Logic Mathematics (132)

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Chapter 5: Function

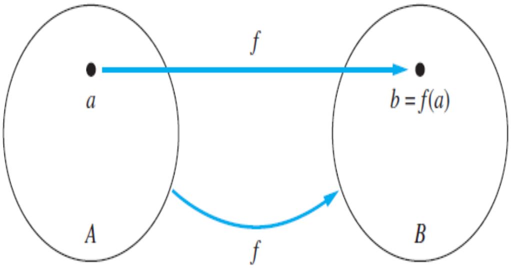
Function

DEFINITION 1

Let A and B be nonempty sets. A **function** f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

DEFINITION 2

If f is a function from A to B, we say that A is the **domain** of f and B is the **codomain** of f.
If $f(a) = b$, we say that b is the **image** of a and a is a **preimage** of b.
The **range**, or **image**, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f **maps** A to B.



Example 1

Let $f : \mathbb{Z} \to \mathbb{Z}$ assign the square of an integer to this integer. Then, $f(x) = x^2$, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, \ldots\}$.

Function

DEFINITION 3

Let f_1 and f_2 be functions from A to $\mathbb{R}$. Then $f_1 + f_2$ and $f_1 \times f_2$ are also functions from A to $\mathbb{R}$ defined for all $x \in A$ by
$(f_1 + f_2)(x) = f_1(x) + f_2(x)$,
$(f_1 \times f_2)(x) = f_1(x) \times f_2(x)$.

DEFINITION 4

Let f be a function from A to B and let S be a subset of A. The image of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by f (S), so

$$f(S) = \{t \; ; \; \exists s \in S, \;\; (t = f(s))\}.$$

We also use the shorthand $\{f(s) \; ; \; s \in S\}$ to denote this set.

Function

Example 2

Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with f (a) = 2, f (b) = 1, f (c) = 4, f (d) = 1, and f (e) = 1. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

DEFINITION 5

A function f is said to be **one-to-one**, or an **injunction**, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be **injective** if it is one-to-one.

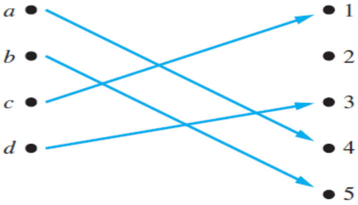
Remark

We can express that f is one-to-one using quantifiers as
$\forall a\, \forall b\, (f(a) = f(b)) \rightarrow a = b$ or equivalently
$\forall a\, \forall b\, (a \neq b \rightarrow f(a) \neq f(b))$, where the universe of discourse is the domain of the function.

Example 3

Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4, f(b) = 5, f(c) = 1$, and $f(d) = 3$ is one-to-one.

Solution: The function f is one-to-one because f takes on different values at the four elements of its domain. This is illustrated in the Figure below.



Example 3

Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.
Solution: The function $f(x) = x + 1$ is a one-to-one function. To demonstrate this, note that $x + 1 \neq y + 1$ when $x \neq y$.

DEFINITION 6

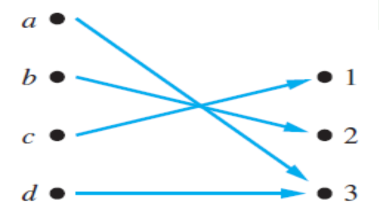
A function f whose domain and codomain are subsets of the set of real numbers is called **increasing** if $f(x) \leqslant f(y)$, and **strictly increasing** if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f.
Similarly, f is called **decreasing** if $f(x) \geqslant f(y)$, and **strictly decreasing** if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f. (The word strictly in this definition indicates a strict inequality.)

DEFINITION 7

A function f from A to B is called **onto**, or a **surjection**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called **surjective** if it is onto.

Example 4

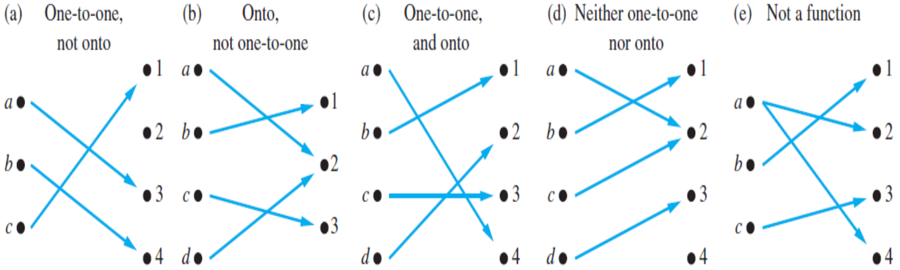
Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3, f(b) = 2, f(c) = 1$, and $f(d) = 3$. Is f an onto function?
Solution: Because all three elements of the codomain are images of elements in the domain, we see that f is onto. This is illustrated in the Figure below.



Example 5

Is the function f (x) = x + 1 from the set of integers to the set of integers onto?

Solution: This function is onto, because for every integer y there is an integer x such that f (x) = y. To see this, note that f (x) = y if and only if x + 1 = y, which holds if and only if x = y - 1.



DEFINITION 8

The function f is a **one-to-one correspondence**, or a **bijection**, if it is both one-to-one and onto. We also say that such a function is **bijective**.

Example 6

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4, f(b) = 2, f(c) = 1$, and $f(d) = 3$. Is f a bijection?
Solution: The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the codomain are images of elements in the domain. Hence, f is a bijection.

Function

Example 6

Let A be a set. The identity function on A is the function $\iota_A : A \to A$, where

$$\iota_A(x) = x$$

for all $x \in A$. In other words, the identity function ι_A is the function that assigns each element to itself. The function ι_A is one-to-one and onto, so it is a bijection. (Note that ι is the Greek letter iota.)

Function

Summarize

Suppose that $f : A \to B$.

To show that f is injective Show that if f (x) = f (y) for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

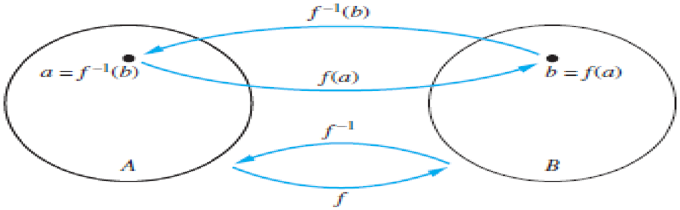
To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Function

DEFINITION 9

Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that f (a) = b. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when f (a) = b.



Function

Example 7

Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2, f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f , so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Example 8

Let $f : \mathbb{Z} \to \mathbb{Z}$ be such that f (x) = x + 1. Is f invertible, and if it is, what is its inverse?

Solution: The function f has an inverse because it is a one-to-one correspondence, as follows from Examples 3 and 5. To reverse the correspondence, suppose that y is the image of x, so that y = x + 1. Then x = y 1. This means that y 1 is the unique element of $\mathbb{Z}$ that is sent to y by f . Consequently, $f^{-1}(y) = y - 1$.

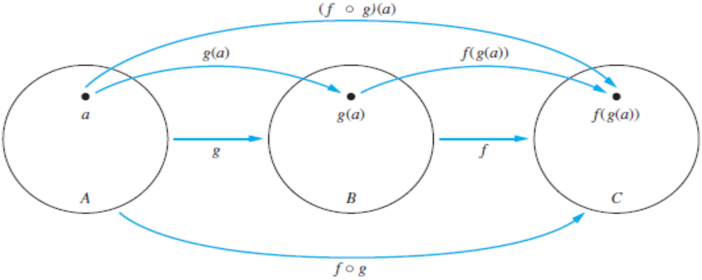
Function

Let f be a function from the set A to the set B. Let S be a subset of B.We define the inverse image of S to be the subset of A whose elements are precisely all pre-images of all elements of S.We denote the inverse image of S by $f^{-1}(S)$, so $f^{-1}(S) = \{a \in A | f(a) \in S\}$. (Beware: The notation f^{-1} is used in two different ways. Do not confuse the notation introduced here with the notation $f^{-1}(y)$ for the value at y of the inverse of the invertible function f . Notice also that $f^{-1}(S)$ the inverse image of the set S, makes sense for all functions f , not just invertible functions.)

DEFINITION 10

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a))$$



Example 9

Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b, g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3, f(b) = 2$, and $f(c) = 1$. What is the composition of $f \circ g$, and what is the composition of $g \circ f$?

Solution: The composition $f \circ g$ is defined by
$(f \circ g)(a) = f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b)) = f(c) = 1$, and
$(f \circ g)(c) = f(g(c)) = f(a) = 3$.

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Function

Example 10

Let f and g be the functions from the set of integers to the set of integers defined by f (x) = 2x + 3 and g(x) = 3x + 2. What is the composition of $f \circ g$? What is the composition of $g \circ f$?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,
$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$ and
$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$

Theorem

Let g be a function from A to B and f be a function from B to C.

1. If both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.
2. If both f and g are onto functions, then $f \circ g$ is also onto.