# Demonic fuzzy operators

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## Abstract

We deal with a relational algebra model to define a refinement fuzzy ordering (*demonic fuzzy inclusion*) and also the associated fuzzy operations which are fuzzy demonic join ($\sqcup_{fuz}$), fuzzy demonic meet ($\sqcap_{fuz}$) and fuzzy demonic composition ($\Box_{fuz}$). We give also some properties of these operations, and illustrate them with simple examples. Our formalism is the relational algebra.

Keywords: Fuzzy sets, demonic operators, demonic fuzzy operators, demonic fuzzy ordering.

## 1 Relation Algebras

Our mathematical tool is abstract relation algebra [8, 28, 30], which we now introduce.

(1) **Definition.** A *(homogeneous) relation algebra* is a structure $(\mathcal{R}, \cup, \cap, ^-, ^\smile, ;)$ over a non-empty set $\mathcal{R}$ of elements, called *relations*. The following conditions are satisfied.

- $(\mathcal{R}, \cup, \cap, ^-)$ is a complete Boolean algebra, with *zero* element $\emptyset$, *universal* element $L$ and ordering $\subseteq$.
- *Composition*, denoted by (;), is associative and has an identity element, denoted by $I$.
- The Schröder rule is satisfied: $P;Q \subseteq R \Leftrightarrow P^\smile;\overline{R} \subseteq \overline{Q} \Leftrightarrow \overline{R};Q^\smile \subseteq \overline{P}$.
- $L;R;L = L \Leftrightarrow R \neq \emptyset$ (Tarski rule).

The relation $R^\smile$ is called the *converse* of $R$. The standard model of the above axioms is the set $\wp(S\times S)$ of all subsets of $S \times S$. In this model, $\cup, \cap, ^-$ are the usual *union, intersection* and *complement*, respectively; the relation $\emptyset$ is the empty relation, the universal relation is $L = S \times S$ and the identity relation is $I = \{(s, s') \mid s' = s\}$. Converse and composition are defined by
$R^\smile = \{(s, s') \mid (s', s) \in R\}$ and $Q;R = \{(s, s') \mid \exists s'' : (s, s'') \in Q \wedge (s'', s') \in R\}$.

The precedence of the relational operators from highest to lowest is the following: $^-$ and $^\smile$ bind equally, followed by ;, then by $\cap$, and finally by $\cup$. From now on, the composition operator symbol ; will be omitted (that is, we write $QR$ for $Q;R$). From Definition 1, the usual rules of the calculus of relations can be derived (see, e.g., [6, 8, 28]). We assume these rules to be known and simply recall a few of them.

(2) **Theorem.** *Let $P, Q, R$ be relations. Then,*

- $\overline{Q \cup R} = \overline{Q} \cap \overline{R}$,
- $\overline{Q \cap R} = \overline{Q} \cup \overline{R}$,
- $Q \cap R \cup \overline{R} = Q \cup ^-$

- $P \cap Q \subseteq R \;\Leftrightarrow\; P \subseteq \overline{Q} \cup R,$
- $Q \subseteq R \;\Leftrightarrow\; \overline{R} \subseteq \overline{Q},$
- $P(Q \cap R) \subseteq PQ \cap PR,$
- $(P \cap Q)R \subseteq PR \cap QR,$
- $P(Q \cup R) = PQ \cup PR,$
- $(P \cup Q)R = PR \cup QR,$
- $Q \subseteq R \;\Rightarrow\; PQ \subseteq PR,$
- $Q \subseteq R \;\Rightarrow\; QP \subseteq RP.$
- $\overline{\overline{R}L}L = \overline{R}L,$
- $PQ \cap R \subseteq P(Q \cap P^{\smile}R),$
- $(P \cap QL)R = PR \cap QL,$
- $(\bigcap_{i \in X} R_i L)L = \bigcap_{i \in X} R_i L.$

# 2 Fuzzy Relation

Fuzzy relations are fuzzy subsets of $A \times B$, that is, mapping from $A \to B$. They have been studied by a number of authors, in particular by Zadeh [38],[39], Kaufmann [20], and Rosenfeld [26]. Applications of fuzzy relations are widespread and important.

(3) **Definition.** Let $A, B \in U$ be universal sets, a fuzzy relation $\tilde{R}$ on $A \times B$ is defined by;
$\tilde{R} = \{((x,y), \mu_{\tilde{R}}(x,y) \mid (x,y) \in A \times B, \mu_{\tilde{R}}(x,y) \in [0,1]\}$ is called a *Fuzzy relation* on $A \times B$.

(4) **Example.**
$\tilde{R} =$ "$x$ considerably larger than $y$, we have: ,

$$\tilde{R} = \begin{pmatrix} 0.8 & 1 & 0.1 & 0.7 \\ 0 & 0.8 & 0 & 0 \\ 0.9 & 1 & 0.7 & 0.8 \end{pmatrix},$$

and, $\tilde{S} =$ "$y$ very close to$x$"

$$\tilde{S} = \begin{pmatrix} 0.4 & 0 & 0.9 & 0.6 \\ 0.9 & 0.4 & 0.5 & 0.7 \\ 0.3 & 0 & 0.8 & 0.5 \end{pmatrix}$$

## 2.1 Basic Operations On Fuzzy Relations

(5) **Definition.** Let $\tilde{R}$ and $\tilde{S}$ be two fuzzy relations on $A \times B$. Then:

- Union: $\mu_{\tilde{R}\cup\tilde{S}}(x,y) = max\{\mu_{\tilde{R}}(x,y), \mu_{\tilde{S}}(x,y)\},$
- Intersection: $\mu_{\tilde{R}\cap\tilde{S}}(x,y) = min\{\mu_{\tilde{R}}(x,y), \mu_{\tilde{S}}(x,y)\},$
- Max-min composition:
  $\tilde{R}\circ\tilde{S} = \{[(x,z), max_y\{min\{\mu_{\tilde{R}}(x,y), \mu_{\tilde{S}}(y,z)\}\}]\},$

(6) **Example.**

- $\tilde{R} = \begin{pmatrix} 0.8 & 1 & 0.1 \\ 0 & 0.8 & 0 \\ 0.9 & 1 & 0.7 \end{pmatrix},$
- $\tilde{S} = \begin{pmatrix} 0.4 & 0 & 0.9 \\ 0.9 & 0.4 & 0.5 \\ 0.3 & 0 & 0.8 \end{pmatrix}$
- $\tilde{R} \cup \tilde{S} = \begin{pmatrix} 0.8 & 1 & 0.9 \\ 0.9 & 0.8 & 0.5 \\ 0.9 & 1 & 0.8 \end{pmatrix},$
- $\tilde{R} \cap \tilde{S} = \begin{pmatrix} 0.4 & 0 & 0.1 \\ 0 & 0.4 & 0 \\ 0.3 & 0 & 0.7 \end{pmatrix},$
- $\tilde{R} \circ \tilde{S} = \begin{pmatrix} 0.9 & 0.4 & 0.8 \\ 0.8 & 0.4 & 0.5 \\ 0.9 & 0.4 & 0.9 \end{pmatrix}$

(7) **Theorem.** *Let $\tilde{R}$ be a fuzzy relation on $A \times A$.*

- *$\tilde{R}$ is reflexive [39] iff $\mu_{\tilde{R}}(x,x) = 1 \;\forall x \in A$*
- *$\tilde{R}$ is* $\varepsilon$-reflective *[40] iff $\mu_{\tilde{R}}(x,x) \geq \varepsilon \;\forall x \in A$*
- *$\tilde{R}$ is* weakly reflexive *[40] iff*
  $\mu_{\tilde{R}}(x,y) \leq \mu_{\tilde{R}}(x,x) \;\forall x,y \in A$
  $\mu_{\tilde{R}}(y,x) \leq \mu_{\tilde{R}}(x,x) \;\forall x,y \in A$
- *$\tilde{R}$ is symmetric iff $\tilde{R}(x,y) = \tilde{R}(y,x)$.*

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- $\tilde{R}$ *is antisymmetric [20] iff for* $x \neq y$ *either* $\mu_{\tilde{R}}(x,y) \neq \mu_{\tilde{R}}(y,x)$ *or* $\mu_{\tilde{R}}(x,y) = \mu_{\tilde{R}}(y,x) = 0$ , $\forall x, y \in A$ .
- $\tilde{R}$ *is* perfectly antisymmetric *[39] iff for* $x \neq y$ *whenever*

  $\mu_{\tilde{R}}(x,y) > 0$ *then* $\mu_{\tilde{R}}(y,x) = 0$ , $\forall x, y \in A$ .

# 3 A demonic fuzzy order refinement

We will give the definition of domain of fuzzy relations $\tilde{R}$

(8) **Definition.** Let $\tilde{R} = \{[(x,y), \mu_{\tilde{R}}(x,y)] \mid (x,y) \in A \times B\}$ be fuzzy relation and $\tilde{R}^{(1)} = \{(x, max_y \mu_{\tilde{R}}(x,y) \mid (x,y) \in A \times B\}$ be the first projection of $\tilde{R}$;
Then:
The domain of fuzzy relation $\tilde{R}L$ is the first projection of $\tilde{R}$, denoted by $\pi_\vee \tilde{R}$;
$\pi_\vee \tilde{R} = \{(x, max_y \mu_{\tilde{R}}(x,y) \mid (x,y) \in A \times B\}$, i.e;

$$\pi_\vee \tilde{R} = \tilde{R}L \qquad .$$

Now, we will give the definition of fuzzy ordering

(9) **Definition.** We say that a fuzzy relation $\tilde{Q}$ *fuzzy refines* a fuzzy relation $\tilde{R}$, denoted by $\tilde{Q} \sqsubseteq_{fuz} \tilde{R}$, iff

$$\pi_\vee \tilde{R} \subseteq \pi_\vee \tilde{Q} \text{ and } \tilde{Q} \cap \pi_\vee \tilde{R} \subseteq \tilde{R}$$

In other words, $\tilde{Q}$ refines $\tilde{R}$ if and only if the prerestriction of $\tilde{Q}$ to the domain of $\tilde{R}$ is included in $\tilde{R}$ : this means that $\tilde{Q}$ must not produce results not allowed by $\tilde{R}$ for those states that are in the domain of $\tilde{R}$.

(10) **Example.**

$$\begin{pmatrix} 0.3 & 0.2 & 0.4 \\ 0.7 & 0.8 & 0.8 \\ 0.3 & 0.5 & 0.6 \end{pmatrix} \sqsubseteq_{fuz} \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.4 & 0.5 & 0.9 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}$$

and

$$\begin{pmatrix} 0.1 & 0.2 & 0.4 \\ 0.5 & 0.7 & 0.9 \end{pmatrix} \not\sqsubseteq_{fuz} \begin{pmatrix} 0.2 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.8 \end{pmatrix}$$

## 3.1 Fuzzy Demonic operators

In this subsection, we will present fuzzy demonic operators and also some of their properties.

To clarify the ideas, take two relations $\tilde{Q}$ and $\tilde{R}$:

- Their supremum is

  $$\tilde{Q} \sqcup_{fuz} \tilde{R} = min\{max\{\tilde{Q}, \tilde{R}\}, \pi_\vee \tilde{Q}, \pi_\vee \tilde{R}\}$$

  and satisfies

  $$\pi_\vee(\tilde{Q} \sqcup_{fuz} \tilde{R}) = \pi_\vee \tilde{Q} \cap \pi_\vee \tilde{R}.$$

  Then, $\tilde{Q} \sqcup_{fuz} \tilde{R}$ is exactly the relational expression of the *fuzzy demonic union.*

(11) **Example.** Let

$$\tilde{Q} = \begin{pmatrix} 0.1 & 0 & 0.2 \\ 0.3 & 0.8 & 1 \\ 0 & 1 & 0.7 \end{pmatrix}, \tilde{R} = \begin{pmatrix} 0 & 1 & 0 \\ 0.3 & 0.5 & 0.4 \\ 0.9 & 0.7 & 0.2 \end{pmatrix}$$

Then;

$$\tilde{Q} \sqcup_{fuz} \tilde{R} = \begin{pmatrix} 0.1 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.5 \\ 0.9 & 0.9 & 0.7 \end{pmatrix}$$

- Their infimum, if it exists, is

  $$\tilde{Q} \sqcap_{fuz} \tilde{R} = max\{min\{\tilde{Q}, \tilde{R}\}, \\ min\{\tilde{Q}, 1 - \pi_\vee \tilde{R}\}, min\{\tilde{R}, 1 - \pi_\vee \tilde{Q}\}\}$$

  and it satisfies

  $$\pi_\vee(\tilde{Q} \sqcap_{fuz} \tilde{R}) = \pi_\vee \tilde{Q} \cup \pi_\vee \tilde{R}.$$

  The operator $\sqcap_{fuz}$ is called *fuzzy demonic intersection.* For $\tilde{Q} \sqcap_{fuz} \tilde{R}$ to exist, we have to verify $\pi_\vee \subseteq \pi_\vee(\tilde{Q} \cup \pi_\vee \bar{\tilde{Q}} \cap \tilde{R} \cup \pi_\vee \bar{\tilde{R}})$. This condition is equivalent to $\pi_\vee \tilde{Q} \cap \pi_\vee \tilde{R} \subseteq \pi_\vee(\tilde{Q} \cap \tilde{R})$, which can be interpreted as follows: the existence condition simply means that on the intersection of their domains, $\tilde{Q}$ and $\tilde{R}$ have to agree for at least one value.

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Let

$$\tilde{Q} = \begin{pmatrix} 0.1 & 0 & 0.2 \\ 0.3 & 0.8 & 1 \\ 0 & 1 & 0.7 \end{pmatrix}, \tilde{R} = \begin{pmatrix} 0 & 1 & 0 \\ 0.3 & 0.5 & 0.4 \\ 0.9 & 0.7 & 0.2 \end{pmatrix}$$

Then;

$$\tilde{Q} \sqcap_{fuz} \tilde{R} = \begin{pmatrix} 0 & 0.8 & 0 \\ 0.3 & 0.5 & 0.5 \\ 0 & 0.7 & 0.2 \end{pmatrix}$$

In what follows, we will give the definition of the fuzzy demonic composition.

(12) **Definition.** The *fuzzy demonic composition* of relations $\tilde{Q}$ and $\tilde{R}$ is

$$\tilde{Q} \,\square_{fuz} \tilde{R} = min\{\tilde{Q}\tilde{R}, 1 - \tilde{Q}\overline{\pi_{\vee}\tilde{R}}\}$$

.

(13) **Example.**

$$\begin{pmatrix} 0.1 & 0 & 0.2 \\ 0.3 & 0.8 & 1 \\ 0 & 1 & 0.7 \end{pmatrix} \square_{fuz} \begin{pmatrix} 0 & 1 & 0 \\ 0.3 & 0.5 & 0.4 \\ 0.9 & 0.7 & 0.2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.2 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.4 \\ 0.5 & 0.5 & 0.4 \end{pmatrix}$$

### 3.2 Properties of fuzzy demonic operators

The fuzzy demonic operators $\sqcap_{fuz}, \sqcup_{fuz}$ and $\square_{fuz}$, have the same properties as $\sqcap, \sqcup$ and $\square$, but the fuzzy demonic intersections have to be defined. Let us give some of them.

(14) **Theorem.** *Let $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}$ be fuzzy relations. Then,*

- $\tilde{P} \sqcap_{fuz} (\tilde{Q} \sqcup_{fuz} \tilde{R}) = (\tilde{P} \sqcap_{fuz} \tilde{Q}) \sqcup_{fuz} (\tilde{P} \sqcap_{fuz} \tilde{R})$,
- $\tilde{P} \sqcup_{fuz} (\tilde{Q} \sqcap_{fuz} \tilde{R}) = (\tilde{P} \sqcup_{fuz} \tilde{Q}) \sqcap_{fuz} (\tilde{P} \sqcup_{fuz} \tilde{R})$,
- $\tilde{R} \,\square_{fuz} I = I \,\square_{fuz} \tilde{R} = \tilde{R}$,
- $\tilde{Q} \sqsubseteq_{fuz} \tilde{R} \Rightarrow \tilde{P} \,\square_{fuz} \tilde{Q} \sqsubseteq_{fuz} \tilde{P} \,\square_{fuz} \tilde{R}$,
- $\tilde{P} \sqsubseteq_{fuz} \tilde{Q} \Rightarrow \tilde{P} \,\square_{fuz} \tilde{R} \sqsubseteq_{fuz} \tilde{Q} \,\square_{fuz} \tilde{R}$,
- $\tilde{P} \,\square_{fuz} (\tilde{Q} \sqcup_{fuz} \tilde{R}) = \tilde{P} \,\square_{fuz} \tilde{Q} \sqcup_{fuz} \tilde{P} \,\square_{fuz} \tilde{R}$,
- $(\tilde{P} \sqcup_{fuz} \tilde{Q}) \,\square_{fuz} \tilde{R} = \tilde{P} \,\square_{fuz} \tilde{R} \sqcup_{fuz} \tilde{Q} \,\square_{fuz} \tilde{R}$,
- $\tilde{P} \,\square_{fuz} (\tilde{Q} \sqcap_{fuz} \tilde{R}) \sqsubseteq_{fuz} \tilde{P} \,\square_{fuz} \tilde{Q} \sqcap_{fuz} \tilde{P} \,\square_{fuz} \tilde{R}$,
- $\tilde{P} \,\square_{fuz} (\tilde{Q} \,\square_{fuz} \tilde{R}) = (\tilde{P} \,\square_{fuz} \tilde{Q}) \,\square_{fuz} \tilde{R}$,
- $(\tilde{P} \sqcap_{fuz} \tilde{Q}) \,\square_{fuz} \tilde{R} \sqsubseteq_{fuz} \tilde{P} \,\square_{fuz} \tilde{R} \sqcap_{fuz} \tilde{Q} \,\square_{fuz} \tilde{R}$.

(15) **Proposition.**

- $\tilde{Q}$ *deterministic* $\Rightarrow \tilde{Q} \,\square_{fuz} \tilde{R} = \tilde{Q}\tilde{R}$,
- $\tilde{P}$ *deterministic* $\Rightarrow \tilde{P} \,\square_{fuz} (\tilde{Q} \sqcap_{fuz} \tilde{R}) = \tilde{P}\tilde{Q} \sqcap_{fuz} \tilde{P}\tilde{R}$,
- $\tilde{R}$ *total* $\Rightarrow \tilde{Q} \,\square_{fuz} \tilde{R} = \tilde{Q}\tilde{R}$,
- $\pi_{\vee}\tilde{P} \sqcap_{fuz} \pi_{\vee}\tilde{Q} = \emptyset \Rightarrow (\tilde{P} \sqcup_{fuz} \tilde{Q}) \,\square_{fuz} \tilde{R} = \tilde{P} \,\square_{fuz} \tilde{R} \cup \tilde{Q} \,\square_{fuz} \tilde{R}$,
- $\pi_{\vee}\tilde{P} \sqcap_{fuz} \pi_{\vee}\tilde{Q} = \emptyset \Rightarrow \tilde{P} \sqcap_{fuz} \tilde{Q} = \tilde{P} \sqcup_{fuz} \tilde{Q}$.

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