtained

stage	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	9	16	23	30	37	44
2	0	10	19	26	33	40	47
3	0	10	19	28	35	42	49



Network representation (some weights not shown for more clarity)

### Inventory problem

A company must meet demand in the next four months as follows: month 1:  $d_1 = 1$  unit, month 2:  $d_2 = 3$  units, month 3:  $d_3 = 2$  units, month 4:  $d_4 = 4$  units. At the beginning of each month it has to be determined how many units to produce. Production has a setup cost of \$3 and then \$1 for each unit. At the end of the month there is holding cost \$0.50 for each unit at hand. The company's warehouse can store up to 4 units from month to month. The capacity

of the production line allows at most 5 units to be produced each month. Initially no products at hand. Determine the minimum cost of production that meets the demand.

**stages:** production until (and including) month  $i$

**states:** number of units at hand at the end of month  $i$

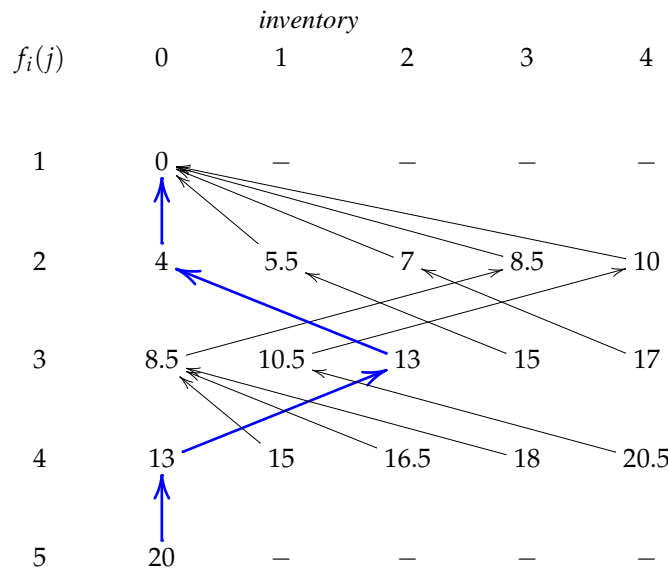
**values:**  $f_i(j)$  = the minimum cost of production that ends in month  $i$  with  $j$  units in the inventory

For example, if we have 4 units at the end of month 2, then we meet month 3 demand of 3 units if we either

- do not produce and are left with 1 unit, held for \$0.50 **total cost \$0.50**
- pay setup cost \$3, produce 1 units for \$1, and are left with 2 units, held for \$1 **total cost \$5.00**
- pay setup cost \$3, produce 2 units for \$2, and are left with 3 units, held for \$1.50 **total cost \$6.50**
- pay setup cost \$3, produce 3 units for \$3, and are left with 4 units, held for \$2 **total cost \$8.00**

Note that in our calculation with already include the holding cost.

$$f_i(j) = \underbrace{j \times \$0.50}_{\text{holding cost}} + \min \begin{cases} f_{i-1}(j + d_i) & \text{no production } k = 0 \\ \underbrace{\$3 + k \times \$1}_{\text{production cost}} + f_{i-1}(j + d_i - k) & \text{production } k \in \{1, 2, \dots, j + d_i\} \end{cases}$$



### Shortest paths

**All-pairs Shortest Paths:** (Floyd-Warshall) given a network  $G = (V, E)$  where each edge  $(u, v) \in E$  has length/cost  $c_{uv}$ , determine the distance between every pair of nodes.

We can solve this problem using dynamic programming as follows. We label the vertices  $v_1, v_2, \dots, v_n$ .

**stages:** in stage  $i$  consider only shortest paths going through intermediate nodes  $v_1, \dots, v_i$

**values:**  $f_i(u, v)$  = minimum length of path from  $u$  to  $v$  whose all intermediate nodes are among  $v_1, \dots, v_i$ .

$$f_0(u, w) = \begin{cases} 0 & u = w \\ c_{uw} & uw \in E \\ \infty & uw \notin E \end{cases}$$

$$f_i(u, w) = \min \{ f_{i-1}(u, w), f_{i-1}(u, v_i) + f_{i-1}(v_i, w) \} \quad \text{for } i \geq 1$$

**Single-source Shortest Paths:** (Bellman-Ford) find distances from a fixed source  $s$  to all other vertices

**stages:** in stage  $i$  consider only shortest paths using at most  $i$  edges

**values:**  $f_i(u)$  = minimum length of a path going from  $s$  to  $u$  and having at most  $i$  edges

We have  $f_0(s) = 0$  and  $f_0(u) = \infty$  for all  $u \neq s$ , since we cannot use any edge at this stage. For later stages:

$$f_i(u) = \min \left\{ f_{i-1}(u), \min_{\substack{v \in V \\ vu \in E}} \{ f_{i-1}(v) + c_{vu} \} \right\}$$

## Knapsack revisited

$$\begin{aligned} \max \quad & c_1x_1 + c_2x_2 + \dots + c_nx_n \\ & d_1x_1 + d_2x_2 + \dots + d_nx_n \leq B \\ & x_1, \dots, x_n \in \{0, 1\} \end{aligned}$$

$f_i(j)$  = optimal solution using first  $i$  items and bag of size  $j$

$$f_i(j) = \begin{cases} -\infty & j < 0 \\ 0 & i = 0 \\ \max \{ f_{i-1}(j), c_i + f_{i-1}(j - d_i) \} & i \geq 1 \end{cases}$$

**Alternative recursion** (only for unbounded Knapsack)

Item	Weight (lbs)	Price (\$)
1	4	11
2	3	7
3	5	12

→ fill a 10 pound bag

$$\begin{aligned} \max \quad & 11x_1 + 7x_2 + 12x_3 \\ & 4x_1 + 3x_2 + 5x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0 \text{ and integer} \end{aligned}$$

$g(j)$  = the maximum total price obtained by filling a  $j$  pound bag

**recursion formula:**  $g(j) = \max_{\substack{i \in \{1,2,3\} \\ j \geq d_i}} \{ c_i + g(j - d_i) \}$

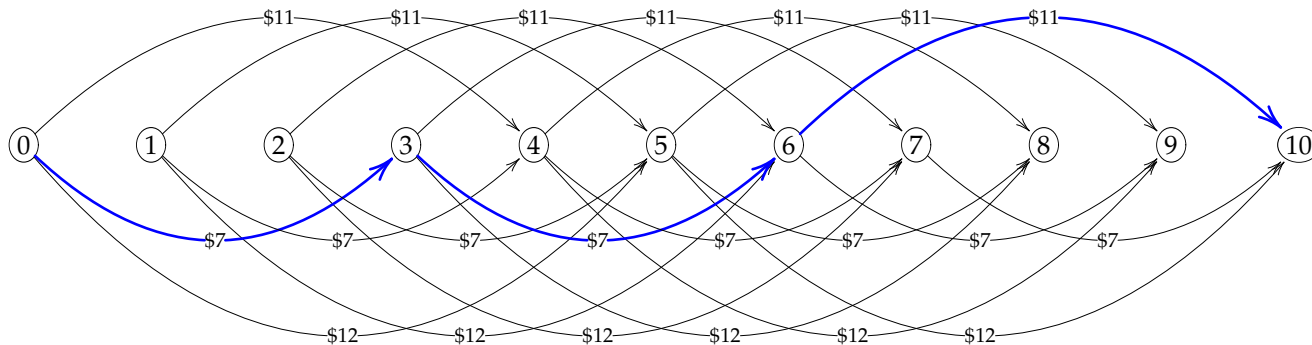
We try to put item  $i$  into the bag and fill the rest as best as possible (note that here we need that we have unlimited number of each item rather than just one; it would not work for binary Knapsack).

$j$	0	1	2	3	4	5	6	7	8	9	10
$g(j)$	0	0	0	7	11	12	14	18	22	23	25

$$\begin{aligned} g(0) &= 0 \\ g(1) &= 0 \\ g(2) &= 0 \\ g(3) &= \max\{c_2 + g(3 - d_2)\} = c_2 + g(0) = 7 + 0 = 7 \\ g(4) &= \max \begin{cases} c_1 + g(0) = 11 + 0 = 11^* \\ c_2 + g(1) = 7 + 0 = 7 \end{cases} \\ g(5) &= \max \begin{cases} c_1 + g(1) = 11 + 0 = 11 \\ c_2 + g(2) = 7 + 0 = 7 \\ c_3 + g(0) = 12 + 0 = 12^* \end{cases} \\ g(6) &= \max \begin{cases} c_1 + g(2) = 11 + 0 = 11 \\ c_2 + g(3) = 7 + 7 = 14^* \\ c_3 + g(1) = 12 + 0 = 12 \end{cases} \\ g(7) &= \max \begin{cases} c_1 + g(3) = 11 + 7 = 18^* \\ c_2 + g(4) = 7 + 11 = 18^* \\ c_3 + g(2) = 12 + 0 = 12 \end{cases} \\ g(8) &= \max \begin{cases} c_1 + g(4) = 11 + 11 = 22^* \\ c_2 + g(5) = 7 + 12 = 19 \\ c_3 + g(3) = 12 + 7 = 19 \end{cases} \\ g(9) &= \max \begin{cases} c_1 + g(5) = 11 + 12 = 23^* \\ c_2 + g(6) = 7 + 14 = 21 \\ c_3 + g(4) = 12 + 11 = 23^* \end{cases} \\ g(10) &= \max \begin{cases} c_1 + g(6) = 11 + 14 = 25^* \\ c_2 + g(7) = 7 + 18 = 25^* \\ c_3 + g(5) = 12 + 12 = 24 \end{cases} \end{aligned}$$

Optimal solution:  $g(10) = c_1 + g(6) = c_1 + (c_2 + g(3)) = c_1 + c_2 + (c_2 + g(0)) = c_1 + c_2 + c_2$

Best solution for a 10 pound bag: pick 2 items #2 and 1 item #1 for a total value of \$25.



Optimal solution (longest path) shown in blue color.

### Scheduling

We have 7 jobs  $j_1, j_2, \dots, j_7$  with processing times  $p_1, p_2, \dots, p_7$  given as follows:

job $j_i$	$j_1$	$j_2$	$j_3$	$j_4$	$j_5$	$j_6$	$j_7$
processing time $p_i$	10	8	6	5	4	4	3

We wish to schedule all 7 jobs on 2 machines. The goal is to minimize the completion time of all jobs.

$$\begin{array}{l} n \text{ jobs } j_1, \dots, j_n \\ \text{processing times } p_1, \dots, p_n \\ m \text{ machines} \\ x_{ij} = \begin{cases} 1 & \text{if job } i \text{ scheduled on machine } j \\ 0 & \text{otherwise} \end{cases} \end{array} \quad \min z \begin{array}{l} \sum_j x_{ij} = 1 \quad i = 1, \dots, n \\ \sum_i p_i x_{ij} \leq z \quad j = 1, \dots, m \\ x_{ij} \in \{0, 1\} \end{array}$$

**stages:** at stage  $i$  we schedule first  $i$  jobs  $j_1, \dots, j_i$

**state:** pair  $(t_1, t_2)$  denoting used up time  $t_1$  on machine #1 and time  $t_2$  on machine #2

**value:**  $f_i(t_1, t_2) = 1$  if it is possible to schedule first  $i$  jobs on the two machines using  $t_1$  time on machine #1 and  $t_2$  time on machine #2;

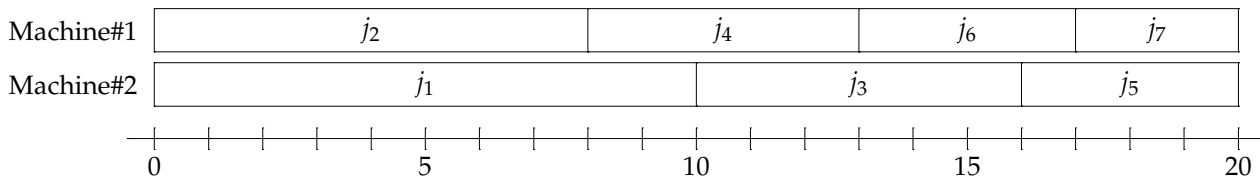
$f_i(t_1, t_2) = 0$  if otherwise

(Note that the order of the jobs does not matter here; only on which machine each job is executed.)

$$\text{recursion: } f_i(t_1, t_2) = \begin{cases} 0 & t_1 < 0 \text{ or } t_2 < 0 \\ 1 & i = 0 \\ \max \{ f_{i-1}(t_1 - p_i, t_2), f_{i-1}(t_1, t_2 - p_i) \} & i \geq 1 \end{cases}$$

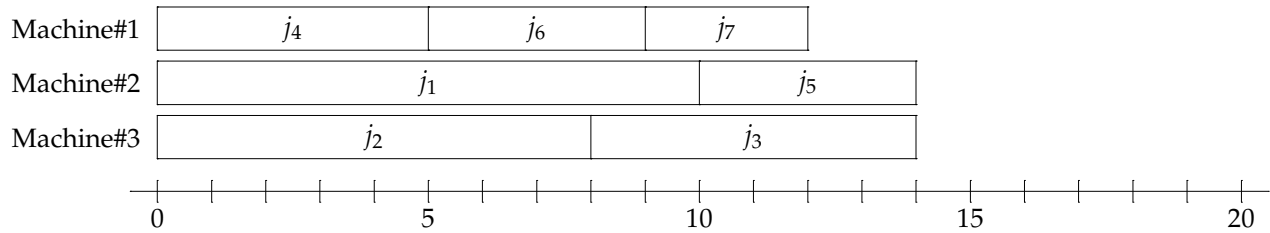
We either execute job  $j_i$  on machine #1 or on machine #2. We pick better of the two options.

**answer:** is given by taking smallest  $t$  such that  $f_7(t, t) = 1$

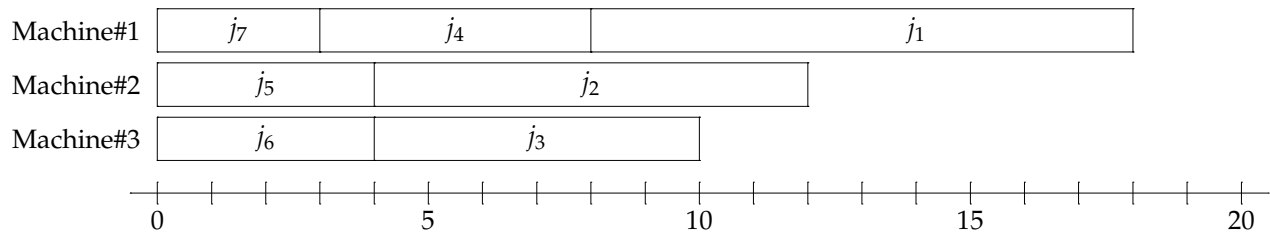


(How would you formulate the problem for 3 machines instead of 2?)





**Average Completion time:** longest job last, assign to machines in a round robin fashion  $\rightarrow$  optimal



Average completion time =  $8\frac{3}{7} \sim 8.43$  (note that moving  $j_7$  to any other machine does not change the average)

## Traveling Salesman

A small company delivers from New York to its customers in neighboring cities (Washington, Pittsburgh, Buffalo, and Boston). The distances (in miles) between cities are as follows.

New York (NY)	228	372	396	211
Washington (WA)	244	381	437	
Pittsburgh (PI)	219	571		
Buffalo (BU)	452			
Boston (BO)				

The company owns one delivery truck. In order to deliver to customers in all the 4 cities, the truck departs from New York and visits each city in turn before heading back to New York. To save on fuel, the company wants to find a route that minimizes the total distance the truck has to travel.

We can think of the route as follows. At each point of our journey, we are in some city  $v$  and before coming to  $v$  we have visited (traveled through) all cities in  $S$  (a subset of cities). Note that neither city  $v$  nor New York (our starting city) is in the set  $S$ .

**stage:** in stage  $S$ , we have travelled through all cities in  $S$  (a selected subset of cities)

**state:** city  $v$ , our current position on the route (after visiting all cities in  $S$ )

**value:**  $f(S, v)$  = the minimum cost of travelling from New York to  $v$  while visiting all cities in  $S$

Our goal is to find  $f(\{WA, PI, BU, BO\}, NY)$ . That is, we visit all cities and come back to New York.

In order to do so, we compute  $f(S, v)$  for all possible values of  $S$  and  $v$ . **How do we do that?** Observe that if we visited all cities in  $S$  and then arrived to  $v$ , we must have arrived to  $v$  from some city  $u$  in the set  $S$ . The total distance of such a journey is then the distance from  $u$  to  $v$  plus the minimum distance we need to travel in order to arrive to  $u$  while visiting all cities in  $S \setminus \{u\}$ . We try all possible choices for  $u$  and select the one that minimizes the total distance.

Let  $c(u, v)$  denote the distance from city  $u$  to city  $v$ . Then the recursion is as follows:

$$\text{recursion: } f(S, v) = \min_{u \in S} \left( f(S \setminus \{u\}, u) + c(u, v) \right)$$

$$\text{initial conditions: } f(\emptyset, v) = c(NY, v)$$

We calculate the values of  $f(S, v)$  for smaller sets first and then gradually for bigger sets. Having done so in this order, we can **reuse** the values we calculated earlier (for smaller sets). This is the **main principle** of dynamic programming. Let us calculate the values.

$$f(\emptyset, WA) = c(NY, WA) = 228$$

$$f(\emptyset, PI) = c(NY, PI) = 372$$

$$f(\emptyset, BU) = c(NY, BU) = 396$$

$$f(\emptyset, BO) = c(NY, BO) = 211$$

$$f(\{WA\}, PI) = \min\{f(\emptyset, WA) + c(WA, PI)\} \\ = 228 + 244 = 472$$

$$f(\{WA\}, BU) = \min\{f(\emptyset, WA) + c(WA, BU)\} \\ = 228 + 381 = 609$$

$$f(\{WA\}, BO) = \min\{f(\emptyset, WA) + c(WA, BO)\} \\ = 228 + 437 = 665$$

$$f(\{PI\}, WA) = \min\{f(\emptyset, PI) + c(PI, WA)\} \\ = 372 + 244 = 616$$

$$f(\{PI\}, BU) = \min\{f(\emptyset, PI) + c(PI, BU)\} \\ = 372 + 219 = 591$$

$$f(\{PI\}, BO) = \min\{f(\emptyset, PI) + c(PI, BO)\} \\ = 372 + 571 = 943$$

$$f(\{BU\}, WA) = \min\{f(\emptyset, BU) + c(BU, WA)\} = 396 + 381 = 777$$

$$f(\{BU\}, PI) = \min\{f(\emptyset, BU) + c(BU, PI)\} = 396 + 219 = 615$$

$$f(\{BU\}, BO) = \min\{f(\emptyset, BU) + c(BU, BO)\} = 396 + 452 = 848$$

$$f(\{BO\}, WA) = \min\{f(\emptyset, BO) + c(BO, WA)\} = 211 + 437 = 648$$

$$f(\{BO\}, PI) = \min\{f(\emptyset, BO) + c(BO, PI)\} = 211 + 571 = 782$$

$$f(\{BO\}, BU) = \min\{f(\emptyset, BO) + c(BO, BU)\} = 211 + 452 = 663$$

$$f(\{WA, PI\}, BU) = \min \begin{cases} f(\{PI\}, WA) + c(WA, BU) = 616 + 381 = 997 \\ f(\{WA\}, PI) + c(PI, BU) = 472 + 219 = 691^* \end{cases}$$

$$f(\{WA, PI\}, BO) = \min \begin{cases} f(\{PI\}, WA) + c(WA, BO) = 616 + 437 = 1053 \\ f(\{WA\}, PI) + c(PI, BO) = 472 + 571 = 1043^* \end{cases}$$

$$f(\{WA, BU\}, PI) = \min \begin{cases} f(\{BU\}, WA) + c(WA, PI) = 777 + 244 = 1021 \\ f(\{WA\}, BU) + c(BU, PI) = 609 + 219 = 828^* \end{cases}$$

$$f(\{WA, BU\}, BO) = \min \begin{cases} f(\{BU\}, WA) + c(WA, BO) = 777 + 437 = 1214 \\ f(\{WA\}, BU) + c(BU, BO) = 609 + 452 = 1061^* \end{cases}$$

$$f(\{WA, BO\}, PI) = \min \begin{cases} f(\{BO\}, WA) + c(WA, PI) = 648 + 244 = 892^* \\ f(\{WA\}, BO) + c(BO, PI) = 665 + 571 = 1236 \end{cases}$$

$$f(\{WA, BO\}, BU) = \min \begin{cases} f(\{BO\}, WA) + c(WA, BU) = 648 + 381 = 1029^* \\ f(\{WA\}, BO) + c(BO, BU) = 665 + 452 = 1117 \end{cases}$$

$$f(\{PI, BU\}, WA) = \min \begin{cases} f(\{PI\}, BU) + c(BU, WA) = 591 + 381 = 972 \\ f(\{BU\}, PI) + c(PI, WA) = 615 + 244 = 859^* \end{cases}$$

$$f(\{PI, BU\}, BO) = \min \begin{cases} f(\{PI\}, BU) + c(BU, BO) = 591 + 452 = 1043^* \\ f(\{BU\}, PI) + c(PI, BO) = 615 + 571 = 1186 \end{cases}$$

$S \setminus v$	WA	PI	BU	BO	NY
$\emptyset$	228	372	396	211	
{WA}		472	609	665	
{PI}	616		519	943	
{BU}	777	615		848	
{BO}	648	782	663		
{WA, PI}			691	1043	
{WA, BU}		828		1061	
{WA, BO}	892	1029			
{PI, BU}	859			1043	
{PI, BO}	1026		1001		
{BU, BO}	1115	882			
{WA, PI, BU}					1143
{WA, PI, BO}				1111	
{WA, BU, BO}		1248			
{PI, BU, BO}	1126				
{WA, PI, BU, BO}					1354

$$f(\{PI, BO\}, WA) = \min \begin{cases} f(\{PI\}, BO) + c(BO, WA) = 943 + 437 = 1380 \\ f(\{BO\}, PI) + c(PI, WA) = 782 + 244 = 1026^* \end{cases}$$

$$f(\{PI, BO\}, BU) = \min \begin{cases} f(\{PI\}, BO) + c(BO, BU) = 943 + 452 = 1395 \\ f(\{BO\}, PI) + c(PI, BU) = 782 + 219 = 1001^* \end{cases}$$

$$f(\{BU, BO\}, WA) = \min \begin{cases} f(\{BU\}, BO) + c(BO, WA) = 848 + 437 = 1285 \\ f(\{BO\}, BU) + c(BU, WA) = 663 + 452 = 1115^* \end{cases}$$

$$f(\{BU, BO\}, PI) = \min \begin{cases} f(\{BU\}, BO) + c(BO, PI) = 848 + 571 = 1419 \\ f(\{BO\}, BU) + c(BU, PI) = 663 + 219 = 882^* \end{cases}$$

$$f(\{WA, PI, BU\}, BO) = \min \begin{cases} f(\{WA, PI\}, BU) + c(BU, BO) = 691 + 452 = 1143^* \\ f(\{WA, BU\}, PI) + c(PI, BO) = 828 + 571 = 1399 \\ f(\{PI, BU\}, WA) + c(WA, BO) = 859 + 437 = 1296 \end{cases}$$

$$f(\{WA, PI, BO\}, BU) = \min \begin{cases} f(\{WA, PI\}, BO) + c(BO, BU) = 1043 + 452 = 1495 \\ f(\{WA, BO\}, PI) + c(PI, BU) = 892 + 219 = 1111^* \\ f(\{PI, BO\}, WA) + c(WA, BU) = 1026 + 381 = 1407 \end{cases}$$

$$f(\{WA, BU, BO\}, PI) = \min \begin{cases} f(\{WA, BU\}, BO) + c(BO, PI) = 1061 + 571 = 1632 \\ f(\{WA, BO\}, BU) + c(BU, PI) = 1029 + 219 = 1248^* \\ f(\{BU, BO\}, WA) + c(WA, PI) = 1115 + 244 = 1359 \end{cases}$$

$$f(\{PI, BU, BO\}, WA) = \min \begin{cases} f(\{PI, BU\}, BO) + c(BO, WA) = 1043 + 437 = 1480 \\ f(\{PI, BO\}, BU) + c(BU, WA) = 1001 + 381 = 1382 \\ f(\{BU, BO\}, PI) + c(PI, WA) = 882 + 244 = 1126^* \end{cases}$$

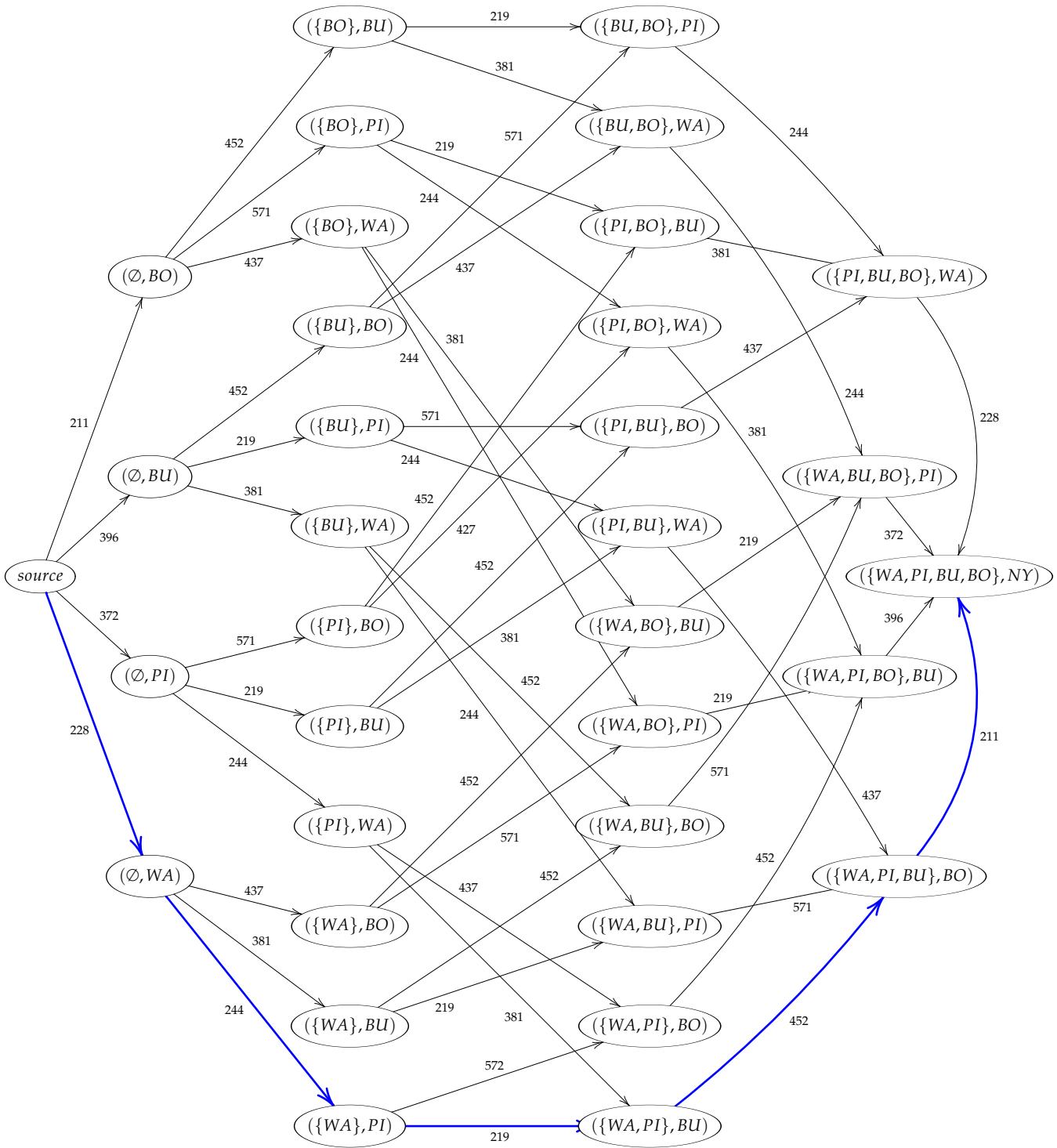
$$f(\{WA, PI, BU, BO\}, NY) = \min \begin{cases} f(\{WA, PI, BU\}, BO) + c(BO, NY) = 1143 + 211 = 1354^* \\ f(\{WA, PI, BO\}, BU) + c(BU, NY) = 1111 + 396 = 1507 \\ f(\{WA, BU, BO\}, PI) + c(PI, NY) = 1248 + 372 = 1620 \\ f(\{PI, BU, BO\}, WA) + c(WA, NY) = 1126 + 228 = 1354^* \end{cases}$$

Optimal solution:

$$\begin{aligned} f(\{WA, PI, BU, BO\}, NY) &= f(\{WA, PI, BU\}, BO) + c(BO, NY) \\ &= f(\{WA, PI\}, BU) + c(BU, BO) + c(BO, NY) \\ &= f(\{WA\}, PI) + c(PI, BU) + c(BU, BO) + c(BO, NY) \\ &= f(\emptyset, WA) + c(WA, PI) + c(PI, BU) + c(BU, BO) + c(BO, NY) \\ &= c(NY, WA) + c(WA, PI) + c(PI, BU) + c(BU, BO) + c(BO, NY) \end{aligned}$$

Travel from New York to Washington to Pittsburgh to Buffalo to Boston and back.

The **total travel distance** 1354 miles.



The cost on edges indicates the increase in distance when that particular action (decision) is taken. Optimal solution shown in blue – longest path from *source* to  $(\{WA, PI, BU, BO\}, NY)$ .

## Analysis of efficiency

In the previous chapters, we have discussed various computational methods for optimization problems such as finding optimal solutions to linear programs (Simplex), finding optimal paths and flows in networks (Dijkstra, Ford-Fulkerson, Network Simplex), solving integer linear programs (Cutting planes, Branch-and-Bound), and finally optimization using recursive methods leveraging memory (dynamic programming).

Some of these methods allowed us to solve the same problem using different approaches (for instance, we can solve shortest paths using the Simplex method, or Dijkstra's algorithm, or dynamic programming). In order to understand the benefits of these different methods, we need to be able to compare them in a uniform way. (We will be deliberately vague; a rigorous treatment of this subject is beyond the scope of this text.)

Let us first discuss in rough numbers the number of steps (operations) of each of the methods.

### Dynamic Programming

- **stages**  $1, 2, \dots, T$
  - **states**  $1, 2, \dots, M$  (for each stage)
  - **next** stage computed from values of **previous** stage
  - **recursive formula** for the value of each state
  - **objective** is the value of a specific state in the **last** stage  $T$
  - using the recursive formula, we calculate the value for **every stage and every state**
    - $M$  stages
    - $T$  states
    - altogether  $M \times T$  states to evaluate
    - for each state and stage we look at all  $M$  states of the previous stage and calculate the best answer  $\rightarrow M$  calculations
- altogether  $M^2 \times T$  calculations using  $M^2$  memory

### Simplex algorithm

The number of steps is proportional to the number of **bases** (dictionaries) we go through. To do so, we need to make sure not to encounter the same basis twice during the calculation (for instance, by Bland's rule).

- $n$  variables
- $m$  equations ( $n \geq m$ )
- each basis consists of a selection of  $m$  variables  $\rightarrow$  at most  $\binom{n}{m}$  different bases

Therefore Simplex method takes at most  $\binom{n}{m}$  pivoting steps. This is roughly  $n^m$  for small  $m$ , but around  $2^n$  for large  $m$  (say  $m = n/2$ ). Note that this seems like a very loose (and pessimistic) estimate. Does the worst-case really happen? Can we get a better general estimate? Unfortunately, there are examples which exhibit this worst-case behaviour  $\rightarrow$

**Klee-Minty examples:**

$$\max \sum_{j=1}^n 10^{n-j} x_j$$

$$\text{s.t. } \left( 2 \sum_{j=1}^{i-1} 10^{i-j} x_j \right) + x_i \leq 100^{i-1} \quad (i = 1, 2, \dots, n)$$

$$x_j \geq 0 \quad (j = 1, 2, \dots, n)$$

for  $n = 3$ , this looks as follows

$$\max 100x_1 + 10x_2 + x_3$$

$$\text{s.t. } \begin{array}{rcl} x_1 & & \leq 1 \\ 20x_1 + x_2 & & \leq 100 \\ 200x_1 + 20x_2 + x_3 & & \leq 10,000 \\ x_1, x_2, x_3 & & \geq 0 \end{array}$$

If at every step the entering variable is chosen to be the one with **largest coefficient** in  $z$ , then the Klee-Minty examples go through  $2^n - 1$  bases before finding the optimum.

**Branch-and-Bound**

- each subproblem is a linear program (solved by Simplex or other methods)
- only bounds on variables change  $\rightarrow$  **size** of the LP is the **same** in each subproblem
- possibly  $2^n$  subproblems (unavoidable in general, even if we branch exactly once on each variable)

**Cutting Planes**

- at each point we have **only one** linear program (no subproblems)
- each step adds one new constraint and one new variable
- the linear program **grows** at each step
- possibly  $2^n$  steps before optimum reached (unavoidable in general)
- could be much worse than Branch-and-Bound once the size of the LP becomes too big (recall that with BnB the LP remains the same size in all subproblems)

**Network Algorithms****Dijkstra**

- $n$  nodes,  $m$  edges  $\rightarrow n$  steps (one for each node)
- each step involves: finding a smallest value  $d(u)$ , and updating other values  $d(v)$   
 $\rightarrow$  roughly  $\approx 2n$  calculations

altogether  $\approx 2n^2$  operations (can be improved to  $\approx m \log n$  with special data structures)

**Ford-Fulkerson (Augmenting path algorithm)**

- $n$  nodes,  $m$  edges
- each step **constructs** the residual network, finds an **augmenting path** and augments the flow  
 $\rightarrow$  roughly  $\approx 2(n + m)$  operations for each step
- at most  $n \times m$  steps needed if shortest augmenting path is used (Edmonds-Karp)

altogether  $\approx 2n^2 m$  operations needed (can be improved to  $\approx nm$  by additional tricks)

**Network Simplex (Cycle cancelling algorithm)**

- $n$  nodes,  $m$  edges
- each step **calculates** shadow prices and reduced costs, finds a cycle (**loop**) and adjusts the flow  
 $\rightarrow$  roughly  $\approx 2n + m$  operations for each step
- at most  $\approx nm \log(Cn)$  steps needed if minimum mean-cost cycle is used (Goldberg-Tarjan)  
where  $C$  is largest cost of an edge

altogether  $\approx 2n^2 m \log(Cn)$  operations needed (can be improved to  $\approx nm$  by additional tricks)

**Other Network Problems**

$n$  nodes,  $m$  edges

1. **Transportation Problem:** same as Network Simplex  $\approx nm$
2. **Assignment Problem:** the number of required steps can be shown to be at most  $\sqrt{n}$   
 $\rightarrow$  altogether  $\approx \sqrt{nm}$  operations need
3. **All-pairs shortest paths (Floyd-Warshall):**
  - nodes are labeled  $v_1, \dots, v_n$
  - at each step  $i$  we improve estimate  $d(u, v)$  on the distance between  $u$  and  $v$  by considering paths from  $u$  to  $v$  passing through  $v_i$  (and some of the  $v_1, \dots, v_{i-1}$ )  $\rightarrow n^3$  calculations in each step altogether  $n^4$  operations
4. **Single-source shortest path (Bellman-Ford)** with general edge costs (negative costs allowed)
  - at step  $i$  we consider paths using  $i$  edges  $\rightarrow m$  calculations in each step altogether  $n \times m$  operations

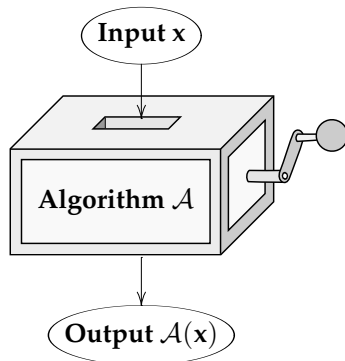
## 14.1 Algorithmic Complexity

a way to compare efficiency (speed) of different computational methods (algorithms) and corresponding computational problems. **How to measure efficiency of algorithms?** Standard practice is to ask:

“How does the algorithm scale when inputs get large?”

i.e., how much **time** or **memory**  $f(n)$  it takes to compute a solution if the input data has size  $n$ .

### Algorithm



An *algorithm*  $\mathcal{A}$

- takes an **input**  $\mathbf{x}$  (numbers),
- performs a certain number of operations (additions, multiplications),
- produces an **output**  $\mathcal{A}(\mathbf{x})$  (a number, answer yes/no)

**time complexity** = number of (elementary) operations performed

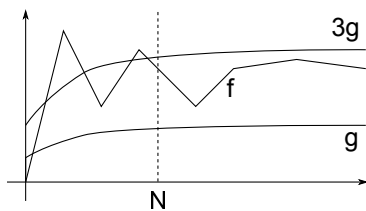
**space complexity** = size of memory used during the computation

**time**  $\geq$  **space**

$f(n)$  = **worst case** time complexity of  $\mathcal{A}$  for inputs of size  $n$   
 (maximum number of operations that  $\mathcal{A}$  performs on any input of size  $n$ )

### Asymptotic notation

We are interested in the behaviour of  $f(n)$  as  $n$  goes to infinity. We use the following asymptotic notation.



For two functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ , we say that  $f$  is  $O(g)$  “the order of  $g$ ”, and write  $f = O(g)$ , if there exist constants  $c > 0$  and  $N > 0$  such that

$$f(n) \leq c \cdot g(n) \text{ for all } n > N.$$

**Example:** let  $f(n) = n^2 + n + 1$  and  $g(n) = n^2$ . Then  $f = O(g)$ , since for  $c = 2$  and  $n > N = 2$  we have

$$f(n) = n^2 + n + 1 \leq 2 \cdot g(n) = 2n^2$$

### Growth of functions

We need to highlight the growth of various complexity functions we shall encounter.





<b>Inventory problem</b> (Dynamic program): $n$ months, warehouse size $M$	$\left  \begin{array}{l} O(n \cdot M^2) \\ O(M^n) \end{array} \right $	$\left  \begin{array}{l} O(n \cdot M) \\ O(n) \end{array} \right $
<b>Inventory problem</b> (Integer LP)		

**Important note:** recall that when we solving linear programs we have no guarantee that the solution we obtain will be an integer point, even if there exists an optimal integer point. Finding an integer solution to a linear program is in general difficult.

However, this is not the case for Network problems (Shortest paths, Maximum flow, Minimum-cost flow).

**Theorem 6.** *If all capacities/source supplies/destination demands/node net supplies are integers, then*

- (i) Every **basic feasible solution** to the Linear Program for Shortest path Problem (Maximum flow Problem, Transportation Problem, Minimum-cost flow Problem) is integral (all variables are integers).
- (ii) There exists an **optimal integer** solution (to the above problems) because one such a solution is basic.
- (iii) Moreover, Dijkstra's (Ford-Fulkerson, Transportation Simplex, Network Simplex) algorithm finds this solution efficiently (in polynomial time).

So if your problem is a Network problem (or you can turn your LP into an equivalent Network problem), then optimal integer solution can be found efficiently (unlike using purely ILP techniques).