PHYS 551-505

HANDOUT 2 – On the algebraic method for the angular momentum

1. Prove the relations:

$$\left[l_z, l_+\right] = l_+, \quad \left[l_z, l_-\right] = -l_-$$

2. Prove the relations:

$$l_+\left|lm\right\rangle = \hbar\sqrt{l\left(l+1\right)-m\left(m+1\right)}\left|l,m+1\right\rangle$$
$$l_-\left|lm\right\rangle = \hbar\sqrt{l\left(l+1\right)-m\left(m-1\right)}\left|l,m-1\right\rangle$$

3. Prove the relations:

$$l_-l_+ = \mathbf{l}^2 - l_z\left(l_z+\hbar\right), \quad l_+l_- = \mathbf{l}^2 - l_z\left(l_z-\hbar\right), \quad \left[l_+, l_-\right] = 2\hbar l_z$$

4. Use the algebraic techniques to show that, on a generic state Y_l^m:

$$\left\langle l_x\right\rangle = \left\langle l_y\right\rangle = 0, \quad \left\langle l_x^2\right\rangle = \left\langle l_y^2\right\rangle = \hbar^2\left[l\left(l+1\right)-m(m+1)\right]/2.$$

5. Find the eigenvalues and eigenfunctions of the operators:

(a) $l_x^2 + l_y^2$

(b) $l_x^2 + l_y^2 - l_z^4$

(c) $l_-l_+^2l_-$

6. Find the average value of the operator l_x^4 at the maximum projection state Y_l^l.

7. We know that $Y_2^1\left(\theta,\phi\right) = -\sqrt{15/8\pi}\sin\theta\cos\theta e^{i\phi}$. Apply the raising operator to find $Y_2^2\left(\theta,\phi\right)$. Hint: You will need the position representation of the operators l_x, l_y .

$$l_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\tan\theta}\frac{\partial}{\partial\phi}\right), \quad l_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \frac{\sin\phi}{\tan\theta}\frac{\partial}{\partial\phi}\right)$$

8. The action of the raising operator l_+ on the maximum projection state Y_l^l gives zero, i.e. $l_+ Y_l^l = 0$. Use this property to find the analytical form of Y_l^l. Hint: You will need the position representation of the operators l_x, l_y .

9. For a state with angular momentum l = 1, find the matrix representation for the operators: $\mathbf{l}^2$, l_x, l_y, l_z.

10. Since the components of the angular momentum operator do not commute, their simultaneous measurement is not possible. Show that in a state $|lm\rangle$ the greatest accuracy of measurement of the components l_x, l_y is obtained $|m| = l$.