Subject: 7.1

1, 2, 3, 4, 5

1) Find $\|x\|_\infty$ and $\|x\|_2$ for the following vectors:-

a. $x = (3, -4, 0, \frac{3}{2})^t$.

$$\|x\|_2 = \sqrt{9 + 16 + 0 + \frac{9}{4}} = \sqrt{27.25} \simeq 5.2202.$$

$$\|x\|_\infty = \max\{|3|, |-4|, |0|, |\tfrac{3}{2}|\} = 4.$$

c. $x = (\sin k, \cos k, 2^k)^t$ for a fixed positive integer k.

$$\|x\|_2 = \sqrt{\sin^2 k + \cos^2 k + 2^{2k}} = \sqrt{1 + 2^{2k}}$$

$$\|x\|_\infty = \max\{|\sin k|, |\cos k|, |2^k|\} = 2^k.$$

since $|\sin k| \leq 1$, $|\cos k| \leq 1$.

d. $x = (\frac{4}{(k+1)}, \frac{2}{k^2}, k^2 e^{-k})^t$ for a fixed positive integer k.

$$\|x\|_2 = \sqrt{\frac{16}{(k+1)^2} + \frac{4}{k^4} + k^4 e^{-2k}}$$

$$\|x\|_\infty = \max\left\{\frac{4}{k+1}, \frac{2}{k^2}, \frac{k^2}{e^k}\right\}$$

Subject:

Date: / /

Subject: Date: / /

2) a. verify that the function $\|\cdot\|_1$, defined on $\mathbb{R}^n$ by,

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|,$$

is a norm on $\mathbb{R}^n$.

① $|x_i| \geq 0 \Rightarrow \|x\|_1 \geq 0$.

② $\|x\| = 0 \Leftrightarrow \sum |x_i| = 0 \Leftrightarrow |x_i| = 0, \forall i$
$\Leftrightarrow x = 0$

③ $\|\alpha x\| = \sum |\alpha x_i| = \sum |\alpha||x_i|$
$= |\alpha| \sum |x_i|$
$= |\alpha| \, \|x\|$

④ $\|x+y\| = \sum |x_i + y_i| \leq \sum (|x_i| + |y_i|)$
$= \sum |x_i| + \sum |y_i|$
$= \|x\| + \|y\|$

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b. Find $\|x\|_1$ for the vectors given in Ex-1.

حل (تطبيق مباشر)

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c. Prove that for all $x \in \mathbb{R}^n$, $\|x\|_1 \geq \|x\|_2$.

$x = (x_1, x_2, \ldots, x_n)$.

إمّا بالإستقراء أو بأي طريقة أخرى

$$\|x\|_2 = \left(\sum |x_i|^2\right)^{1/2}$$

$$\|x\|_2^2 = \sum |x_i|^2$$

NNB

Subject: ____________ Date: / /

NNB

Subject: ______________ Date: / /

$$\|x\|_2^2 = \sum |x_i|^2$$

$$\leq \left(\sum |x_i|\right)^2$$

$$= \|x_1\|^2$$

$$\|x_2\|^2 \leq \|x_1\|^2$$

$$\Longrightarrow \|x_2\| \leq \|x_1\|$$

we want to prove that:

$$\sum_{i=1}^{n} |x_i|^2 \leq \left(\sum_{i=1}^{n} |x_i|\right)^2$$

- بالاستقراء الرياضي -

① if $n=2$:

$$\sum_{i=1}^{2} |x_i|^2 = |x_1|^2 + |x_2|^2$$

$$\left(\sum_{i=1}^{2} |x_i|\right)^2 = (|x_1| + |x_2|)^2 = |x_1|^2 + 2|x_1||x_2| + |x_2|^2$$

$$\geq \sum_{i=1}^{2} |x_i|^2.$$

② نفرض صحة العلاقة عند k

i.e. $$\sum_{i=1}^{k} |x_i|^2 \leq \left(\sum_{i=1}^{k} |x_i|\right)^2$$

Subject: ____________________ Date: / /

Subject: Date: / /

③ (k+1) نثبت صحة العبارة عند

$$\sum_{i=1}^{k+1} |x_i|^2 \overset{?!!}{\leq} \left(\sum_{i=1}^{k+1} |x_i|\right)^2.$$

$$\sum_{i=1}^{k+1} |x_i|^2 = |x_1|^2 + |x_2|^2 + \dots + |x_k|^2 + |x_{k+1}|^2$$

$$= \sum_{i=1}^{k} |x_i|^2 + |x_{k+1}|^2$$

$$\leq \left(\sum_{i=1}^{k} |x_i|\right)^2 + |x_{k+1}|^2$$

$$\leq \left(\sum_{i=1}^{k+1} |x_i|\right)^2.$$

---

3. prove that the following sequences are convergent, and find their limits...

a. $x^{(k)} = \left(\frac{1}{k}, e^{1-k}, \frac{-2}{k^2}\right)^t$.

$$\lim_{k\to\infty} \frac{1}{k} = 0$$

$$\lim_{k\to\infty} e^{1-k} = \lim_{k\to\infty} e \cdot e^{-k} = \lim_{k\to\infty} \frac{e}{e^k} = 0$$

$$\lim_{k\to\infty} \frac{-2}{k^2} = 0$$

$$\therefore x^{(k)} \longrightarrow (0,0,0).$$

NNB

Subject: ______________________ Date: / /

Subject: Date: / /

b. $x^{(k)} = (e^{-k}\cos k, k\sin(\frac{1}{k}), 3+k^{-2})^t$.

$$\lim_{k\to\infty} e^{-k}\cos k = \lim_{k\to\infty} \frac{\cos k}{e^k} = 0$$

$$-1 \leq \cos k \leq 1$$

$$\frac{-1}{e^k} \leq \frac{\cos k}{e^k} \leq \frac{1}{e^k}$$

$$\lim_{k\to\infty} k\sin\left(\frac{1}{k}\right) = \lim_{\frac{1}{k}\to 0} \frac{\sin(\frac{1}{k})}{\frac{1}{k}} = 1.$$

$$\lim_{k\to\infty} 3+k^{-2} = \lim_{k\to\infty}\left(3+\frac{1}{k^2}\right) = 3+0 = 3.$$

$$\therefore x^{(k)} \longrightarrow (0,1,3).$$

c. $x^{(k)} = (k e^{-k^2}, \frac{\cos k}{k}, \sqrt{k^2+k}-k)^t$.

$$\lim_{k\to\infty} k e^{-k^2} = \lim_{k\to\infty} \frac{k}{e^{k^2}} = \lim_{k\to\infty} \frac{1}{2ke^{k^2}} = 0$$

$$\lim_{k\to\infty} \frac{\cos k}{k} = 0$$

$$\lim_{k\to\infty} \frac{\sqrt{k^2+k}-k \times \sqrt{k^2+k}+k}{\sqrt{k^2+k}+k}$$

$$= \lim_{k\to\infty} \frac{k^2+k-k^2}{\sqrt{k^2+k}+k} = \lim_{k\to\infty} \frac{k}{\cancel{k}\left(\frac{\sqrt{k^2+k}}{\sqrt{k^2}}+1\right)}$$

$$= \lim_{k\to\infty} \frac{1}{\sqrt{1+\frac{1}{k}}+1} = \frac{1}{2}$$

$$x^k \longrightarrow (0,0,\tfrac{1}{2}).$$

NNB

Subject:

Date: / /

Subject: $\|A\|_\infty = \max \sum |a_{ij}|$ Date ((thm 7.11))

4. Find $\|\cdot\|_\infty$ for the following matrices:-

a. $\begin{bmatrix} 10 & 15 \\ 0 & 1 \end{bmatrix}$

$\|A\|_\infty = \max\{25, 1\} = 25$

b. $\begin{bmatrix} 10 & 0 \\ 15 & 1 \end{bmatrix}$

$\|A\|_\infty = \max\{10, 16\} = 16$

c. $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$\|A\|_\infty = \max\{3, 4, 3\} = 4.$

5. The following linear systems $Ax=b$ have x as the actual solution and $\tilde{x}$ as an approximate solution. Compute $\|x-\tilde{x}\|_\infty$ and $\|A\tilde{x}-b\|_\infty$.

a. $\frac{1}{2}x_1 + \frac{1}{3}x_2 = \frac{1}{63}$

$\frac{1}{3}x_1 + \frac{1}{4}x_2 = \frac{1}{168}$

$x = \left(\frac{1}{7}, \frac{-1}{6}\right)^t$

$\tilde{x} = (0.142, -0.166)^t.$

Subject: ____________ Date: / /

Subject:

Date:

$$\|x - \tilde{x}\|_\infty = \left\|\left(\frac{1}{7} - 0.142, \frac{-1}{6} + 0.166\right)\right\|_\infty$$

$$= 0.000857$$

$$\|A\tilde{x} - b\|_\infty = \left\| \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0.142 \\ -0.166 \end{bmatrix} - \begin{bmatrix} \frac{1}{63} \\ \frac{1}{168} \end{bmatrix} \right\|_\infty$$

$$= \left\| \begin{bmatrix} -0.000206 \\ -0.00012 \end{bmatrix} \right\|_\infty$$

$$= \max\{0.000206, 0.00012\}$$

$$= 0.000206.$$

Subject:

Date: / /

Subject: 7.2 Date: / /

1, 2, 3, 4, 6

1. compute the eigenvalues and associated eigenvectors of the following matrices...

a. $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)^2 - 1 = 0$

$4 - 4\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{4}}{2}$$

$$= \frac{4 \pm 2}{2}$$

$\lambda_1 = 3$ or $\lambda_2 = 1$

NNB

Subject: ______________

Date:

Subject: Date: / /

① if $\lambda=1$:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$\therefore x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

Let $x_2 = t \in \mathbb{R}$.

$\therefore x_1 = t$.

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} ; t\in\mathbb{R}.$$

$\therefore$ eigen vector $(1,1)^t$.

② if $\lambda=3$:

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \xrightarrow[-1R_1]{} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

Let $x_2 = t \in \mathbb{R}$, $x_1 = -t$.

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix} ; t\in\mathbb{R}.$$

$\therefore$ eigenvector $(-1,1)^t$.

Subject: ____________

Date: / /

Subject: Date: / /

2. Find the spectral radius for each matrix in Exercise 1...

(a) $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$\Rightarrow \lambda = 1$ or $\lambda = 3$.

$\therefore \rho(A) = \max |\lambda| = \max\{1, 3\} = 3$.

3. which of the matrices in Ex. 1 are convergent?

a. $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ "Thm. 7.17"

since $\rho(A) = 3 > 1 \Rightarrow$ not convergent.

4. Let $A_1 = \begin{bmatrix} 1 & 0 \\ 1/4 & 1/2 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1/2 & 0 \\ 16 & 1/2 \end{bmatrix}$. show that A_1 is not convergent, but A_2 is convergent.

$\begin{vmatrix} 1-\lambda & 0 \\ 1/4 & \frac{1}{2}-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(\frac{1}{2}-\lambda) = 0$

$\cancel{\Rightarrow \frac{1}{2} - \lambda - \frac{1}{2}\lambda + \lambda^2 = 0}$

$\cancel{\Rightarrow \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = 0}$

$\Rightarrow \lambda = 1$ or $\lambda = \frac{1}{2}$

$\rho(A_1) = \max\{1, \frac{1}{2}\} = 1 \not< 1 \Rightarrow$ not convergent.

NNB

Subject: ____________________ Date: / /

NNB

Subject: Date: / /

$$\begin{vmatrix} \frac{1}{2}-\lambda & 0 \\ 16 & \frac{1}{2}-\lambda \end{vmatrix} = 0 \Rightarrow \left(\frac{1}{2}-\lambda\right)^2 = 0$$

$$\Rightarrow \lambda = \frac{1}{2}, \frac{1}{2}$$

$$\therefore \rho(A_2) = \max\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2} < 1.$$

$\therefore A_2$ is Convergent.

6. Show that if λ is an eigen value of a matrix A and $\|\cdot\|$ is a vector norm, then an eigenvector x associated with λ exists with $\|x\| = 1$.

- Let x be the eigenvector associated with λ, with $\|x\| = a$.

then $a \neq 0$, since $x \neq 0$.

now, consider the vector $\frac{x}{a}$, then $\frac{x}{a} \neq 0$

and $(A-\lambda I)\left(\frac{x}{a}\right) = \frac{1}{a}\left[(A-\lambda I)x\right]$

$$= \frac{1}{a}[0] = 0$$

hence, $\frac{x}{a}$ is an eigenvector, and:

$$\left\|\frac{x}{a}\right\| = \frac{1}{a}\|x\| = \frac{1}{\|x\|}\|x\| = 1.$$

NNB

Subject: Date: / /

Date:

Subject: 7.4

1 – 2 – 3

1) Compute the condition numbers of the following matrices relative to $\|\cdot\|_\infty$.

(a) $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$

$$\|A\|_\infty = \max\left\{\frac{1}{2}+\frac{1}{3}, \frac{1}{3}+\frac{1}{4}\right\} = \max\left\{\frac{5}{6}, \frac{7}{12}\right\} = \frac{5}{6}$$

$$A^{-1} = \frac{1}{\frac{1}{8}-\frac{1}{9}} \begin{bmatrix} \frac{1}{4} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{1}{2} \end{bmatrix} = \frac{1}{\frac{1}{72}} \begin{bmatrix} \frac{1}{4} & \\ \frac{-1}{3} & \end{bmatrix}$$

$$= 72 \begin{bmatrix} \frac{1}{4} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 18 & -24 \\ -24 & \end{bmatrix}$$

$$\|A^{-1}\|_\infty = \max\{42, 60\} = 60$$

$$\therefore K(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty$$

$$= \frac{5}{\cancel{6}}\left(\overset{10}{\cancel{60}}\right)$$

$$= 50$$

6

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

Date:

$\|A\|_\infty = \max\{3, 2, 1\} = 3.$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[1R_2+R_1]{-1R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$\downarrow$

A^{-1}

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$$

$\max\{4, 2, 1\} = 4.$

$\|_\infty \cdot \|A^{-1}\|_\infty = 3 \times 4 = 12.$

NNB

Date:

Subject:

2) The following linear systems $Ax=b$ have x as the actual solution and $\tilde{x}$ as an approximation solution. Using the results of Ex. 1, compute $\|x-\tilde{x}\|_\infty$ and $K_\infty(A)\frac{\|b-A\tilde{x}\|_\infty}{\|A\|_\infty}$ ...

ⓐ $\frac{1}{2}x_1+\frac{1}{3}x_2=\frac{1}{63}$

$\frac{1}{3}x_1+\frac{1}{4}x_2=\frac{1}{168}$

$x=\left(\frac{1}{7},\frac{-1}{6}\right)^t$

$\tilde{x}=(0.142,-0.166)^t$

$\|A\|_\infty=\frac{5}{6}$.

$K_\infty(A)=50$.

$\cancel{\|}x-\tilde{x}\cancel{\|_\infty}=(0.000857,-0.000666)$

0.000857.

$\Longrightarrow \|x-\tilde{x}\|_\infty=0.000857$

$b-A\tilde{x}=\begin{bmatrix}\frac{1}{63}\\ \frac{1}{168}\end{bmatrix}-\begin{bmatrix}\frac{1}{2} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{4}\end{bmatrix}\begin{bmatrix}0.142\\ -0.16\end{bmatrix}$

$=\begin{bmatrix}\frac{1}{63}\\ \frac{1}{168}\end{bmatrix}-\begin{bmatrix}0.015666666\\ 0.005833333\end{bmatrix}$

Subject:

Date: / /

$$\frac{-1}{0.0002}\begin{bmatrix} 2 & -2 \\ -1.0001 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -10000 & +10000 \\ 5000.5 & -5000 \end{bmatrix}$$

Date: / /

Subject:

$$b - A\tilde{x} = \begin{bmatrix} 0.000206349203 \\ 0.00011904 76524 \end{bmatrix}.$$

$$\therefore \|b - A\tilde{x}\|_\infty = 0.000206$$

$$\therefore K_\infty(A) \frac{\|b - A\tilde{x}\|_\infty}{\|A\|_\infty} = (50)\frac{(0.000206)}{\left(\frac{5}{6}\right)}$$

$$= \frac{(\cancel{50}^{10})(0.000206)(6)}{\cancel{5}_{1}}$$

$$= 0.01236.$$

3) The linear system:

$$\begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.0001 \end{bmatrix}.$$

has solution $(1, 1)^t$. change A slightly to

$$\begin{bmatrix} 1 & 2 \\ 0.9999 & 2 \end{bmatrix}$$

and consider the linear system:

$$\begin{bmatrix} 1 & 2 \\ 0.9999 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.0001 \end{bmatrix}$$

Subject: Date: / /

(7.24):

$$\frac{\|x-\tilde{x}\|}{\|x\|} \leq \frac{K(A)\|A\|}{\|A\|-K(A)\|\delta A\|}\left(\frac{\|\delta b\|}{\|b\|}+\frac{\|\delta A\|}{\|A\|}\right)$$

$$\frac{\|x-\tilde{x}\|_\infty}{\|x\|_\infty} = \frac{\max\{2,1\}}{\max\{1,1\}} = \frac{2}{1} = 2$$

$$\delta(A) = \begin{bmatrix} 0 & 0 \\ -0.0002 & 0 \end{bmatrix}, \delta(b) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\|\delta(A)\|_\infty = 0.0002 \quad , \|\delta(b)\|_\infty = 0$$

$$\therefore \frac{K(A)\|A\|}{\|A\|-K(A)\|\delta(A)\|}\left(\frac{\|\delta b\|}{\|b\|}+\frac{\|\delta(A)\|}{\|A\|}\right)$$

$$= \frac{(60000)(3)}{3-\underbrace{(60000)(0.0002)}_{=12}}\left(\frac{0}{3.0001}+\frac{0.0002}{3}\right)$$

$$= \frac{(60000)\cancel{(3)}^{1}}{-9/_3}\left(\frac{1}{15000}\right)$$

$$= \frac{-20000}{15000} = \frac{-4}{3}.$$

Subject: Date: / /

$$2 < \frac{-4}{3}$$

(X)

:. العلاقة (7,24) ليست صحيحة.

Subject: Date: / /