PHYS 454

HANDOUT 9 – *The algebraic theory of a quantum mechanical simple harmonic oscillator*

1. For an eigenstate of a quantum SHO prove the following results:

a) The expectation values of the position and momentum are zero.
b) The expectation values of the potential and kinetic energies are equal.
c) The uncertainties in position and momentum satisfy the relation $\Delta x \Delta p = \left(n + 1/2\right)\hbar$, where n is the quantum number of the state.

2. Show that $\left[N,\ a\right] = -a$, $\left[N,\ a^{\dagger}\right] = a^{\dagger}$, $\left[N,\ a^2\right] = -2a^2$.

3. Show that $\left[N,\ aa^{\dagger}a\right] = -aa^{\dagger}a$.

4. The wave function of a SHO at a certain time instant is given by $\psi\left(\xi\right) = \left(m\omega / \hbar\pi\right)^{1/4} \exp\left[-\left(\xi - a\right)^2 / 2\right]$. Show that the probabilities to find anyone of the even or odd eigenvalues are given by

$$P_{\pm}\left(a\right) = \frac{1 \pm e^{-a^2}}{2}$$

Do they change with time?

5. Compute the quantities $\left\langle n\middle|x^2\middle|m\right\rangle$ and $\left\langle n\middle|p^2\middle|m\right\rangle$ for the one-dimensional harmonic oscillator.

6. Show that the function $u(x) = e^{-x^2/4}$ is an eigenfunction of the operator $\left(\frac{d^2}{dx^2} - \frac{1}{4}x^2\right)$. Find its eigenvalue.

7. A particle moves under in a potential $V(x) = \left(1/2\right)kx^2$ and at a certain moment its state is given by the wave function $\psi(x) = N\exp\left(-\lambda x^2 / 2\right)$. Calculate the average value of the energy. Calculate the value λ for which this energy is minimum.

8. In a harmonic oscillator consider the wave function

$$\psi(x) = \left(ax^2 + bx + c\right)e^{-x^2/2}.$$

Find the constants a, b and c so the above function is an eigenfunction of the quantum SHO. Calculate its energy.

9. A particle of mass m is inside a harmonic potential $V(x) = \frac{1}{2} m\omega^2 x^2$. At a certain moment the particle captures another particle of the same mass. What is the probability the new composite particle to stay in the ground state?

10. For a simple harmonic oscillator, consider the set of *coherent states* defined as:

$$|x\rangle \equiv e^{-x^2/2} \sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n!}} |n\rangle$$

(a) Show that they are normalized. Prove that they are eigenstates of the annihilation operator a with eigenvalue x.

(b) Calculate the expectation value $\langle N \rangle$ of the operator $N = a^\dagger a$ and the uncertainty ΔN in such a state. Show that $\lim_{N\to\infty} \Delta N / N = 0$.

(c) Suppose that the oscillator is initially in such a state at $t = 0$. Calculate the probability of finding the system in this state at a later time $t > 0$. Prove that the evolved state is still an eigenstate of the annihilation operator with a time-dependent eigenvalue. Calculate $\langle N \rangle$ and $\langle N^2 \rangle$ in this state and prove that they are time independent.

11. Estimate the minimum energy of a quantum simple harmonic oscillator from Heisenberg's Uncertainty Principle and check if it coindiced with the real value of the ground state energy.

12. Find the eigenstates of the annihilation operator $\hat{a}$: $\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$.

13. Prove the following commutation relation: $\left[a, (a^\dagger)^n\right] = n(a^\dagger)^{n-1}$.