PHYS 551-505
HANDOUT 5 – Composition of angular momenta

1. Show in the expression for the Clebsch-Gordan coefficients that $m = m_1 + m_2$.

2. Show the orthonormality relation:

$$\sum_{j,m}\left\langle j_1 j_2 m_1' m_2' \middle| j_1 j_2 jm \right\rangle \cdot \sum_{j,m}\left\langle j_1 j_2 jm \middle| j_1 j_2 m_1 m_2 \right\rangle = \delta_{m_1' m_1}\delta_{m_2' m_2}$$

3. Show that the Clebsch-Gordan coefficient is non-zero if $j_1' = j_1$ and $j_2' = j_2$.

4. Show the orthonormality relation:

$$\sum_{m_1, m_2}\left\langle j_1 j_2 jm \middle| j_1 j_2 m_1 m_2 \right\rangle \cdot \left\langle j_1 j_2 m_1 m_2 \middle| j_1 j_2 j' m' \right\rangle = \delta_{jj'}\delta_{mm'}$$

5. Construct the eigenstates of total angular momentum of a hydrogen atom at the excited state *2p.*

6. Find the Clebsch-Gordan coefficients for two *p* electrons.

7. In the external orbit of the carbon C atom there are two electrons with *l*=1 for each of them and of course with spin *s*=1/2. For reasons that have a deep physical meaning, but we do not explain them here the combination of these for angular momenta is done in two stages as follows: a) first the partial angular momenta $\mathbf{l}_1$, $\mathbf{l}_2$ are composed to give the total orbital angular momentum **l**. b) after that the partial angular momenta $\mathbf{s}_1$, $\mathbf{s}_2$ are composed to give the total spin **s.** Then we get the total angular momentum **j**=**l**+**s**.

   Follow the above steps to calculate all the possible values of the total angular momentum *j*. Confirm that the number of states before and after the composition remains the same.

8. Show that the product $\mathbf{l}\cdot\mathbf{s}$ has well defined eigenvalues at states with definite total angular momentum **j**.

9. Two particles with spin $s_1 = 3/2$ and interact with the Hamiltonian $H = A\mathbf{s}_1\cdot\mathbf{s}_2$ where *A* is a given constant. Calculate the energy eigenvalues of the system and the degree of degeneracy of the system.

10. For two particles with spins $s_1 = s_2 = 1/2$ find the common eigenstates of the operators $\mathbf{S}^2$, $\mathbf{S}_z$, $\mathbf{s}_1^2$, $\mathbf{s}_2^2$ as a linear combination of the eigenstates of the operators s_{1z}, s_{2z}, $\mathbf{s}_1^2$, $\mathbf{s}_2^2$, if you are given the following table of Glebsch-Gordan coefficients (Menis 333):

		J=1			
		M=1	J=1	J=0	
$m_{s1}=1/2,\ m_{s2}=-1/2$		1	M=0	M=0	
	$m_{s1}=1/2,\ m_{s2}=-1/2$		$1/\sqrt{2}$	$1/\sqrt{2}$	J=1 M=-1
	$m_{s1}=-1/2,\ m_{s2}=1/2$		$1/\sqrt{2}$	$-1/\sqrt{2}$	
			$m_{s1}=-1/2,\ m_{s2}=-1/2$		1

11. Show that for the particles of the previous problem:
$\left|0,0;\frac{1}{2},\frac{1}{2}\right\rangle=\frac{1}{\sqrt{2}}\left|\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle-\frac{1}{\sqrt{2}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle$ (Menis 337).

12. In the hydrogen atom there is an additional spin-orbit interaction which is described by the Hamiltonian:

$$H^{'}=\frac{2a}{\hbar^2}\mathbf{L}\cdot\mathbf{S}$$

Calculate the energy spectrum of the hydrogen atom (Lag 68).

13. Two atoms have angular momenta $j_1=j_2=2$ and move in one dimension with a potential given by $V(x)=\lambda\mathbf{J}_1\cdot\mathbf{J}_2x^2$ with $\lambda>0$. The two atoms can form a molecule if their potential energy is given by $V(x)=ax^2$ with $a>0$. Find the energies for which a molecule may be formed (Lag 70).

14. Three particles with spins $s_1=s_2=s_3=1/2$ interact through the so called Heisenberg Hamiltonian given by (Lag 78):

$$\hat{H}=A\sum_{i<j}\mathbf{S}_i\cdot\mathbf{S}_j \qquad (i,j=1,2,3)\,.$$

15. You are given the matrices:

$$\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}5 & 4\\ 2 & 1\end{pmatrix}.$$

Find the tensorial product $\mathbf{A}\otimes\mathbf{B}$.

16. Form the states $|++\rangle$, $|+-\rangle$, $|-+\rangle$, $|--\rangle$ where (menis 341):

$$|+\rangle=\begin{pmatrix}1\\ 0\end{pmatrix},\quad |-\rangle=\begin{pmatrix}0\\ 1\end{pmatrix}.$$

Where $\left|+\right\rangle$, $\left|-\right\rangle$ the known eigenvectors of the spin operators $\mathbf{s}^2$, s_z.

17. For a particle with spin *s*=1/2 we know that:

$$\mathbf{s}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find the matrices which represent the operators: $\mathbf{s}_{1z} = \mathbf{s}_z \otimes \mathbf{I}$ and $\mathbf{s}_{2z} = \mathbf{I} \otimes \mathbf{s}_z$ and verify their action on the common eigenvectors (Menis 342).

18. For a particle with spin *s*=1/2 we know that:

$$\mathbf{s}^2 = \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Find the matrices which represent the operators: $\mathbf{s}_1^2 = \mathbf{s}^2 \otimes \mathbf{I}$ and $\mathbf{s}_1^2 = \mathbf{I} \otimes \mathbf{s}^2$. (Menis 343).

19. Find the matrices which represent the operators: $\mathbf{s}_{1x}$, $\mathbf{s}_{1y}$, $\mathbf{s}_{2x}$, $\mathbf{s}_{2y}$ in the space of two particles of spin *s*=1/2 (Menis 344).

20. Find the matrices which represent the operators: $\mathbf{s}^2$, $\mathbf{s}_z$ in the space of two particles of spin *s*=1/2 (Menis 345).

21. What are the eigenstates of $\mathbf{s}^2$, $\mathbf{s}_z$ in the space of two particles of spin *s*=1/2 (Menis 346).

22. The Hamiltonian of a system of two electrons is given by:

$$\mathbf{H} = \left(2\varepsilon_0 / \hbar^2\right)\mathbf{s}_1 \cdot \mathbf{s}_2 .$$

The time instant *t*=0, the system is at the state:

$$\left|\psi(0)\right\rangle = \frac{1}{\sqrt{3}}\left|\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}\right\rangle.$$

a) Find the energy spectrum and the degeneracy.
b) Find the state at $t > 0$.
c) What is the probability the system to be found at the states of total spin.