Hindawi
Journal of Function Spaces
Volume 2020, Article ID 8106494, 12 pages
https://doi.org/10.1155/2020/8106494

Research Article

A Note on Conformable Double Laplace Transform and Singular Conformable Pseudoparabolic Equations

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Received 10 September 2019; Accepted 18 October 2019; Published 17 February 2020

Academic Editor: Maria Alessandra Ragusa

In this work, we combine conformable double Laplace transform and Adomian decomposition method and present a new approach for solving singular one-dimensional conformable pseudoparabolic equation and conformable coupled pseudoparabolic equation. Furthermore, some examples are given to show the performance of the proposed method.

1. Introduction

Fractional partial differential equations have attracted much attention in applied sciences and engineering such as acoustics, control, and viscoelasticity. The parabolic equation appeared in different fields of applied mathematics, such as heat conduction and fluid mechanics (for instance, see [1–4]). The authors in [5, 6] studied the fractional diffusion equations problems by using the Adomian decomposition method and series expansion method. Many papers exist in the literature, which are related to conformable fractional derivative with its properties and applications [7, 8]. This new method was quickly generalized by Katugampola [9, 10]. The authors in [11] investigated existence and uniqueness theorems for sequential linear conformable fractional differential equations. The authors in [12] revisited the Grünwald Letnikov, Riemann–Liouville, and Caputo fractional derivatives and analysed under the light of the proposed criteria. The nonhomogeneous nonlocal theory has been presented based on conformable derivatives (CD) to study the critical point instability of micro/nanobeams under a distributed variable-pressure force (see [13]). The authors in [14] proposed a new fractional nonlocal model and its application in free vibration of Timoshenko and Euler–Bernoulli beams. Recently, several researchers applied the conformable Laplace transform method to solve different types of fractional differential equation (see [15, 16]). Many exact solutions in various wave forms for the nonlinear conformable time-fractional parabolic equation with exponential nonlinearity are formally constructed in [17]. The goal of this paper is to investigate the solution of singular conformable fractional pseudoparabolic equation and conformable coupled pseudoparabolic equation by conformable double Laplace transform decomposition methods (CDLTDMs). Moreover, we are able to prove some theorems related to this work.

1.1. Conformable Partial Derivatives

Definition 1 (see [18]). Given a function $f(x,t)$: $R\times(0,\infty)\longrightarrow R$, the conformable space fractional partial derivative of order α of the function $f(x,t)$ is denoted by

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}f(x,t)=\lim_{\varepsilon\longrightarrow 0}\frac{f(x+\varepsilon x^{1-\alpha},t)-f(x,t)}{\varepsilon},\quad x,t>0,\ 0<\alpha\leq 1. \tag{1}$$

Definition 2 (see [18]). Given a function $f(x,t)$: $R\times(0,\infty)\longrightarrow R$, the conformable time partial derivative of order β of the function $f(x,t)$ is defined as

$$\frac{\partial^{\beta}}{\partial t^{\beta}}f(x,t)=\lim_{\sigma\longrightarrow 0}\frac{f(x,t+\sigma t^{1-\beta})-f(x,t)}{\varepsilon},\quad x,t>0,\ 0<\beta\leq 1, \tag{2}$$

where $\partial^\alpha/\partial x^\alpha$ and $\partial^\beta/\partial t^\beta$ are called the fractional derivatives of order α and β, respectively.

In Theorem 1, the connection between the conformable derivatives and the first derivative can be represented as follows.

Theorem 1. *Let $\alpha, \beta \in (0,1]$ and $f(x,t)$ be α and β differentiable at a point $x, t > 0$.*

Then,

$$\frac{\partial^\alpha f(x,t)}{\partial x^\alpha} = x^{1-\alpha}\frac{\partial f(x,t)}{\partial x},$$
$$\frac{\partial^\beta f(x,t)}{\partial t^\beta} = t^{1-\beta}\frac{\partial f(x,t)}{\partial x}. \tag{3}$$

Proof. By using definitions 1 and 2 and $h = \epsilon x^{1-\alpha}$ in equation (1), we have

$$\begin{aligned}\frac{\partial^\alpha f(x,t)}{\partial x^\alpha} &= \lim_{\varepsilon\longrightarrow 0}\frac{f(x+\varepsilon x^{1-\alpha},t)-f(x,t)}{\varepsilon}\\ &= \lim_{h\longrightarrow 0}\frac{f(x+h,t)-f(x,t)}{hx^{\alpha-1}}\\ &= x^{1-\alpha}\lim_{h\longrightarrow 0}\frac{f(x+h,t)-f(x,t)}{h}\\ &= x^{1-\alpha}\frac{\partial f(x,t)}{\partial x}.\end{aligned} \tag{4}$$

Similarly, we prove equation (2).

In the next example, we introduce the conformable derivative of specific functions, by using Theorem 1 as follows. □

Example 1. Let $\alpha, \beta \in (0,1]$ and a, b, m, n, λ, and $\mu \in \mathbb{R}$; then

(1) $(\partial^\alpha/\partial x^\alpha)(au(x,t)+bv(x,t)) = a(\partial^\alpha u(x,t)/\partial x^\alpha) + b(\partial^\alpha v(x,t)/\partial x^\alpha)$
(2) $(\partial^{\alpha+\beta}/\partial x^\alpha \partial t^\beta)(x^\mu t^\lambda) = \mu\lambda x^{\kappa-\alpha}t^{\lambda-\beta}$, $\lambda, \mu \in \mathbb{R}$
(3) $(\partial^\alpha/\partial x^\alpha)(e^{\lambda(x^\alpha/\alpha)+(\tau t^\beta/\beta)}) = \lambda e^{\lambda(x^\alpha/\alpha)+(\tau t^\beta/\beta)}$, $(\partial^\beta/\partial t^\beta)(e^{\lambda(x^\alpha/\alpha)+(\tau t^\beta/\beta)}) = \tau e^{\lambda(x^\alpha/\alpha)+(\tau t^\beta/\beta)}$
(4) $(\partial^\beta/\partial t^\beta)(x^\alpha/\alpha)(t^\beta/\beta) = (x^\alpha/\alpha)$, $(\partial^\beta/\partial t^\beta)(x^\alpha/\alpha)^n(t^\beta/\beta)^m = m(x^\alpha/\alpha)^n(t^\beta/\beta)^{m-1}$
(5) $(\partial^\alpha/\partial x^\alpha)(x^\alpha/\alpha)^n(t^\beta/\beta) = n(x^\alpha/\alpha)^{n-1}(t^\beta/\beta)$, $(\partial^\beta/\partial t^\beta)(x^\alpha/\alpha)^n(t^\beta/\beta)^m = m(x^\alpha/\alpha)^n(t^\beta/\beta)^{m-1}$
(6) $(\partial^\beta/\partial t^\beta)(\sin(x^\alpha/\alpha)\sin(t^\beta/\beta)) = \sin(x^\alpha/\alpha)\cos(t^\beta/\beta)$, $(\partial^\alpha/\partial x^\alpha)(\sin a(x^\alpha/\alpha)\sin(t^\beta/\beta)) = \sin(x^\alpha/\alpha)\cos(t^\beta/\beta)$

2. Some Properties of the Conformable Laplace Transform

Here, we work with the single conformable Laplace transform and conformable double Laplace transform (CDLT) which are defined, respectively as follows.

Definition 3 (see [7, 19, 20]). Let $f: [a,\infty) \longrightarrow \mathbb{R}$ and $0 < \beta \le 1$. Then, the fractional Laplace transform of order β is defined by

$$L_t^\beta(f(t)) = \int_0^\infty e^{-s(t^\beta/\beta)}f(t)t^{\beta-1}dt. \tag{5}$$

Definition 4 (see [21]). Let $u(x,t)$ be a piecewise continuous function on the interval $[a,\infty)\times[a,\infty)$ of exponential order. Consider for some $a, b \in \mathbb{R}$ sup $x, t > 0$, $(|u(x,t)|/e^{(ax^\alpha/\alpha)+(bt^\beta/\beta)})$ in these conditions (CDLT) is defined by

$$L_x^\alpha L_t^\beta(u(x,t)) = \int_0^\infty\int_0^\infty e^{-p(x^\alpha/\alpha)-s(t^\beta/\beta)}u(x,t)t^{\beta-1}x^{\alpha-1}dt\,dx, \tag{6}$$

where $p, s \in \mathbb{C}$, $0 < \alpha, \beta \le 1$, and the integrals are by means of conformable integral with respect to t and x, respectively.

Theorem 2. *If $L_x^\alpha L_t^\beta[f(x^\alpha/\alpha, t^\beta/\beta)] = F_{\alpha,\beta}(p,s)$, then*

$$\begin{aligned}&L_x^\alpha L_t^\beta\left[f\left(\frac{x^\alpha}{\alpha}-\frac{\zeta^\alpha}{\alpha},\frac{t^\beta}{\beta}-\frac{\eta^\beta}{\beta}\right)H\left(\frac{x^\alpha}{\alpha}-\frac{\zeta^\alpha}{\alpha},\frac{t^\beta}{\beta}-\frac{\eta^\beta}{\beta}\right)\right]\\ &= e^{-p(\zeta^\alpha/\alpha)-s(\eta^\beta/\beta)}F_{\alpha,\beta}(p,s),\end{aligned} \tag{7}$$

where $H(x,t)$ is the Heaviside unit step function defined by $H((x^\alpha/\alpha)-a, (t^\beta/\beta)-b) = 1$ when $x > a$ and $t > b$ and $H((x^\alpha/\alpha)-a, (t^\beta/\beta)-b) = 0$ when $x < a$ and $t < b$.

Proof. By applying the definition of double conformable Laplace transform,

$$\begin{aligned}&L_x^\alpha L_t^\beta\left[f\left(\frac{x^\alpha}{\alpha}-\frac{\zeta^\alpha}{\alpha},\frac{t^\beta}{\beta}-\frac{\eta^\beta}{\beta}\right)H\left(\frac{x^\alpha}{\alpha}-\frac{\zeta^\alpha}{\alpha},\frac{t^\beta}{\beta}-\frac{\eta^\beta}{\beta}\right)\right]\\ &= \int_0^\infty\int_0^\infty e^{-p(x^\alpha/\alpha)-s(t^\beta/\beta)}f\left(\frac{x^\alpha}{\alpha}-\frac{\zeta^\alpha}{\alpha},\frac{t^\beta}{\beta}-\frac{\eta^\beta}{\beta}\right)\\ &\quad\cdot H\left(\frac{x^\alpha}{\alpha}-\frac{\zeta^\alpha}{\alpha},\frac{t^\beta}{\beta}-\frac{\eta^\beta}{\beta}\right)t^{\beta-1}x^{\alpha-1}dt\,dx\\ &= \int_{\zeta^\alpha/\alpha}^\infty\int_{\eta^\beta/\beta}^\infty e^{-p(x^\alpha/\alpha)-s(t^\beta/\beta)}f\left(\frac{x^\alpha}{\alpha}-\frac{\zeta^\alpha}{\alpha},\frac{t^\beta}{\beta}-\frac{\eta^\beta}{\beta}\right)t^{\beta-1}x^{\alpha-1}dt\,dx,\end{aligned} \tag{8}$$

which is, by putting $(x^\alpha/\alpha)-(\zeta^\alpha/\alpha) = \tau^\alpha/\alpha$, $(t^\beta/\beta)-(\eta^\beta/\beta) = \upsilon^\beta/\beta$, we have

$$\begin{aligned}&L_x^\alpha L_t^\beta\left[f\left(\frac{x^\alpha}{\alpha}-\frac{\zeta^\alpha}{\alpha},\frac{t^\beta}{\beta}-\frac{\eta^\beta}{\beta}\right)H\left(\frac{x^\alpha}{\alpha}-\frac{\zeta^\alpha}{\alpha},\frac{t^\beta}{\beta}-\frac{\eta^\beta}{\beta}\right)\right]\\ &= e^{-p(\zeta^\alpha/\alpha)-s(\eta^\beta/\beta)}\int_0^\infty\int_0^\infty e^{-p(\tau^\alpha/\alpha)-s(\upsilon^\beta/\beta)}f\left(\frac{\tau^\alpha}{\alpha},\frac{\upsilon^\beta}{\beta}\right)\upsilon^{\beta-1}\tau^{\alpha-1}d\upsilon\,d\tau\\ &= e^{-p(\zeta^\alpha/\alpha)-s(\eta^\beta/\beta)}F_{\alpha,\beta}(p,s).\end{aligned} \tag{9}$$

In the next example, we reported that some conformable Laplace transforms of definite functions are important in this study. □

Example 2

(1) $L_x^\alpha L_t^\beta[(x^\alpha/\alpha)^n \quad (t^\beta/\beta)^m] = L_x L_t[(x)^n (t)^m] = n!m!/p^{n+1}s^{m+1}$, where m and n are positive integers
(2) $L_x^\alpha L_t^\beta[e^{\lambda(x^\alpha/\alpha)+(\tau t^\beta/\beta)}] = L_x L_t[e^{\lambda x+\tau t}] = 1/((p-\lambda)(s-\tau))$
(3) $L_x^\alpha L_t^\beta[\sin(\lambda(x^\alpha/\alpha))\sin(\tau(t^\beta/\beta))] = L_x L_t[\sin(x)\cos(t)] = 1/((p^2+\lambda^2)(s^2+\tau^2))$

Theorem 3. *Let f be piecewise continuous on $[a,\infty)\times[a,\infty)$; the (CDLT) of the conformable partial derivatives of orders α-th and β-th, $\partial^\alpha u/\partial x^\alpha, \partial^\beta u/\partial t^\beta, \partial^{2\alpha} u/\partial x^{2\alpha}$, and $\partial^{2\beta} u/\partial t^{2\beta}$ is given by*

$$L_x^\alpha L_t^\beta\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right) = pU(p,s) - U(0,s), \quad (10)$$

$$L_x^\alpha L_t^\beta\left(\frac{\partial^\beta u}{\partial t^\beta}\right) = sU(p,s) - U(p,0), \quad (11)$$

$$\begin{aligned} L_x^\alpha L_t^\beta\left(\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}\right) &= p^2U(p,s) - pU(0,s) - U_x(0,s), \\ L_x^\alpha L_t^\beta\left(\frac{\partial^\beta u}{\partial t^\beta}\right) &= s^2U(p,s) - sU(p,0) - U_t(p,0). \end{aligned} \quad (12)$$

Proof. By using definition (CDLT) for $\partial^\alpha u/\partial x^\alpha$, we have

$$\begin{aligned} L_x^\alpha L_t^\beta\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right) &= \int_0^\infty \int_0^\infty e^{-p(x^\alpha/\alpha)-s(t^\beta/\beta)} \frac{\partial^\alpha u}{\partial x^\alpha} t^{\beta-1} x^{\alpha-1} dt\, dx \\ &= \int_0^\infty e^{-s(t^\beta/\beta)} t^{\beta-1}\left(\int_0^\infty e^{-p(x^\alpha/\alpha)} \frac{\partial^\alpha u}{\partial x^\alpha} x^{\alpha-1} dx\right) dt. \end{aligned} \quad (13)$$

By applying Theorem $1, \partial^\alpha u/\partial x^\alpha = x^{1-\alpha}(\partial u(x,t)/\partial x)$ equation (13) becomes

$$L_x^\alpha L_t^\beta\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right) = \int_0^\infty e^{-s(t^\beta/\beta)} t^{\beta-1}\left(\int_0^\infty e^{-p(x^\alpha/\alpha)} \frac{\partial u(x,t)}{\partial x} dx\right) dt. \quad (14)$$

The integral inside bracket given by

$$\int_0^\infty e^{-p(x^\alpha/\alpha)} \frac{\partial u(x,t)}{\partial x} dx = pU(p,t) - U(0,t). \quad (15)$$

By substituting equation (15) into equation (14), we obtain

$$L_x^\alpha L_t^\beta\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right) = pU(p,s) - U(0,s). \quad (16)$$

In the same manner, the (CDLT) of $\partial^\beta u/\partial t^\beta, \partial^{2\alpha} u/\partial x^{2\alpha}$, and $\partial^{2\beta} u/\partial t^{2\beta}$ can be obtained.

Double Laplace transform of the function $(x^\alpha/\alpha)^n(\partial^\beta f/\partial t^\beta)$ and $(x^\alpha/\alpha)f(x,t)$ are studied in the next theorem. □

Theorem 4. *If the (CDLT) of the conformable partial derivatives $(\partial^\beta/\partial t^\beta)u$ is given by equation (11), then double Laplace transform of $(x^\alpha/\alpha)^n(\partial^\beta/\partial t^\beta)u(x,t)$ and $(x^\alpha/\alpha)u(x,t)$ are given by*

$$(-1)^n \frac{d^n}{dp^n}\left(L_x^\alpha L_t^\beta[u(x,t)]\right) = L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^n u(x,t)\right], \quad (17)$$

$$(-1)^n \frac{d^n}{dp^n}\left(L_x^\alpha L_t^\beta\left[\frac{\partial^\beta u}{\partial t^\beta}\right]\right) = L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^n \frac{\partial^\beta u}{\partial t^\beta}\right], \quad (18)$$

where $n = 1, 2, 3, \ldots$.

Proof. By applying the nth derivative with respect to p for both sides of equation (6), we get equation (17) as follows:

$$\begin{aligned} \frac{d^n}{dp^n}\left(L_x^\alpha L_t^\beta[u(x,t)]\right) &= \int_0^\infty \int_0^\infty \frac{d^n}{dp^n}\left(e^{-p(x^\alpha/\alpha)-s(t^\beta/\beta)} u(x,t)\right) \\ &\quad \cdot t^{\beta-1} x^{\alpha-1} dt\, dx \\ &= (-1)^n \int_0^\infty \int_0^\infty \left(\frac{x^\alpha}{\alpha}\right)^n e^{-p(x^\alpha/\alpha)-s(t^\beta/\beta)} \\ &\quad \cdot t^{\beta-1} x^{\alpha-1} u(x,t) dt\, dx \\ &= (-1)^n L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^n u(x,t)\right]. \end{aligned} \quad (19)$$

We obtain

$$(-1)^n \frac{d^n}{dp^n}\left(L_x^\alpha L_t^\beta[u(x,t)]\right) = L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^n u(x,t)\right]. \quad (20)$$

Similarly, we can prove equation (18). □

3. Singular One-Dimensional Conformable Fractional Pseudoparabolic Equation

The conformable double Laplace decomposition methods (CDLTDMs) are an efficient technique which is used to obtain the solution linear and nonlinear singular pseudoparabolic equation.

Problem. We consider $0<\alpha\le 1$ and $0<\beta\le 1$ as singular one-dimensional pseudoparabolic equations with initial conditions in the form

$$\frac{\partial^\beta u}{\partial t^\beta} - \frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u\right) - \frac{\alpha}{x^\alpha}\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u\right) = f(x,t), \quad (21)$$

subject to

$$u(x,0) = f_1(x), \quad (22)$$

where, the term, $(\alpha/x^\alpha)(\partial^\alpha u/\partial x^\alpha)((x^\alpha/\alpha)(\partial^\alpha/\partial x^\alpha))$ is called conformable Bessel's operator and $f(x,t)$ and $f_1(x)$ are known functions. In order to solve equation (21), we apply the following steps:

Step 1: multiplying equation (21) by x^α/α:

$$\frac{x^\alpha}{\alpha}\frac{\partial^\beta u}{\partial t^\beta}-\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u\right)-\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u\right)=\frac{x^\alpha}{\alpha}f(x,t). \tag{23}$$

Step 2: using Lemma 1 and equation (18) for equations in step 1 and single conformable Laplace transform for equation (22), we obtain

$$\begin{aligned}\frac{d}{dp}U(p,s)&=\frac{1}{s}\frac{d}{dp}L_x^\alpha[f_1(x)]+\frac{1}{s}\frac{d}{dp}\left[L_x^\alpha L_t^\beta[f(x,t)]\right]\\&-\frac{1}{s}L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u\right)\right.\\&\left.+\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u\right)\right],\end{aligned} \tag{24}$$

where the symbol $L_x^\alpha L_t^\beta$ indicates (CDLT) with respect to x and t.

Step 3: applying the integral for both sides of equation (24), from 0 to p with respect to p, we have

$$\begin{aligned}U(p,s)&=\frac{1}{s}\int_0^p\left(\frac{d}{dp}L_x^\alpha[f_1(x)]\right)dp-\frac{1}{s}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\right.\right.\\&\left.\left.\cdot\frac{\partial^\alpha}{\partial x^\alpha}u\right)+\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u\right)\right]dp\\&+\frac{1}{s}\int_0^p\left(\frac{d}{dp}\left[L_x^\alpha L_t^\beta[f(x,t)]\right]\right)dp.\end{aligned} \tag{25}$$

Step 4: next, the (CDLTDM) consists of representing the solution of the singular pseudoparabolic equation as $u(x,t)$ by the infinite series

$$u(x,t)=\sum_{n=0}^\infty u_n(x,t). \tag{26}$$

Step 5: working with the double Laplace transform on both sides of equation (25) and using equation (26), we receive

$$\begin{aligned}\sum_{n=0}^\infty u_n(x,t)&=f_1(x)+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(\frac{d}{dp}\left(L_x^\alpha L_t^\beta[f(x,t)]\right)\right)dp\right]-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\right.\right.\right.\right.\\&\left.\left.\left.\left.\cdot\left(\sum_{n=0}^\infty u_n(x,t)\right)\right)\right]\right)dp\right]-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^\infty u_n(x,t)\right)\right)\right]\right)dp\right].\end{aligned} \tag{27}$$

We define the following recursive formula:

$$u_0(x,t)=f_1(x)+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(\frac{d}{dp}\left(L_x^\alpha L_t^\beta[f(x,t)]\right)\right)dp\right]. \tag{28}$$

The rest of the terms can be written as follows:

$$\begin{aligned}u_{n+1}(x,t)&=-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^\infty u_n(x,t)\right)\right)\right]\right)dp\right]-L_p^{-1}L_s^{-1}\\&\cdot\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^\infty u_n(x,t)\right)\right)\right]\right)dp\right],\end{aligned} \tag{29}$$

where $L_p^{-1}L_s^{-1}$ indicates double inverse Laplace transform with respect to p and s.

Here, we assume that double inverse Laplace transform with respect to p and s exists for each terms in equations (28) and (29). To confirm our method, we solve the next example.

Example 3. Consider the following nonhomogeneous form of a singular one-dimensional pseudoparabolic equation:

$$\frac{\partial^\beta u}{\partial t^\beta} - \frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u\right) - \frac{\alpha}{x^\alpha}\frac{\partial^{\alpha+\beta}u}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u\right)$$
$$= -\left(\frac{x^\alpha}{\alpha}\right)^2 \sin\left(\frac{t^\beta}{\beta}\right) + 4\sin\left(\frac{t^\beta}{\beta}\right) - 4\cos\left(\frac{t^\beta}{\beta}\right), \tag{30}$$
$$0 \le x < \infty,\ 0 \le t < \infty,$$
$$0 < \alpha \le 1,\ 0 < \beta \le 1,$$

with the condition

$$u(x,t) = x^2. \tag{31}$$

By applying the above steps and Theorem 1, we obtain

$$u_0(x,t) = \left(\frac{x^\alpha}{\alpha}\right)^2 \cos\left(\frac{t^\beta}{\beta}\right) - 4\cos\left(\frac{t^\beta}{\beta}\right) - 4\sin\left(\frac{t^\beta}{\beta}\right) + 4,$$
$$u_{n+1}(x,t) = -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_n}{\partial x^\alpha}\right)\right]\right)\mathrm{d}p\right]$$
$$- L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_n}{\partial x^\alpha}\right)\right]\right)\mathrm{d}p\right]. \tag{32}$$

Based on the (CDLTDM), we obtain

$$u_1(x,t) = -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x L_t\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u_0\right)\right.\right.\right.$$
$$\left.\left.\left. + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u_0\right)\right]\right)\mathrm{d}p\right] \tag{33}$$
$$= -4 + 4\cos\left(\frac{t^\beta}{\beta}\right) + 4\sin\left(\frac{t^\beta}{\beta}\right).$$

In a similar manner, we obtain that

$$u_2(x,t) = -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x L_t\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u_1\right)\right.\right.\right.$$
$$\left.\left.\left. + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u_1\right)\right]\right)\mathrm{d}p\right] \tag{34}$$
$$= 0,$$
$$u_3 = 0, u_4 = 0, \ldots.$$

By adding all the terms, we get

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots. \tag{35}$$

Thus, the exact solution is obtained as follows:

$$u(x,t) = \left(\frac{x^\alpha}{\alpha}\right)^2 \cos\left(\frac{t^\beta}{\beta}\right). \tag{36}$$

By taking $\alpha = 1$ and $\beta = 1$, the fractional solution becomes

$$u(x,t) = x^2\cos(t). \tag{37}$$

Problem. Consider the following nonlinear singular one-dimensional pseudoparabolic equation:

$$\frac{\partial^\beta u}{\partial t^\beta} - \frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) - \frac{\alpha}{x^\alpha}\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right)$$
$$= -2u\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{x^\alpha}{\alpha}\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)^2 + f(x,t), \tag{38}$$
$$0 < \alpha \le 1,\ 0 < \beta \le 1,$$
$$0 \le x < \infty, 0 \le t < \infty,$$

subject to

$$u(x,0) = f_1(x). \tag{39}$$

Using our method, we get

$$u_0(x,t) = f_1(x) + L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(\frac{\mathrm{d}}{\mathrm{d}p}\left(L_x^\alpha L_t^\beta\left[f(x,t)\right]\right)\right)\mathrm{d}p\right]. \tag{40}$$

The rest of the terms are given by

$$u_{n+1}(x,t) = -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_n}{\partial x^\alpha}\right)\right]\right)\mathrm{d}p\right]$$
$$- L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_n}{\partial x^\alpha}\right)\right]\right)\mathrm{d}p\right]$$
$$+ L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\left(2\frac{x^\alpha}{\alpha}A_n\right)\right]\right)\mathrm{d}p\right]$$
$$- L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\left(\left(\frac{x^\alpha}{\alpha}\right)^2 B_n\right)\right]\right)\mathrm{d}p\right], \tag{41}$$

where A_n and B_n are the so-called Adomian polynomials, given by

$$A_n = \sum_{n=0}^{\infty} u_n u_{nx}, \qquad B_n = \sum_{n=0}^{\infty} (u_{nx})^2. \tag{42}$$

The nonlinear terms uu_x and $(u_x)^2$ are represented as

$$\begin{aligned}
A_0 &= u_0 u_{0x}, \\
A_1 &= u_0 u_{1x} + u_1 u_{0x}, \\
A_2 &= u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}, \\
A_3 &= u_0 u_{3x} + u_1 u_{2x} + u_2 u_{1x} + u_3 u_{0x}, \\
B_0 &= (u_{0x})^2, \\
B_1 &= 2u_{0x} u_{1x}, \\
B_2 &= 2u_{0x} u_{2x} + (u_{1x})^2, \\
B_3 &= 2u_{0x} u_{3x} + 2u_{1x} u_{2x}.
\end{aligned} \tag{43}$$

To illustrate this method for nonlinear problem, we consider the following example.

Example 4. Consider the following nonlinear pseudoparabolic equation:

$$\frac{\partial^\beta u}{\partial t^\beta} - \frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) - \frac{\alpha}{x^\alpha}\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) = -2u\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{x^\alpha}{\alpha}\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)^2 - \left(\frac{x^\alpha}{\alpha}\right)^2 e^{-\left(t^\beta/\beta\right)}, \quad (44)$$

$$0 \le x < \infty, 0 \le t < \infty,$$
$$0 < \alpha \le 1, 0 < \beta \le 1,$$

subject to

$$u(x,0) = \left(\frac{x^\alpha}{\alpha}\right)^2. \quad (45)$$

By applying the aforesaid conformable double Laplace decomposition method and Theorem 1, we have

$$u_0(x,t) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{-\left(t^\beta/\beta\right)},$$

$$u_1(x,t) = -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_n}{\partial x^\alpha}\right) + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\cdot\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_n}{\partial x^\alpha}\right)\right]\right)\mathrm{d}p\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\cdot\left[\left(2\frac{x^\alpha}{\alpha}A_n\right) - \left(\frac{x^\alpha}{\alpha}\right)^2 B_n\right]\right)\mathrm{d}p\right] = 0. \quad (46)$$

Proceeding in a similar manner, we have

$$u_2 = 0,$$
$$u_3 = 0, \quad (47)$$
$$u_4 = 0, \dots.$$

So according to equation (26), we have

$$u(x,t) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{-\left(t^\beta/\beta\right)}, \quad (48)$$

which is the exact solution of equation (44).

4. Conformable Double Laplace Transform Method and Singular Conformable Coupled Pseudoparabolic Equation

The purpose of this part is to examine the use of the (CDLTDM) for the linear one-dimensional conformable coupled pseudoparabolic equation. We consider the following conformable coupled pseudoparabolic equations:

$$\frac{\partial^\beta u}{\partial t^\beta} - \left(\frac{\alpha}{x^\alpha}\right)\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) - \frac{\alpha}{x^\alpha}\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) + \zeta v = f(x,t),$$

$$\frac{\partial^\beta v}{\partial t^\beta} - \left(\frac{\alpha}{x^\alpha}\right)\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha v}{\partial x^\alpha}\right) - \frac{\alpha}{x^\alpha}\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha v}{\partial x^\alpha}\right) + \zeta u = g(x,t), \quad (49)$$

with conditions

$$u(x,0) = f_1(x),$$
$$v(x,0) = g_1(x), \quad (50)$$

where $f(x,t), g(x,t), f_1(x)$, and $g_1(x)$ are the known functions and ζ is the coupling parameter. One can get the solution of equation (49), by using (CDLTDM); this method consists of the following steps:

(1) Multiply both sides of equation (49) by x^α/α leading to the following equation:

$$\frac{x^\alpha}{\alpha}\frac{\partial^\beta u}{\partial t^\beta} - \frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) - \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) + \zeta\frac{x^\alpha}{\alpha}v = \frac{x^\alpha}{\alpha}f(x,t),$$

$$\frac{x^\alpha}{\alpha}\frac{\partial^\beta v}{\partial t^\beta} - \frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha v}{\partial x^\alpha}\right) - \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha v}{\partial x^\alpha}\right) + \zeta\frac{x^\alpha}{\alpha}u = \frac{x^\alpha}{\alpha}g(x,t). \quad (51)$$

(2) Applying (CDLT) on both sides of equation (51) and single conformable Laplace transform for equation (50), we get

$$L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^\beta u}{\partial t^\beta}\right] = L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) - \zeta\frac{x^\alpha}{\alpha}v + \frac{x^\alpha}{\alpha}f(x,t)\right],$$

$$L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^\beta v}{\partial t^\beta}\right] = L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha v}{\partial x^\alpha}\right) + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha v}{\partial x^\alpha}\right) - \zeta\frac{x^\alpha}{\alpha}u + \frac{x^\alpha}{\alpha}g(x,t)\right]. \quad (52)$$

On using Theorem 1 and Theorem 2, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}U(p,s)&=\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}p}L_x^{\alpha}\left[f_1(x)\right]-\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}u\right)\right.\\&\left.+\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}u\right)\right]-\zeta\frac{x^{\alpha}}{\alpha}v+\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}p}\left[L_x^{\alpha}L_t^{\beta}\left[f(x,t)\right]\right],\\\frac{\mathrm{d}}{\mathrm{d}p}V(p,s)&=\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}p}L_x^{\alpha}\left[g_1(x)\right]-\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}v\right)\right.\\&\left.+\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}v\right)\right]-\zeta\frac{x^{\alpha}}{\alpha}u+\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}p}\left[L_x^{\alpha}L_t^{\beta}\left[g(x,t)\right]\right].\end{aligned}\tag{53}$$

(3) By integrating both sides of equation (53) from 0 to p with respect to p, we have

$$\begin{aligned}U(\mathrm{p},\mathrm{s})&=\frac{1}{s}\int_0^p\left(\frac{\mathrm{d}}{\mathrm{d}p}L_x^{\alpha}\left[f_1(x)\right]\right)\mathrm{d}p-\frac{1}{s}\int_0^p L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}u\right)+\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}u\right)-\zeta\frac{x^{\alpha}}{\alpha}v\right]\mathrm{d}p+\frac{1}{s}\int_0^p\left(\frac{\mathrm{d}}{\mathrm{d}p}\left[L_x^{\alpha}L_t^{\beta}\left[f(x,t)\right]\right]\right)\mathrm{d}p,\\V(\mathrm{p},\mathrm{s})&=\frac{1}{s}\int_0^p\left(\frac{\mathrm{d}}{\mathrm{d}p}L_x^{\alpha}\left[g_1(x)\right]\right)\mathrm{d}p-\frac{1}{s}\int_0^p L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}v\right)+\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}v\right)-\zeta\frac{x^{\alpha}}{\alpha}u\right]\mathrm{d}p+\frac{1}{s}\int_0^p\left(\frac{\mathrm{d}}{\mathrm{d}p}\left[L_x^{\alpha}L_t^{\beta}\left[g(x,t)\right]\right]\right)\mathrm{d}p.\end{aligned}\tag{54}$$

By applying double inverse Laplace transform for equation (54), we have

$$\begin{aligned}u(x,t)&=f_1(x)+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(\frac{\mathrm{d}}{\mathrm{d}p}\left(L_x^{\alpha}L_t^{\beta}\left[f(x,t)\right]\right)\right)\mathrm{d}p\right]-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\right]\right)\mathrm{d}p\right]\\&-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)-\zeta\frac{x^{\alpha}}{\alpha}v\right]\right)\mathrm{d}p\right],\end{aligned}\tag{55}$$

$$\begin{aligned}v(x,t)&=g_1(x)+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(\frac{\mathrm{d}}{\mathrm{d}p}\left(L_x^{\alpha}L_t^{\beta}\left[g(x,t)\right]\right)\right)\mathrm{d}p\right]-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}v}{\partial x^{\alpha}}\right)\right]\right)\mathrm{d}p\right]\\&-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}v}{\partial x^{\alpha}}\right)-\zeta\frac{x^{\alpha}}{\alpha}u\right]\right)\mathrm{d}p\right].\end{aligned}\tag{56}$$

The (CDLTDM) defines the solutions of conformable coupled pseudoparabolic equations as $u(x,t)$ and $v(x,t)$ by the infinite series:

$$\begin{aligned}u(x,t)&=\sum_{n=0}^{\infty}u_n(x,t),\\v(x,t)&=\sum_{n=0}^{\infty}v_n(x,t).\end{aligned}\tag{57}$$

By substituting equation (57) into equations (55) and (56), we get

(4) Working with the double Laplace transform on both sides of equation (25) and using equation (26), we receive

$$\sum_{n=0}^{\infty} u_n(x,t) = f_1(x) + L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(\frac{d}{dp}\left(L_x^{\alpha}L_t^{\beta}[f(x,t)]\right)\right)dp\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\sum_{n=0}^{\infty}u_n\right)\right)\right]\right)dp\right]$$
$$- L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\sum_{n=0}^{\infty}u_n\right)\right)\right]\right)dp\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\zeta\frac{x^{\alpha}}{\alpha}\sum_{n=0}^{\infty}v_n\right]dp\right],$$
$$\sum_{n=0}^{\infty} v_n(x,t) = g_1(x) + L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(\frac{d}{dp}\left(L_x^{\alpha}L_t^{\beta}[g(x,t)]\right)\right)dp\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\sum_{n=0}^{\infty}v_n\right)\right)\right]\right)dp\right]$$
$$- L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\sum_{n=0}^{\infty}v_n\right)\right)\right]\right)dp\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\zeta\frac{x^{\alpha}}{\alpha}\sum_{n=0}^{\infty}u_n\right]dp\right]. \tag{58}$$

This technique suggests that the zeroth components u_0 and v_0 are identified by the initial conditions and from source terms as follows:

$$u_0 = f_1(x) + L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(\frac{d}{dp}\left(L_x^{\alpha}L_t^{\beta}[f(x,t)]\right)\right)dp\right],$$
$$v_0 = g_1(x) + L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(\frac{d}{dp}\left(L_x^{\alpha}L_t^{\beta}[g(x,t)]\right)\right)dp\right]. \tag{59}$$

The rest of the terms are given by

$$u_{n+1} = -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}u_n}{\partial x^{\alpha}}\right)\right]\right)dp\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}u_n}{\partial x^{\alpha}}\right)\right]\right)dp\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\zeta\frac{x^{\alpha}}{\alpha}v_n\right]dp\right], \tag{60}$$

$$v_{n+1} = -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}v_n}{\partial x^{\alpha}}\right)\right]\right)dp\right]$$
$$- L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}v_n}{\partial x^{\alpha}}\right)\right]\right)dp\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\zeta\frac{x^{\alpha}}{\alpha}u_n\right]dp\right]. \tag{61}$$

In order to ensure the four techniques for solving the conformable fractional coupled pseudoparabolic equation, we will consider the following example.

Example 5. Consider the following homogeneous form of conformable coupled pseudoparabolic equations:

$$\frac{\partial^{\beta}u}{\partial t^{\beta}} - 2\frac{\alpha}{x^{\alpha}}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right) - \frac{\alpha}{x^{\alpha}}\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right) + 2v = 0,$$
$$\frac{\partial^{\beta}v}{\partial t^{\beta}} - 2\frac{\alpha}{x^{\alpha}}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}v}{\partial x^{\alpha}}\right) - \frac{\alpha}{x^{\alpha}}\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}v}{\partial x^{\alpha}}\right) + 2u = 0, \tag{62}$$

where

$$0 \le x < \infty, \quad 0 \le t < \infty, \quad 0 < \alpha \le 1, \quad 0 < \beta \le 1, \tag{63}$$

with conditions

$$u(x,0) = \left(\frac{x^{\alpha}}{\alpha}\right)^2, \quad v(x,0) = \left(\frac{x^{\alpha}}{\alpha}\right)^2. \tag{64}$$

By applying the above method and Theorem 1 for equation (62) and (64) and using equations (59), (60), and (61), we obtain

$$\begin{aligned}
u_0 &= \left(\frac{x^\alpha}{\alpha}\right)^2, \\
v_0 &= \left(\frac{x^\alpha}{\alpha}\right)^2, \\
u_1 &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[2\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u_0\right)\right.\right. \\
&\quad \left.\left. + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u_0\right) - 2\frac{x^\alpha}{\alpha}v_0\right]\right] \\
&= 8\frac{t^\beta}{\beta} - 2\left(\frac{x^\alpha}{\alpha}\right)^2\frac{t^\beta}{\beta}, \\
v_1 &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[2\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}v_0\right)\right.\right. \\
&\quad \left.\left. + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}v_0\right) - 2\frac{x^\alpha}{\alpha}u_0\right]\right] \\
&= 8\frac{t^\beta}{\beta} - 2\left(\frac{x^\alpha}{\alpha}\right)^2\frac{t^\beta}{\beta}, \\
u_2 &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[2\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u_1\right)\right.\right. \\
&\quad \left.\left. + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u_1\right) - 2\frac{x^\alpha}{\alpha}v_1\right]\right] \\
&= -16\left(\frac{t^\beta}{\beta}\right)^2 - 8\frac{t^\beta}{\beta} + 2\left(\frac{x^\alpha}{\alpha}\right)^2\left(\frac{t^\beta}{\beta}\right)^2, \\
v_2 &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[2\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}v_1\right)\right.\right. \\
&\quad \left.\left. + \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial t^\beta}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}v_1\right) - 2\frac{x^\alpha}{\alpha}u_1\right]\right] \\
&= -16\left(\frac{t^\beta}{\beta}\right)^2 - 8\frac{t^\beta}{\beta} + 2\left(\frac{x^\alpha}{\alpha}\right)^2\left(\frac{t^\beta}{\beta}\right)^2, \\
u_3 &= 16\left(\frac{t^\beta}{\beta}\right)^3 + 16\left(\frac{t^\beta}{\beta}\right)^2 - \frac{4}{3}\left(\frac{x^\alpha}{\alpha}\right)^2\left(\frac{t^\beta}{\beta}\right)^3, \\
v_3 &= 16\left(\frac{t^\beta}{\beta}\right)^3 + 16\left(\frac{t^\beta}{\beta}\right)^2 - \frac{4}{3}\left(\frac{x^\alpha}{\alpha}\right)^2\left(\frac{t^\beta}{\beta}\right)^3, \\
u_4 &= -\frac{32}{3}\left(\frac{t^\beta}{\beta}\right)^4 - 16\left(\frac{t^\beta}{\beta}\right)^3 + \frac{2}{3}\left(\frac{x^\alpha}{\alpha}\right)^2\left(\frac{t^\beta}{\beta}\right)^4, \\
v_4 &= -\frac{32}{3}\left(\frac{t^\beta}{\beta}\right)^4 - 16\left(\frac{t^\beta}{\beta}\right)^3 + \frac{2}{3}\left(\frac{x^\alpha}{\alpha}\right)^2\left(\frac{t^\beta}{\beta}\right)^4,
\end{aligned} \tag{65}$$

and similarly for the rest components. Using equation (57), the series solutions are therefore given by

$$\begin{aligned}
u &= u_0 + u_1 + u_2 + \cdots = \left(1 - \left(2\frac{t^\beta}{\beta}\right) + \frac{\left(2\left(t^\beta/\beta\right)\right)^2}{2!}\right. \\
&\quad \left. - \frac{\left(2\left(t^\beta/\beta\right)\right)^3}{3!} + \frac{\left(2\left(t^\beta/\beta\right)\right)^4}{4!} - \cdots\right)\left(\frac{x^\alpha}{\alpha}\right)^2, \\
v &= v_0 + v_1 + v_2 + \cdots = \left(1 - \left(2\frac{t^\beta}{\beta}\right) + \frac{\left(2\left(t^\beta/\beta\right)\right)^2}{2!}\right. \\
&\quad \left. - \frac{\left(2\left(t^\beta/\beta\right)\right)^3}{3!} + \frac{\left(2\left(t^\beta/\beta\right)\right)^4}{4!} - \cdots\right)\left(\frac{x^\alpha}{\alpha}\right)^2.
\end{aligned} \tag{66}$$

Hence,

$$\begin{aligned}
u(x,t) &= \left(\frac{x^\alpha}{\alpha}\right)^2 e^{-2\left(t^\beta/\beta\right)}, \\
v(x,t) &= \left(\frac{x^\alpha}{\alpha}\right)^2 e^{-2\left(t^\beta/\beta\right)}.
\end{aligned} \tag{67}$$

The exact solution is obtained by taking $\alpha = 1$ and $\beta = 1$, as follows:

$$\begin{aligned}
u(x,t) &= x^2 e^{-2t}, \\
v(x,t) &= x^2 e^{-2t}.
\end{aligned} \tag{68}$$

5. Numerical Result

In this section, we discuss the precision and efficiency of the (CDLTDM) by numerical results of $u(x,t)$ for the exact solution when $(\alpha = \beta = 1)$ and approximate solutions at α and β taking different fractional values for conformable pseudoparabolic equation and conformable coupled pseudoparabolic equation. The solutions of equation (30) are depicted in Figures 1 and 2, respectively.

In Figure 1, the approximate solutions of equation (30) at $(t = 1)$ and $(\alpha = \beta)$, taking different fractional values, are compared and we found that the numerical solution becomes close to the exact solution when the fractional value increases:

$$(\alpha = \beta = 1). \tag{69}$$

Figure 2 indicates that the exact solution at $(\alpha\beta = 1)$ of equation (30) and the approximate solution of equation (30) decrease at the fractional derivative values $(\alpha\ 0.99)$ and $(\beta\ 0.98, 0.96, 0.94)$. Similarly, the exact solution and approximate solution of equation (44) are demonstrated in

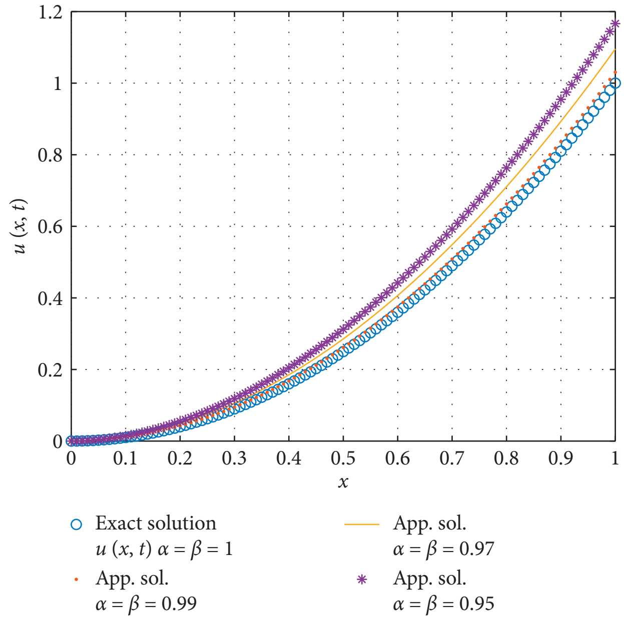

Figure 1: The exact and approximate solutions of $u(x,t)$ for Example 3, when $\alpha = \beta$.

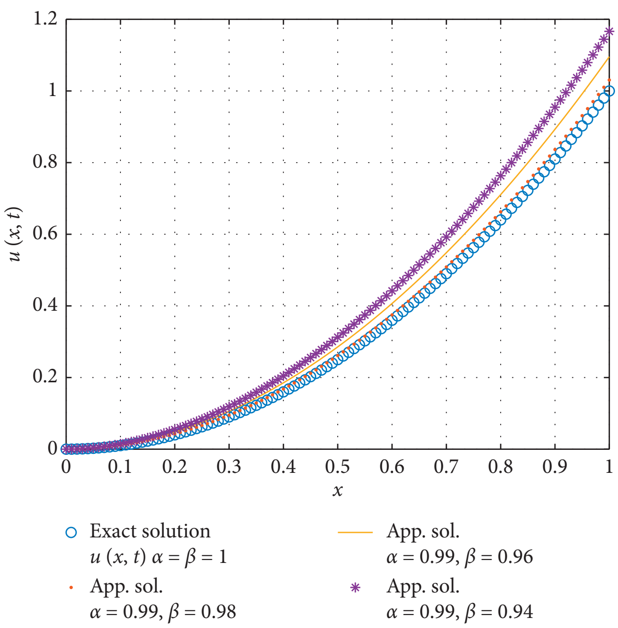

Figure 2: The exact and approximate solutions of $u(x,t)$, for Example 3, when we take different values of fractional order β and $\alpha = 0.99$.

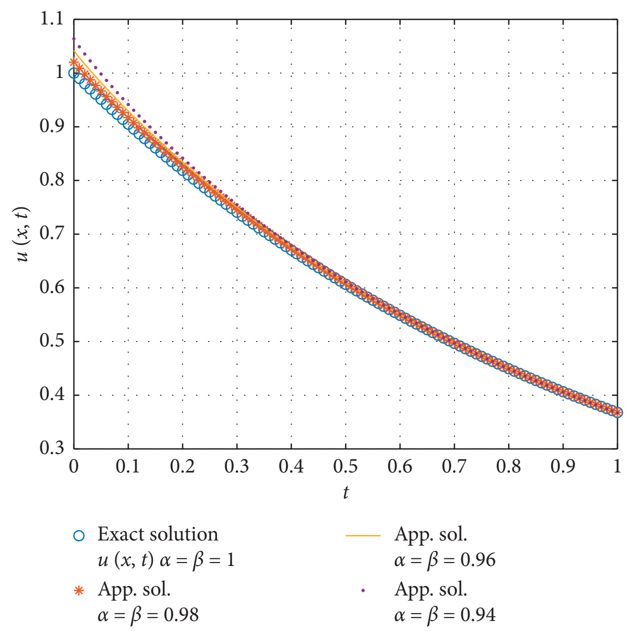

Figure 3: The exact and approximate solutions of $u(x,t)$ for Example 4, when $\alpha = \beta$.

Figures 3 and 4. Figure 3 give the plots of the behaviour of equation (44) when $(t = 1)$ and $(\alpha = \beta)$ with different fractional values taken in this case; the solution $u(x,t)$ becomes close to the exact solution at $(\alpha = \beta)$ close to one.

Figure 4 shows the approximate solution of equation (44) with $(0 < \alpha \leq 1)$, $(\beta = 0.99)$, and $(t = 1)$; in such a case, the function $u(x,t)$ gradually decreases. Finally, Figure 5 suggests that in the solutions of equation (62) at $(t = 1)$ and $(0 < \alpha = \beta \leq 1)$, we find that the numerical solution becomes close to the exact solution when the fractional value increases.

Figure 6 demonstrates that the exact solution at $(\alpha = \beta = 1)$ of equation (62) and the approximate solution of

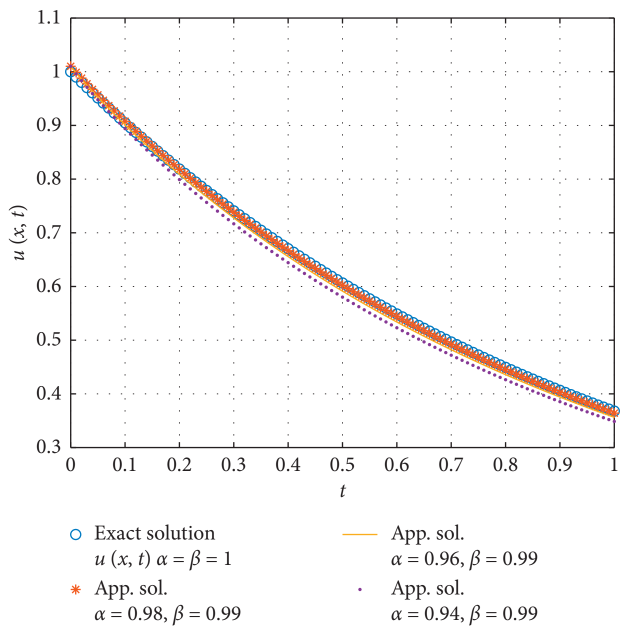

FIGURE 4: The exact and approximate solutions of $u(x,t)$, for Example 4, when we take different values of fractional order α and $\beta = 0.99$.

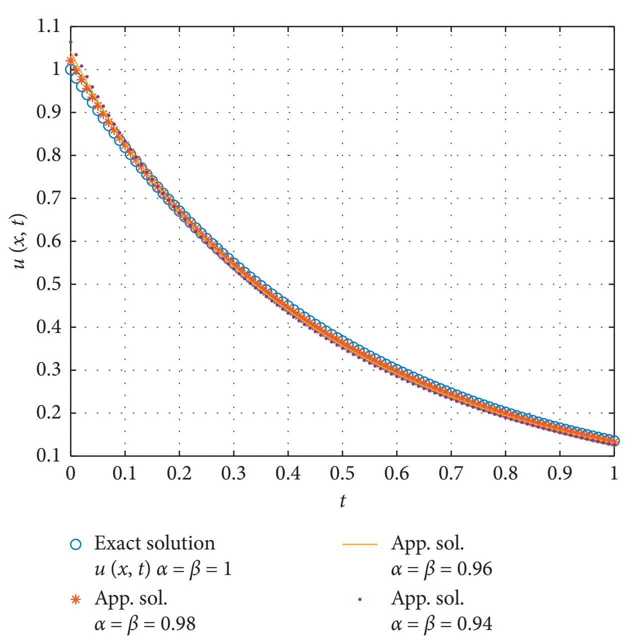

FIGURE 5: The exact and approximate solutions of $u(x,t)$ for Example 5, when $\alpha = \beta$.

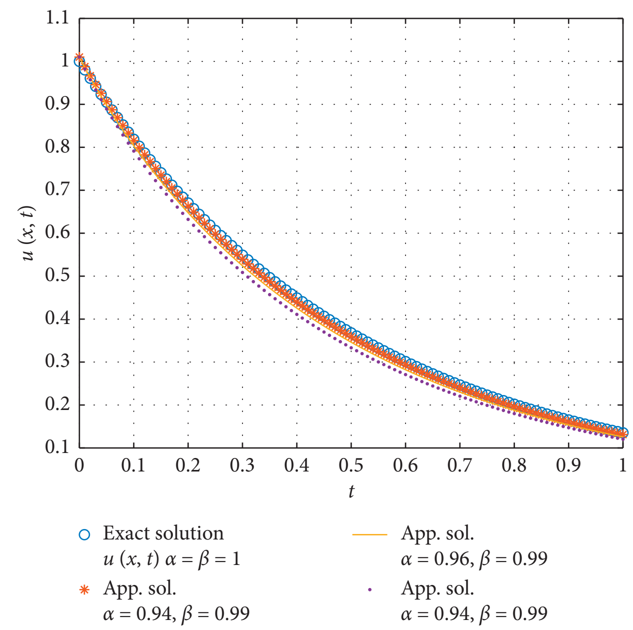

FIGURE 6: The exact and approximate solutions of $u(x,t)$, for Example 5, when we take different values of fractional order α and $\beta = 0.99$.

equation (62) are concave upward at the fractional derivative increasing when $(0 < \alpha \leq 1)$ and (β) fixed.

6. Conclusion

In this work, singular one-dimensional conformable pseudoparabolic equation and conformable coupled pseudoparabolic equation have been considered. Then, new conformable double Laplace transform decomposition methods have been applied to the problems. Finally, we gave three differential examples to show that this method is applicable and valid. The suggested method can also be applied for systems with more than two linear and nonlinear partial differential equations. In addition, if we let $\alpha = 1$ and $\beta = 1$ in Examples 3 and 4, we get the solution which is considered in [22]. All figure results are obtained by using Matlab.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this research group (RG-1440-030).

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