PHYS 551-505
HANDOUT 4 – On spin, angular momentum and rotations

1. Prove the identities for the Pauli matrices:
   i) Find the matrices $\sigma_x\sigma_y,\ \sigma_y\sigma_x,\ \sigma_z\sigma_y,\ \sigma_y\sigma_z,\ \sigma_x\sigma_z,\ \sigma_z\sigma_x$.
   ii) $\sigma_x\sigma_y+\sigma_y\sigma_x=\sigma_z\sigma_y+\sigma_y\sigma_z=\sigma_x\sigma_z+\sigma_z\sigma_x=0$
   iii) Prove that $\sigma_x^2=\sigma_y^2=\sigma_z^2=1$.
   iv) $(\sigma\cdot\mathbf{A})(\sigma\cdot\mathbf{B})=\mathbf{A}\cdot\mathbf{B}+i\sigma(\mathbf{A}\times\mathbf{B})$, where $\sigma=(\sigma_x,\sigma_y,\sigma_z)$ is the vector operator made up by Pauli matrices.

2. Show that if the rotation around z-axis at an angle ϕ is represented by the matrix:
$$R_z(\phi)=\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix},$$
   then the rotation around x-axis is described by the matrix:
$$R_x(\phi)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi & \cos\phi\end{pmatrix}.$$

3. Show that if we consider very small angles of rotation (call them ε) then, for the corresponding rotation matrices R we have approximately:
$$R_x(\varepsilon)R_y(\varepsilon)-R_y(\varepsilon)R_x(\varepsilon)=R_z(\varepsilon^2)-\mathbf{1}.$$
   Thus show that infinitesimal rotations about different axes commute if we ignore terms of order ε^2 or higher.

4. a) For a particle with spin $1/2$ write the projection of the spin $S_n=\hat{S}\hat{\mathbf{n}}$ on an axis n (where $\hat{\mathbf{n}}$ the unit vector) in a matrix form in Cartesian coordinates. b) Show that the projection has eigenvalues $\pm\hbar/2$. c) Express the operator in spherical coordinates. d) Find the eigenvectors of this operator. e) What is the probability in a measurement of the projection of the spin along the z-axis to find the values $S_n=\pm\hbar/2$.

5. A particle with spin $1/2$ is found at the state $|X\rangle=\frac{1}{\sqrt{2}}|+\rangle+\frac{1+i}{2}|-\rangle$. In which direction of space the uncertainty of spin is zero?

6. In the basis of the eigenvectors of S_z prove the closure relation $\sum_n |n\rangle\langle n| = \mathbf{I}$

7. The state of an electron is described by the wavefunction:

$$\psi(\mathbf{r},\text{spin}) = \varphi(\mathbf{r})\cdot X = A\left[\sqrt{x^2+y^2+z^2} - 2\sqrt{3}x + \sqrt{3}z\right]e^{-a\sqrt{x^2+y^2+z^2}}\begin{pmatrix}0\\1\end{pmatrix}$$

a) What will be the outcome of a measurement of the magnitude of the angular momentum and of its projection along the *z*-axis.
b) What will be the outcome of a measurement of the spin projection along the *z*-axis and what of the spin projection along the *y*-axis.

8. Examine the effect of rotation operator $D_z(\phi) = \exp(-iS_z\phi/\hbar)$ on a general ket in the space of a particle with spin $s=1/2$.

9. If an operator commutes with two of the components of the angular momentum operator, then it commutes also with the third.

10. Show that a for a system in the eigenstate $|jm\rangle$ of the operator J_z, the mean value of the component of angular momentum along the direction z', which makes an angle θ with the *z*-axis, is equal to $m\hbar\cos\theta$.

11. Since the components of the angular momentum operator do not commute, their simultaneous measurement is not possible. Show that in a state $|jm\rangle$ the greatest accuracy of measurement of the components J_x and J_y is obtained when $|m| = j$.

12. Obtain expression for the operators l_z by starting from the fact that these operators are related to infitesimal rotation operators.

13. Let $\mathbf{l}$ be the orbital angular momentum and $\mathbf{P}$ the so called parity operator, which performs a reflection about the origin, so that any function of the form $F(r,\theta,\phi)$ is given by the relation: $\mathbf{P}F(r,\theta,\phi) = F(r,\pi-\theta,\phi+\pi)$. Show that $[\mathbf{P},\,\mathbf{l}] = 0$