التكاملات الثنائية

تعريف:

مثال 3 ص 164: احسب التكامل: $I=\int_0^1\int_x^{2x}(2x+3y^2)\,dy\,dx$

الحل:

$$I=\int_0^1\int_x^{2x}(2x+3y^2)\,dy\,dx \qquad R_x=\{(x,y): 0\le x\le 1,\ x\le y\le 2x\}$$

$$=\int_0^1\left(\int_x^{2x}(2x+3y^2)\,dy\right)dx$$

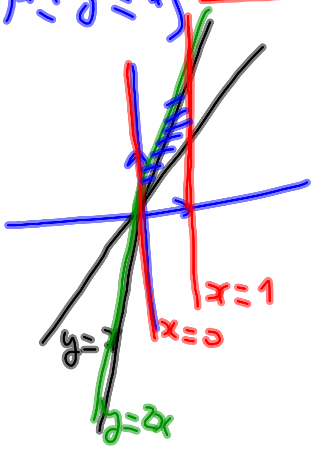
$$=\int_0^1 \left[2xy+y^3\right]_x^{2x}dx$$

$$=\int_0^1\left(\left(2x(2x)+(2x)^3\right)-\left(2x\cdot x+x^3\right)\right)dx$$

$$=\int_0^1\left(4x^2+8x^3-2x^2-x^3\right)dx=\int_0^1\left(2x^2+7x^3\right)dx$$

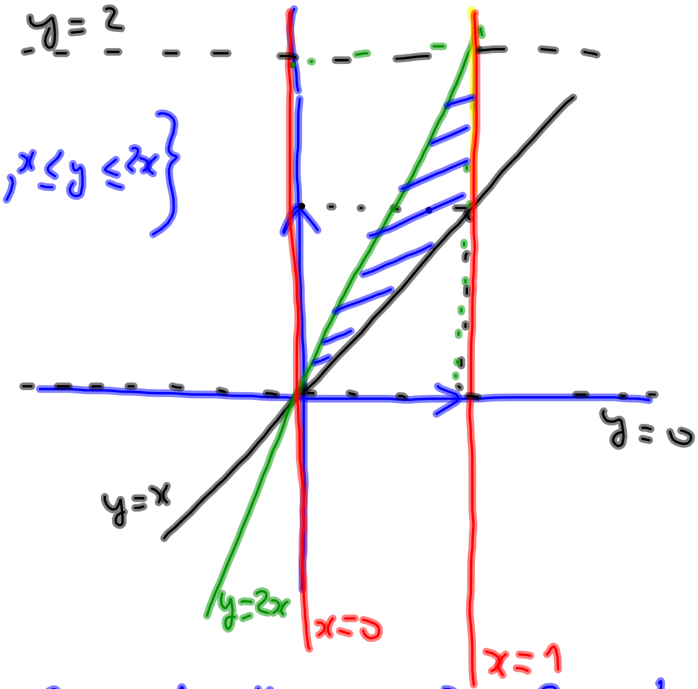
$$=\left[\frac{2}{3}x^3+\frac{7}{4}x^4\right]_0^1=\left(\frac{2}{3}(1^3)+\frac{7}{4}(1^4)\right)-\left(\frac{2}{3}(0^3)+\frac{7}{4}(0^4)\right)$$

$$\boxed{I=\frac{2}{3}+\frac{7}{4}=\frac{8+21}{12}=\frac{29}{12}}$$



ملحوظة:

$R = R_x = \{(x,y) : 0 \le x \le 1 , x \le y \le 2x\}$



$y = 2x$

$x = \frac{y}{2}$

$R = R_y = \{(x,y) : 0 \le y \le 1 , \frac{y}{2} \le x \le y\} \cup \{(x,y) : 1 \le y \le 2 , \frac{y}{2} \le x \le 1\}$

$I = \iint_{R_y} (2x + 3y^2)\, dx\, dy = \int_0^1 \int_{\frac{y}{2}}^{y} (2x + 3y^2)\, dx\, dy + \int_1^2 \int_{\frac{y}{2}} (2x + 3y^2)\, dx\, dy$

مثال 4 ص 164 : احسب التكامل

$$I = \int_0^{\frac{\pi}{2}} \int_0^{y} 2x \sin y^3 \, dx \, dy$$

الحل :

$$I = \int_0^{\frac{\pi}{2}} \int_0^{y} 2x \sin y^3 \, dx \, dy = \int_0^{\frac{\pi}{2}} \left(\int_0^{y} 2x \sin y^3 \, dx \right) dy$$

$$= \int_0^{\frac{\pi}{2}} \left. x^2 \sin y^3 \right]_0^{y} dy = \int_0^{\frac{\pi}{2}} \left(y^2 \sin y^3 - 0^2 \sin y^3 \right) dy$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{2}} 3y^2 \sin y^3 \, dy = \frac{1}{3} \left. \left(-\cos y^3 \right) \right]_0^{\frac{\pi}{2}}$$

$$I = \frac{1}{3} \left(-\cos\left(\frac{\pi}{2}\right)^3 + \cos 0^3 \right) = \frac{1}{3} \left(1 - \cos\left(\frac{\pi^3}{8}\right) \right)$$

مثال ٢ ص 16 : احسب التكامل : $I = \iint_R f(x,y)\,dA$

حيث : $f(x,y) = 3x + 2y$ ، $R_x = R = \{ (x,y) : -1 \le x \le 1 ,\ x^3 \le y \le x+1 \}$

$$I = \iint_R (3x+2y)\,dA = \int_{-1}^{1}\left(\int_{x^3}^{x+1} (3x+2y)\,dy\right)dx$$

$$= \int_{-1}^{1} \left[3xy + y^2\right]_{x^3}^{x+1} dx = \int_{-1}^{1} \left(\left(3x(x+1) + (x+1)^2\right) - \left(3x\,x^3 + (x^3)^2\right)\right)dx$$

$$= \int_{-1}^{1} \left(3x^2 + 3x + x^2 + 2x + 1 - 3x^4 - x^6\right)dx$$

$$= \int_{-1}^{1} \left(1 + 5x + 4x^2 - 3x^4 - x^6\right)dx = 2\int_{0}^{1}\left(1 + 4x^2 - 3x^4 - x^6\right)dx$$

$$= \left[x + \frac{5}{2}x^2 + \frac{4}{3}x^3 - \frac{3}{5}x^5 - \frac{1}{7}x^7\right]_{-1}^{1}$$

$$= \left(1 + \frac{5}{2}\cdot 1^2 + \frac{4}{3}\cdot 1^3 - \frac{3}{5}\cdot 1^5 - \frac{1}{7}\cdot 1^7\right) - \left(-1 + \frac{5}{2}(-1)^2 + \frac{4}{3}(-1)^3 - \frac{3}{5}(-1)^5 - \frac{1}{7}(-1 \ldots\right.$$

$$= 2 + \frac{8}{3} - \frac{6}{5} - \frac{2}{7} = \frac{210 + 8(35) - 6(21) - 2(15)}{105}$$

$$\boxed{I = \frac{334}{105}}$$

تذكير: (1) f دالة زوجية على المجال I إذا كان

لكل $x \in I$ فإن $-x \in I$ و $f(-x)=f(x)$

وفي هاته الحالة: $\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx}$

(2) f دالة فردية على المجال I إذا كان

لكل $x \in I$ فإن $-x \in I$ و $f(-x)=-f(x)$

وفي هاته الحالة

$$\boxed{\int_{-a}^{a} f(x)\,dx = 0}$$

مثال 6) ص 165: احسب التكامل $I = \iint_R f(x,y)\,dA$

حيث إن $f(x,y) = x^3 + 4y$, R المنطقة المستوية المحدودة بالمنحنيين

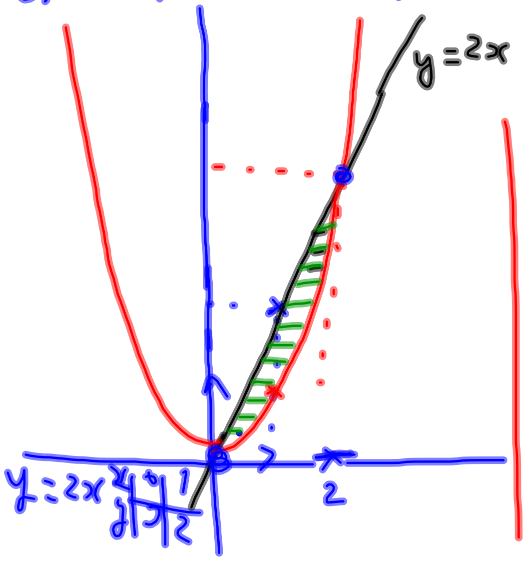
$y = x^2$, $y = 2x$

الحل: لدينا: $x^2 = 2x$

$\Leftrightarrow x^2 - 2x = 0$

$\Leftrightarrow x(x-2) = 0$

$\Leftrightarrow x = 0$ أو $x = 2$



$y = 2x$

x	0	1
y	0	2

$y = x^2$

x	0	$\frac{1}{2}$	1	2
y	0	$\frac{1}{4}$	1	4

$R = R_x = \{(x,y) : 0 \le x \le 2 ,\ x^2 \le y \le 2x\}$

$R = R_y = \{(x,y) : 0 \le y \le 4 ,\ \frac{y}{2} \le x \le \sqrt{y}\}$

ملاحظة:

$y = 2x \Leftrightarrow x = \frac{y}{2}$

$y = x^2 \Leftrightarrow x = \sqrt{y}$

$$I = \iint_R (x^3 + 4y)\, dA = \int_0^2 \left(\int_{x^2}^{2x} (x^3 + 4y)\, dy \right) dx$$

$$= \int_0^2 \left[x^3 y + 2y^2 \right]_{x^2}^{2x} dx$$

$$= \int_0^2 \left(\left(x^3 \cdot 2x + 2(2x)^2 \right) - \left(x^3 \cdot x^2 + 2(x^2)^2 \right) \right) dx$$

$$= \int_0^2 \left(\underline{2x^4} + 8x^2 - x^5 - \underline{2x^4} \right) dx$$

$$= \int_0^2 (8x^2 - x^5)\, dx = \frac{8}{3}x^3 - \frac{x^6}{6} \Big]_0^2$$

$$= \left(\frac{8}{3} 2^3 - \frac{2^6}{6} \right) - 0 = \frac{64}{3} - \frac{2^5}{3} = \frac{64}{3} - \frac{32}{3}$$

$$\boxed{I = \frac{32}{3}}$$

سنا (7 ص66) !: احسب التكامل $I = \iint_R f(x,y)\,dA$

I هو نفس المثال 6، أوجد I بطريقة ثانية.

الحل:

$$I = \int_0^4 \left(\int_{\frac{y}{2}}^{\sqrt{y}} (x^3 + 4y)\,dx \right) dy$$

$$R_y = \left\{ (x,y) : 0 \le y \le 4,\ \frac{y}{2} \le x \le \sqrt{y} \right\}$$

$$I = \iint_R f(x,y)\,dA = \iint_{R_y} (x^3 + 4y)\,dx\,dy$$

$$= \int_0^4 \left[\frac{x^4}{4} + 4xy \right]_{\frac{y}{2}}^{\sqrt{y}} dy$$

$$= \int_0^4 \left(\frac{y^2}{4} + 4\sqrt{y}\cdot y - \frac{\left(\frac{y}{2}\right)^4}{4} - 4\frac{y}{2}y \right) dy$$

$$I = \int_0^4 \left(-\frac{7}{4}y^2 + 4y^{\frac{3}{2}} - \frac{y^4}{64} \right) dy.$$

$$= \left[-\frac{7}{4}\frac{y^3}{3} + 4\cdot\frac{y^{\frac{5}{2}}}{\frac{5}{2}} - \frac{y^5}{5.64} \right]_0^4$$

$$I = -\frac{7}{4}\cdot\frac{4^3}{3} + \frac{8}{5}2^5 - \frac{4^5}{5.64}.$$

$$= -\frac{7.16}{3} + \frac{256}{5} - \frac{16}{5} = -\frac{7.16}{3} + 48 = \frac{-7.16 + 9.16}{3}$$

$$\boxed{I = \frac{16.2}{3} = \frac{32}{3}}$$

$I = \iint_A f(x,y)\, dA$ اكتب التكامل 168 مثال 8 ص

التكامل متعاقب حيث أن f دالة متصلة, المجال R

هو المنطقة المستوية المحدودة بالمنحنيات

$y = \sqrt{x}$ و $y = 0$, $y = \sqrt{3x-18}$

الحل:

$3x - 18 \geq 0$

$\Leftrightarrow x \geq 6$

$x \geq 0$

$y = \sqrt{x}$

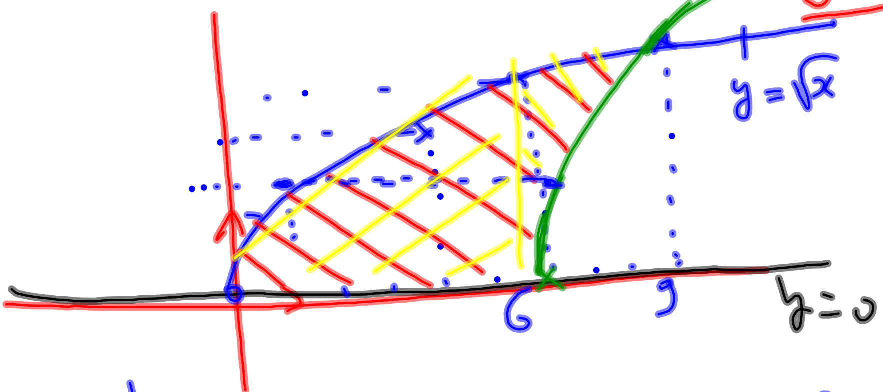
x	0	1	4
y	0	1	2

$y = \sqrt{3x-18}$

x	6	9
y	0	3

$\sqrt{3x-18} = \sqrt{x}$

$\Leftrightarrow 3x - 18 = x \Leftrightarrow \boxed{x = 9}$



$R_y = \left\{(x,y) : 0 \leq y \leq 3 ,\ y^2 \leq x \leq \frac{y^2}{3} + 6\right\}$

$y = \sqrt{x} \Leftrightarrow x = y^2$

$y = \sqrt{3x-18} \Leftrightarrow 3x - 18 = y^2$

$\Leftrightarrow x = \frac{y^2+18}{3} = \frac{y^2}{3} + 6$

ملاحظة:

$$I = \int_0^3 \left(\int_{y^2}^{\frac{y^2}{3}+6} f(x,y)\, dx \right) dy$$

$R_x = \left\{(x,y) : 0 \leq x \leq 6 ,\ 0 \leq y \leq \sqrt{x}\right\} \cup \left\{(x,y) : 6 \leq x \leq 9 ,\ \sqrt{3x-18} \leq y \leq \sqrt{x}\right\}$

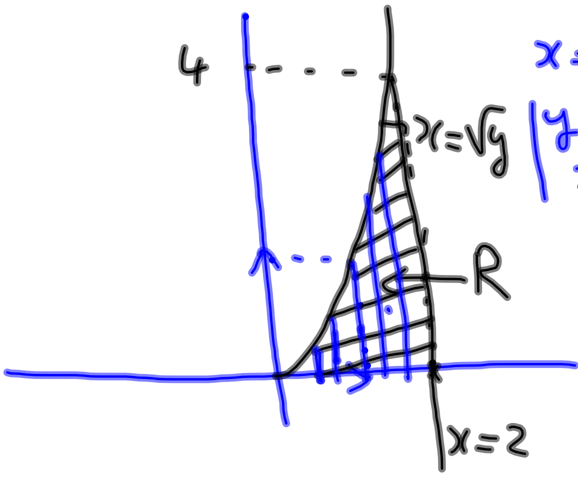
$$I = \int_0^6 \left(\int_0^{\sqrt{x}} f(x,y)\, dy \right) dx + \int_6^9 \left(\int_{\sqrt{3x-18}}^{\sqrt{x}} f(x,y)\, dy \right) dx$$

مثال ص 169 : احسب التكامل : $I = \int_0^4 \int_{\sqrt{y}}^2 y \cos x^5 \, dx \, dy$

الحل :

$R_y = \{(x,y) : 0 \le y \le 4 \, , \sqrt{y} \le x \le 2\}$

$x = \sqrt{y}$ | $y = x^2$, $x > 0$



$R = R_x = \{(x,y) : 0 \le x \le 2 \, , 0 \le y \le x^2\}$

$$I = \int_0^4 \left(\int_{\sqrt{y}}^2 y \cos x^5 \, dx \right) dy$$

$$= \int_0^2 \left(\int_0^{x^2} y \cos x^5 \, dy \right) dx$$

$$= \int_0^2 \left. \frac{y^2}{2} \cos x^5 \right]_0^{x^2} dx$$

$$= \int_0^2 \left(\frac{(x^2)^2}{2} \cos x^5 - 0 \right) dx$$

$$= \frac{1}{5} \int_0^2 5 \frac{x^4}{2} \cos x^5 \, dx$$

$$= \frac{1}{10} \left. \sin x^5 \right]_0^2 = \frac{1}{10} \sin 32 - 0$$

$$\boxed{I = \frac{1}{10} \sin 32}$$

مثال 10 ص 170 : احسب التكامل :

$$I=\int_0^8\int_{\sqrt[3]{x}}^{2}\frac{1}{y^4+1}\,dy\,dx$$

الحل :

$$R_x=\{(x,y):\ 0\le x\le 8,\ \sqrt[3]{x}\le y\le 2\}$$

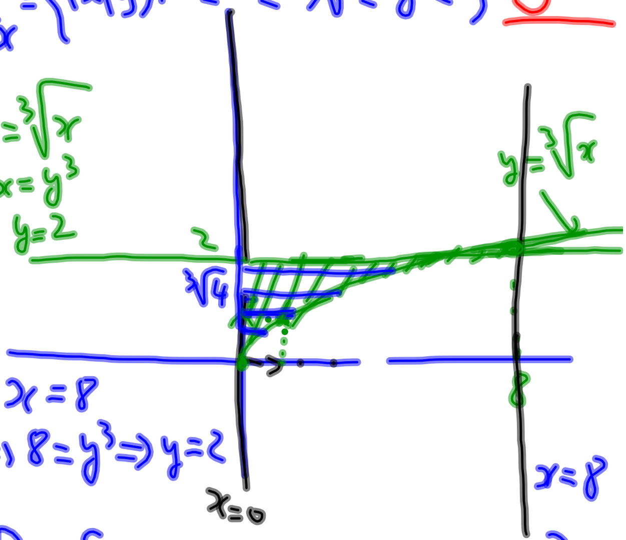
$y=\sqrt[3]{x}$

$\Leftrightarrow x=y^3$

$y=2$

$x=8$

$\Rightarrow 8=y^3\Rightarrow y=2$



$$R_y=\{(x,y):\ 0\le y\le 2,\ 0\le x\le y^3\}$$

$$I=\int_0^8\left(\int_{\sqrt[3]{x}}^{2}\frac{1}{y^4+1}\,dy\right)dx$$

$$I=\int_0^2\left(\int_0^{y^3}\frac{1}{y^4+1}\,dx\right)dy$$

$$=\int_0^2\left.\frac{x}{y^4+1}\right]_0^{y^3}dy$$

$$=\frac{1}{4}\int_0^2\left(\frac{4y^3}{y^4+1}-\frac{0}{y^4+1}\right)dy$$

$$=\frac{1}{4}\Big[\ln|y^4+1|\Big]_0^2$$

$$=\frac{1}{4}\ln(17)$$

$$\boxed{I=\frac{1}{4}\ln 17}$$

مجتهدة :