Introduction to fragmentation processes

lecture notes[1], version: July 18, 2011

Robert Knobloch

[1]This course was given in the summer semester 2011 for students in the Master's degree programme in mathematics at the Goethe–Universität Frankfurt am Main

CONTENTS

1 Homogenous fragmentation processes **1**

1.1 Introductory remarks 1

1.2 Preliminary considerations – The finite activity case 1

1.2.1 Stick–breaking process 2

1.2.2 Inheritance along a Galton–Watson process 2

1.3 Mass fragmentation processes 3

1.4 Exchangeable partition measures 5

1.5 Fragmentation processes with a genealogical structure 11

1.5.1 Partition–valued fragmentation processes 11

1.5.2 Dislocation 15

1.6 Erosion 18

1.6.1 Interval fragmentation processes 19

1.7 Bijections between different classes of fragmentation processes . 21

2 Properties of fragmentation processes **23**

2.1 Connection between fragmentation processes and subordinators . 23

2.1.1 Subordinators 23

2.1.2 Subordinators associated with fragmentations 24

2.2 Many–to–one identities 31

2.3 The intrinsic additive martingale for fragmentation processes . . 32

2.4 Speed of the largest particle 35

2.5 Spine decomposition 36
2.6 Stopped fragmentations 39

3 Energy cost of fragmentations **42**
3.1 Introduction 42
3.2 The case of finite activity 43
3.2.1 Main result 43
3.2.2 Crump–Mode–Jagers processes 45
3.2.3 Proof in the fragmentation setting 46
3.3 The general case 47

Bibliography **53**

CHAPTER 1

HOMOGENOUS FRAGMENTATION PROCESSES

Here we give a brief introduction to the theory of fragmentation processes and related concepts. Our intention is to provide the basic notions that are used throughout this course.

1.1 Introductory remarks

This chapter is devoted to the compilation of a couple of important definitions and results on fragmentation processes that form the foundation of our considerations in this course. The present chapter aims at introducing the various classes of fragmentation processes that we use in this course and to show how they are related to each other. Let us point out that the results presented here as well as additional background information can be found in [Ber01] and [Ber06].

Throughout these notes we denote by $\delta_{(\cdot)}$ the Dirac measure and by $\sharp$ the counting measure on $\mathbb{N}$. In addition, we adopt $\ln(0) := \lim_{x\downarrow 0}\ln(x) = -\infty$.

1.2 Preliminary considerations – The finite activity case

The study of fragmentation processes as mathematical objects in the spirit of the presentation in this course constitutes a relatively recent field of research. Indeed, the first paper introducing this class of processes is [Ber01] and dates back only to the beginning of this millennium. The ideas of introducing fragmentations are partly based on equally recent developments in the theory of coalescent processes, notably [Pit99] and [Sch00], though there are many other influences from coalescence theory that goes back to Kingman [Kin82].

Fragmentation processes are continuous–time Markov processes and they exhibit a close relationship with Poisson point processes and Lévy processes. In some sense Lévy processes can be seen as the continuous–time analogue of random walks and in a similar fashion fragmentation processes extend branching random walks to the continuous–time setting. Some of the mathematical roots of fragmentation processes lay with older families of branching processes such as branching random walks and Crump–Mode–Jagers processes (also known as general branching processes). Such stochastic processes exemplify the phenomena of random splitting according to systematic rules and they may be seen as modelling the growth of special types of multi–particle systems.

Some applications of fragmentation processes related to the mining industry were considered by Bertoin and Martínez in [BM05] as well as by Fontbona, Krell and Martínez in [FKM10]. However, as this involves more advanced concepts, we won't go deeper into this at the moment. In fact, Chapter 3 will be devoted to the main result of [BM05]. Recently, in [KP10] Kyprianou and Pardo established a connection between fragmentation processes and an optimal stopping problem.

Let us motivate the topic by starting with some basic examples to illustrate some basic concepts of fragmentation processes.

1.2.1 Stick–breaking process

The simplest example of a fragmentation process is the stick–breaking process, see Figure 1.1. More precisely, let us consider a stick of unit size and say that after an exponentially distributed time the stick breaks into two pieces of length β and $1-\beta$ respectively. Then each of the resulting smaller sticks independently repeats the procedure and the process continues ad infinitum. The stochastic process $\lambda = (\lambda(t))_{t\in\mathbb{R}_0^+}$, consisting at each time $t \in \mathbb{R}_0^+$ of the decreasingly ordered set of the lengths $(\lambda_n(t))_{n\in\mathbb{N}}$ of the sub–sticks present at time t, constitutes a so–called (conservative) mass fragmentation process (without erosion) In general such processes can have a much more complicated structure. Firstly, the splitting does not need to be binary, that is the stick could break into a random, possibly infinite, number of pieces. Secondly, the time between two splittings does not need to be exponentially distributed with a finite parameter as the splitting times may be dense in $\mathbb{R}_0^+$. We give a rigorous definition of such a process in the following section.

1.2.2 Inheritance along a Galton–Watson process

Consider a genealogical tree given by a continuous–time Galton–Watson process. In addition, assume the ancestor has a certain initial amount of

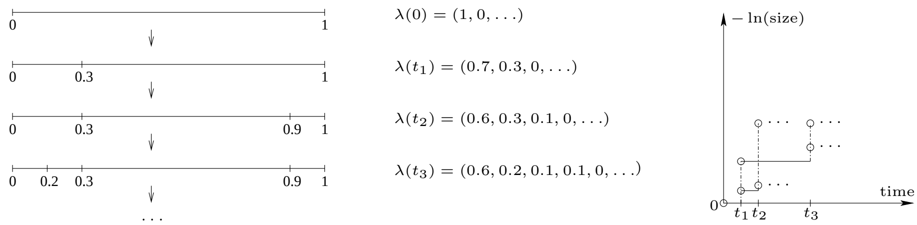


Figure 1.1: *Stick–breaking process* $(\lambda(t))_{t\in\mathbb{R}_0^+}$ *with jump times* $(t_n)_{n\in\mathbb{N}}$*. Note that in the illustration on the left–hand side the time axis points downwards.*

money, say 1, which he spends continuously at some fixed rate $c > 0$. When he dies after some independent exponential timeτ with some wealth that amounts to$e^{-c\tau}$ he has a nonnegative number, say $n \in \mathbb{N}_0$, of children who inherit some random proportions $s_1, \ldots, s_n$, with $\sum_{i=1}^n s_i \leq 1$, of the father's fortune. If he has no children, i.e. if $n = 0$, all his money goes to charity. The evolution of each child (and his part of the total wealth) then follows an independent copy of the ancestor's evolution. Note that the wealth is split into certain proportions, whose distribution does not depend on the amount of the wealth. This description yields a (dissipative) homogenous fragmentation process with erosion and "with finite activity".

1.3 Mass fragmentation processes

In this section we introduce a first kind of fragmentation processes. For this purpose, consider the infinite–dimensional vector space $\mathcal{S}$ of nonincreasing sequences in $[0,1]$ given by

$$\mathcal{S} := \left\{ \mathbf{s} := (s_n)_{n\in\mathbb{N}} : s_1 \geq s_2 \ldots \geq 0, \sum_{n\in\mathbb{N}} s_n \leq 1 \right\}.$$

For any sequence $(x_n)_{n\in\mathbb{N}}$ of nonnegative real numbers we denote by $(x_n)^{\downarrow}_{n\in\mathbb{N}}$ the decreasing reordering of $(x_n)_{n\in\mathbb{N}}$, that is $(x_n)^{\downarrow}_{n\in\mathbb{N}} \in \mathcal{S}$ if and only if $\sum_{n\in\mathbb{N}} x_n \leq 1$. We consider $\mathcal{S}$ to be endowed with the uniform distance. That is to say, we work with the metric space $(\mathcal{S}, d_{\mathcal{S}})$, where the metric $\rho_{\mathcal{S}}$ on $\mathcal{S}$ is given by

$$d_{\mathcal{S}}(\mathbf{s}, \mathbf{u}) = \sup_{n\in\mathbb{N}} |s_n - u_n|$$

for all $\mathbf{s}, \mathbf{u} \in \mathcal{S}$. In what follows we consider continuity in probability of an $\mathcal{S}$–valued stochastic process with respect to the metric $d_{\mathcal{S}}$. That is to say, an $\mathcal{S}$–valued stochastic process $(\lambda(t))_{t\in\mathbb{R}_0^+}$ is continuous in probability if and only if for all $\epsilon > 0$ and any $u \in \mathbb{R}_0^+$ we have

$$\mathbb{P}\left(d_{\mathcal{S}}(\lambda(s), \lambda(u)) > \epsilon\right) \to 0$$

as $s \to u$.

Let us now give our first definition of fragmentation processes.

Definition 1.1 We call an $\mathcal{S}$–valued Markov process $\lambda := (\lambda(t))_{t\in\mathbb{R}_0^+}$, continuous in probability, a *homogenous mass fragmentation process* if

(i) $\lambda(0) = (1, 0, \ldots)$.

(ii) For any $t, u \in \mathbb{R}_0^+$, conditional on $\lambda(t) = (s_n)_{n\in\mathbb{N}}$ the random variable $\lambda(t+u)$ has the same distribution as the random variable obtained by taking the components of $s_n\lambda^{(n)}(u)$ for all $n \in \mathbb{N}$, where the $\lambda^{(n)}$ are i.i.d. copies of λ, and ordering the resulting sequence in the decreasing order to obtain an element of $\mathcal{S}$, i.e.

$$\lambda(t+u) \stackrel{\mathrm{d}}{=} \left(s_n\lambda_k^{(n)}(u)\right)^{\downarrow}_{k,n\in\mathbb{N}},$$

where $\stackrel{\mathrm{d}}{=}$ means equality in distribution.

In the above definition property (i) says that mass fragmentation processes start with exactly one fragment and this fragment has size 1. Property (ii) is called *fragmentation property* and is the analogue of the branching property in the theory of Markov branching processes. See Figure 1.2 for an illustration of a mass fragmentation process. Note that the illustration in Figure 1.2 only depicts a fragmentation process with finite dislocation measure, because a realisation of a fragmentation process with an infinite dislocation measure is much more difficult to visualise.

Remark 1.2 The above–defined fragmentation process is called homogenous, because the fragmentation rate of a block does not depend on the size of that block. It is possible to define more general fragmentation processes, in particular self–similar fragmentation processes, where the fragmentation rate depends on the block–size. An important result due to Bertoin, [Ber02b, Theorem 3 (i)], cf. also Theorem 2 in [Ber02b] and Proposition 1 in [Haa03], says that any self–similar fragmentation processes is a time–changed homogenous fragmentation, and thus in many cases understanding homogenous fragmentations is enough to deduce results for these more general fragmentations. However, often the time–parameter plays a significant role and the time–change cannot easily be applied to uplift results from the homogenous to the self–similar case. ♢

One disadvantage of the class of fragmentations that we defined in this chapter is that it is difficult to obtain some genealogy of the blocks in the process. This will be resolved by considering two different classes of fragmentations. Before we introduce those fragmentation processes, let us first establish some results on exchangeable partition measures that we shall need later on.

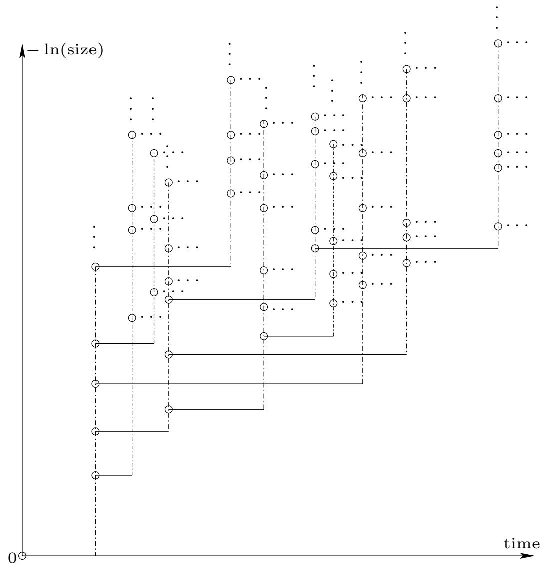


Figure 1.2: *Realisation of a standard (mass) fragmentation process with finite dislocation measure. In this illustration the term* size *refers to the values* $\lambda_n(t)$.

1.4 Exchangeable partition measures

More material on the theory of exchangeable random partitions than presented here can be found in Section 2.3 of [Ber06]. Our exposition is based on Section 4 of [Ber01].

We denote by $\mathcal{P}$ the set of ordered partitions $\pi := (\pi_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$, ordered such that $\inf\pi_1 \le \inf\pi_j$ for all $i \le j \in \mathbb{N}$, with the convention $\inf\emptyset = \infty$. A partition of $\mathbb{N}$ is a sequence of blocks $\pi_n \subseteq \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}}\pi_n = \mathbb{N}$ and $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$. We equip $\mathcal{P}$ with the metric $d_{\mathcal{P}}$ on $\mathcal{P}$ defined as follows, cf. Section 2 of [Ber01]. For any two partitions π_1, π_2 of $\mathbb{N}$ we set

$$d_{\mathcal{P}}(\pi_1,\pi_2) = \begin{cases} 0, & \pi_1 = \pi_2 \\ 2^{-N(\pi_1,\pi_2)}, & \pi_1 \neq \pi_2, \end{cases}$$

where $N(\pi_1,\pi_2) := \sup(\{n \in \mathbb{N} : \pi_1|_{[n]} = \pi_2|_{[n]}\})$. Here $\pi|_{[n]}$ denotes the restriction of a partition $\pi \in \mathcal{P}$ to the subset $[n] \subseteq \mathbb{N}$. i.e. $\pi|_{[n]} = (\pi_k \cap [n])_{k\in\mathbb{N}}$. We remark that the metric space $(\mathcal{P}, d_{\mathcal{P}})$ is compact. For any $E \subseteq \mathbb{N}$ we denote by $\mathcal{P}_E$ the set of ordered partitions of E and define $\mathcal{P}_E^* := \mathcal{P}_E \setminus (\mathbb{N}, \emptyset, \ldots)$. In addition, we set $\mathcal{P}^* := \mathcal{P}_{\mathbb{N}}^*$ and we denote by $\mathcal{P}_n^*$ the subset of $\mathcal{P}$ consisting of those partitions whose restriction to $[n]$ is not trivial, i.e. doesn't equal $([n], \emptyset, \ldots)$. Furthermore, for any partition $\pi = (\pi_n)_{n\in\mathbb{N}}$ of some $E \subseteq \mathbb{N}$ we denote by $(\pi_n)_{n\in\mathbb{N}}^{\uparrow}$ the reordering of π that yields an element of $\mathcal{P}_E$.

Definition 1.3 A permutation $\sigma : \mathcal{P} \to \mathcal{P}$ is called *finite permutation* if $\sigma(\pi_n) = \pi_n$ for all but finitely many $n \in \mathbb{N}$. Then we call a measure μ on $\mathcal{P}$ *exchangeable* if $\mu(\pi) = \mu(\sigma(\pi))$ for all $\pi \in \mathcal{P}$ and any finite permutation σ on $\mathcal{P}$. In addition, we say a measure μ on $\mathcal{P}$ is an *exchangeable partition measure* if

- μ is exchangeable,
- $\mu((\mathbb{N}, \emptyset, \ldots)) = 0$,
- $\mu(\mathcal{P}_2^*) < \infty$.

For our construction of $\mathcal{P}$–valued fragmentation processes we shall use the fact that exchangeable partition measures are σ-finite.

Let us now consider two exchangeable partition measures that will play a crucial role in the construction of fragmentation processes.

Definition 1.4 Let $c \in \mathbb{R}_0^+$. Then we call *erosion measure* the measure

$$\mu_c := c \sum_{n \in \mathbb{N}} \delta_{(\mathbb{N} \setminus \{n\}, \{n\})^\uparrow},$$

where for any $\pi \in \mathcal{P}$ the notation δ_π denotes the Dirac measure at π on $\mathcal{P}$. The constant $c > 0$ is referred to as *rate of erosion.*

For the second example of an exchangeable partition measure recall the space $\mathcal{S}$ that we introduced in the previous section. Further, we shall use the following definition

Definition 1.5 For any open set $U \subseteq (0, 1)$ we call *interval partition* of U the uniquely given sequence of disjoint open intervals of which U is composed. Moreover, for any $\mathbf{s} \in \mathcal{S}$ we call *interval representation* any interval partition of some open set such that the decreasingly ordered sequence of interval lengths coincides with $\mathbf{s}$.

Definition 1.6 Let ϑ be an interval representation of some $\mathbf{s} \in \mathcal{S}$, that is ϑ is an open subset of $(0, 1)$ such that the ranked sequence of the lengths of its interval components is given by $\mathbf{s}$. Let $(U_n)_{n \in \mathbb{N}}$ be an i.i.d. sequence of uniform random variables on $[0, 1]$. We call *Kingman's paint–box* based on $\mathbf{s}$ the random partition $\pi^{\mathbf{s}}$ of $\mathbb{N}$ induced by the following equivalence relation

$$i \overset{\pi^{\mathbf{s}}}{\sim} j \Leftrightarrow (U_i \text{ and } U_j \text{ belong to the same interval component of } \vartheta) \text{ or } (i = j).$$

Note that the alternative on the right–hand side is necessary, because the Lebesgue measure of ϑ may be less than one, and if U_i does not belong to ϑ for some $i \in \mathbb{N}$, then $\{i\}$ is a singleton of $\pi^{\mathbf{s}}$.

The name "paint–box" stems from the following alternative description of the equivalence relation $\pi^{\mathbf{s}}$ described in Definition 1.6. Let us interpret the unit interval as a paint–box in which a different colour is assigned to each interval component of ϑ. Every integer i then receives the colour of the interval to which the random variable U_i belongs, and i does not receive any colour if U_i is not in ϑ. The equivalence classes are then given by the sets of indices with the same colour, where we adopt that indices with no colour form singletons.

Lemma 1.7 *Kingman's paint–box based on some* $\mathbf{s} \in \mathcal{S}$ *is independent of the choice of the interval representation of* $\mathbf{s}$ *and its distribution is an exchangeable random partition measure.*

Proof Exercise.

Definition 1.8 We call *Lévy measure* a measure $\nu \not\equiv 0$ on $\mathcal{S}$ that satisfies

$$\nu(\{(1,0,\ldots)\}) = 0 \quad \text{and} \quad \int_{\mathcal{S}} (1 - s_1)\nu(\mathrm{d}\mathbf{s}) < \infty.$$

Furthermore, we set

$$\mathcal{S}^* := \mathcal{S} \setminus \{(1,0,\ldots)\}.$$

Remark 1.9 The name Lévy measure in the above definition is motivated by fact that, as we will see, the measure ν, as the Lévy measure for Lévy processes, describes the jump structure of fragmentation processes. Moreover, the two conditions in Definition 1.8 bear a strong resemblance to the properties that Lévy measures of subordinators satisfy. That similar measures appear for both types of processes is not surprising as there is a very close relationship between fragmentation processes and Lévy processes. In Section 2.1.2 we will see that this similarity between the two classes of processes gives rise to an underlying Lévy process, more precisly a subordinator, which is used prevalently in the literature on fragmentations. ◊

Definition 1.10 For each Lévy measure ν on $\mathcal{S}$ we call *dislocation mesure* the measure μ_ν defined by

$$\mu_\nu(\mathrm{d}\pi) = \int_{\mathcal{S}} \varrho_{\mathbf{s}}(\mathrm{d}\pi)\nu(\mathrm{d}\mathbf{s})$$

for any $\pi \in \mathcal{P}$, where $\varrho_{\mathbf{s}}$ is the distribution of Kingman's paint paint–box based on $\mathbf{s}$.

Lemma 1.11 (Lemma 2 in [Ber01]) *For each Lévy measure* ν *on* $\mathcal{S}$ *the dislocation measure* μ_ν *is an exchangeable partition measure.*

Proof Exercise. □

This section is devoted to proving that any exchangeable partition measure can be written as a sum of some μ_c and some μ_ν. More specifically, the main result of this section is the following.

Theorem 1.12 *For any random partition measure μ on $\mathcal{P}$ there exist some uniquely defined measure ν on $\mathcal{S}$ and $c > 0$ such that $\mu = \mu_\nu + \mu_c$.*

In order to tackle the proof of Theorem 1.12 we shall need some auxiliary results that we are now going to develop. For this purpose and also later on the following notion of asymptotic frequency turns out to be crucial.

Definition 1.13 We say a partition $\pi \in \mathcal{P}$ possesses *asymptotic frequencies* if

$$|\pi_n| = \lim_{k\to\infty} \frac{\text{card}(\pi_n \cap \{1, \dots, k\})}{k},$$

exist for any block π_n. Furthermore, we set $|\pi| := (|\pi_n|)_{n\in\mathbb{N}}$.

The above definition provides us with a notion of "size" for the blocks of partitions in $\mathcal{P}$. In fact, the notion of asymptotic frequencies will be considered as the size of blocks in the context of partition–valued fragmentation processes as defined in the next section.

Theorem 1.14 (Theorem 2 (i) in [Ber01]) *Let μ be an exchangeable partition measure. Then $|\pi|$ exists μ–almost surely.*

Proof According to Theorem 2 in [Kin82] (see also (11.9) Proposition of [Ald85] for a proof based on de Finetti's theorem rather than on martingale arguments as in [Kin82]) we have that for for any finite exchangeable measure $\tilde{\mu}$ on $\mathcal{P}$ $\tilde{\mu}$–a.e. $\pi \in \mathcal{P}$ possesses asymptotic frequencies that satisfy the following mixture of paint–boxes

$$\tilde{\mu}(\mathrm{d}\pi) = \int_{\mathcal{S}} \rho_{\mathbf{s}}(\mathrm{d}\pi)\tilde{\mu}(\Gamma \in \mathcal{P} : |\Gamma| \in \mathrm{d}\mathbf{s}), \tag{1.1}$$

where $\varrho_{\mathbf{s}}$ is the distribution of Kingman's paint paint–box based on $\mathbf{s}$.

In order to apply the above–mentioned result to our situation of a possibly infinite measure μ, set $\mu_n := \mu \mathbb{1}_{\mathcal{P}_n^*}$. Then $\mu_n(\mathcal{P}) < \infty$ and moreover, $\mu_n \circ \sigma = \mu_n$ for any finite permutation σ on $\mathcal{P}$ with $\sigma|_{[n]} \equiv \text{id}$. Consider the map $\pi \mapsto \pi^{(n)}$ given by

$$i \overset{\pi^{(n)}}{\sim} j \Longleftrightarrow i + n \overset{\pi}{\sim} j + n, \qquad \forall\, i, j \in \mathbb{N}, \tag{1.2}$$

called *n–shift of partitions*, and let $\tilde{\mu}_n$ be the image measure of μ_n under this map. Then $\tilde{\mu}_n$ is a finite exchangeable measure on $\mathcal{P}$, and thus it follows from (1.1) that $\tilde{\mu}_n$–a.e. $\pi \in \mathcal{P}$ possesses asymptotic frequencies satisfying

$$\tilde{\mu}_n(\mathrm{d}\pi) = \int_{\mathcal{S}} \varrho_{\mathbf{s}}(\mathrm{d}\pi)\tilde{\mu}_n(\Gamma \in \mathcal{P} : |\Gamma| \in \mathrm{d}\mathbf{s}). \tag{1.3}$$

The n–shift does not affect asymptotic frequencies, and thus $|\pi|$ exists μ_n–almost surely. Letting $n \to \infty$ this proves the assertion. $\square$

For any exchangeable partition measure μ on $\mathcal{P}$ let ν_μ on $\mathcal{S}$ be the image measure of μ under the map $\pi \mapsto |\pi|$.

Theorem 1.15 (Theorem 2 (ii) in [Ber01]) *Let μ be some exchangeable partition measure on $\mathcal{P}$ and consider the measure ν on $\mathcal{S}$ given by $\nu = \mathbb{1}_{\mathcal{S}^*}\nu_\mu$. Then the changed measure given by the μ–Radon-Nikodým derivative $\pi \mapsto \mathbb{1}_{\{|\pi|\in\mathcal{S}^*\}}$ is a dislocation measure with Lévy measue ν, i.e. it satisfies*

$$\mathbb{1}_{\{|\pi|\in\mathcal{S}^*\}}\mu(d\pi) = \mu_\nu(d\pi) \tag{1.4}$$

for all $\pi \in \mathcal{P}$.

Proof We devide the proof into two parts. The first part proves (1.4), i.e. it shows that $\mathbb{1}_{\{|\pi|\in\mathcal{S}^*\}}\mu(\mathrm{d}\pi)$ can be expressed as a mixture of paint–boxes with respect to ν. The second part shows that ν is a Lévy measure, which proves that $\mathbb{1}_{\{|\pi|\in\mathcal{S}^*\}}\mu(\mathrm{d}\pi)$ is a dislocation measure.

Part I We first show that (1.4) holds true. To this end, let $k \in \mathbb{N}$ and consider a non–trivial partition Γ_k of $\{1,\ldots,k\}$. The MCT tells us that

$$\begin{aligned}&\mu(\pi|_k = \Gamma_k, |\pi| \in \mathcal{S}^*) \\ &= \lim_{n\to\infty} \mu(\pi|_k = \Gamma_k, |\pi| \in \mathcal{S}^*, \pi|_{\{k+1,\ldots,k+n\}} \neq (\{k+1,\ldots,k+n\}, \emptyset, \ldots)).\end{aligned} \tag{1.5}$$

Then we have

$$\begin{aligned}&\mu(\pi|_k = \Gamma_k, |\pi| \in \mathcal{S}^*, \pi|_{\{k+1,\ldots,k+n\}} \neq (\{k+1,\ldots,k+n\}, \emptyset, \ldots)) \\ &= \tilde{\mu}_n^{(k)}(\pi|_k = \Gamma_k, |\pi| \in \mathcal{S}^*),\end{aligned}$$

where $\tilde{\mu}_n^{(k)}$ is defined analogously to $\tilde{\mu}_n$ in the proof of Theorem 1.14, but with

$$i \overset{\pi^{(n)}}{\sim} j \Longleftrightarrow \begin{cases} i \overset{\pi}{\sim} j, & i,j \le k \\ i \overset{\pi}{\sim} j+n, & i \le k, j > k \\ i+n \overset{\pi}{\sim} j+n, & i,j > k. \end{cases}$$

instead of the n–shift used in (1.2). Hence, resorting to (1.1), we deduce from (1.5) that

$$\mu(\pi|_k = \Gamma_k, |\pi| \in \mathcal{S}^*) = \int_{\mathcal{S}} \varrho_{\mathbf{s}}(\pi|_k = \Gamma_k)\nu(\mathrm{d}\mathbf{s}) = \mu_\nu(\pi|_k = \Gamma_k).$$

Part II In view of Part I the proof is complete once we have shown that ν is a Lévy measure on $\mathcal{S}$. As in the proof of Theorem 1.14 set $\mu_n := \mu\mathbb{1}_{\mathcal{P}_n^*}$

and write $\{i \overset{\pi}{\not\sim} j\}$ for the event that $i, j \in \mathbb{N}$ are not in the same block of the partition $\pi \in \mathcal{P}$. According to (1.3) we then have that

$$\begin{aligned} \mu_n\left(\pi \in \mathcal{P} : 1+n \overset{\pi}{\not\sim} 2+n \middle| \, |\pi| = \mathbf{s}\right) &= \varrho_{\mathbf{s}}(1 \overset{\pi}{\not\sim} 2) \\ &= 1 - \sum_{k \in \mathbb{N}} s_k^2 \\ &\geq 1 - s_1 \sum_{k \in \mathbb{N}} s_k \qquad (1.6) \\ &\geq 1 - s_1 \end{aligned}$$

for any $n \in \mathbb{N}$ and $\mathbf{s} \in \mathcal{S}$. Note that above s_k^2 is the probability that 1 and 2 fall into the k–th largest interval component of the interval representation of $\mathbf{s}$. Consider the measure ν_n on $\mathcal{S}$ given by $\nu_n = \mathbb{1}_{\mathcal{S}^*}\nu_{\mu_n}$. Then we infer from (1.6) that

$$\mu_n((1+n) \not\sim (2+n)) \geq \int_{\mathcal{S}} (1 - s_1)\nu_n(\mathrm{d}\mathbf{s}), \qquad (1.7)$$

where $\{i \not\sim j\} := \{\pi \in \mathcal{P} : i \overset{\pi}{\not\sim} j\}$ for any $i, j \in \mathbb{N}$. Moreover, $\nu_n \to \nu$ weakly as $n \to \infty$, i.e.

$$\lim_{n\to\infty} \int_{\mathcal{S}} (1 - s_1)\nu_n(\mathrm{d}\mathbf{s}) = \int_{\mathcal{S}} (1 - s_1)\nu(\mathrm{d}\mathbf{s}).$$

Since, by exchangeability of μ,

$$\mu_n((1+n) \not\sim (2+n)) \leq \mu((1+n) \not\sim (2+n)) = \mu(1 \not\sim 2) = \mu(\mathcal{P}_2^*) < \infty,$$

we conclude in view of (1.7) that $\int_{\mathcal{S}}(1-s_1)\nu(\mathrm{d}\mathbf{s}) < \infty$ and hence ν is a Lévy measure on $\mathcal{S}$. □

Proposition 1.16 (Theorem 2 (iii) in [Ber01]) *Let μ be some exchangeable partition measure on $\mathcal{P}$. Then there exists some $c \in \mathbb{R}_0^+$ such that*

$$\mathbb{1}_{\{|\pi|=(1,0,\ldots)\}}\mu(d\pi) = \mu_c(d\pi)$$

for all $\pi \in \mathcal{P}$.

Proof Let $\hat{\mu}$ be the restriction of μ to the set

$$\{\pi \in \mathcal{P} : 1 \overset{\pi}{\not\sim} 2, |\pi| = (1, 0 \ldots))\} \subseteq \mathcal{P}_2^*.$$

Its image measure $\hat{\mu}_2$ by the 2-shift then yields a finite exchangeable measure on $\mathcal{P}$ with $|\pi| = (1, 0 \ldots)$ $\hat{\mu}_2$–almost surely. Hence, by (1.3), $\hat{\mu}_2$ is proportional to $\delta_{(\mathbb{N},\emptyset,\ldots)}$. Set

$$\Gamma^{(1)} := (\{1\}, \{2, \ldots\})$$

$$\Gamma^{(2)} := (\{2\}, \{1, 3, \ldots\})$$
$$\Gamma^{(3)} := (\{1\}, \{2\}, \{3, \ldots\})$$

and observe that
$$\hat{\mu} = c_1\delta_{\Gamma^{(1)}} + c_2\delta_{\Gamma^{(2)}} + c_3\delta_{\Gamma^{(3)}}$$

for some constants $c_1, c_2, c_3 \in \mathbb{R}_0^+$. Since $\Gamma^{(2)}$ contains three blocks, the exchangeability of μ results in $c_3 = 0$. Indeed, if in $\Gamma^{(3)}$ we permute 1 or 2 with $n \in \mathbb{N}\setminus\{1, 2\}$, then we obtain a $\hat{\mu}$–null set, but if $c_3 > 0$ then $\hat{\mu}(\Gamma^{(3)}) > 0$, which contradicts exchangeability. Moreover, the exchangeability also implies that $c_1 = c_2$. Consequently, there exists some $c > 0$ such that

$$\hat{\mu} = c\delta_{\Gamma^{(1)}\cup\Gamma^{(2)}},$$

which by means of exchangeability proves the assertion. □

Combining Theorem 1.15 with Proposition 1.16 proves the statement of Theorem 1.12.

Proof of Theorem 1.12 It follows from Theorem 1.15 with Proposition 1.16 that

$$\mu(\mathrm{d}\pi) = \mathbb{1}_{\{|\pi|\in\mathcal{S}^*\}}\mu(\mathrm{d}\pi) + \mathbb{1}_{\{|\pi|=(1,0,\ldots)\}}\mu(\mathrm{d}\pi) = (\mu_\nu + \mu_c)(\mathrm{d}\pi)$$

for any $\pi \in \mathcal{P}$, which proves the assertion. □

1.5 Fragmentation processes with a genealogical structure

As mentioned before, one disadvantage of mass fragmentation processes is the lack of a genealogical structure. That is, in a mass fragmentation process it is difficult to define the notion of "ancestor" or "parent" of a given block. In this section we introduce two classes of fragmentation processes which avoid this problem and which are thus more applicable in many situations.

1.5.1 Partition–valued fragmentation processes

In this section we define fragmentation processes that take values in the space $\mathcal{P}$ that we considered in the previous section. To this end, we call a $\mathbb{P}$–valued process Π exchangeable if Π has the same distribution as $\sigma(\Pi)$ for any finite permutation σ, where $\sigma(\Pi) := (\sigma(\Pi(t)))_{t\in\mathbb{R}_0^+}$. The definition of fragmentation processes in this setting then reads as follows:

Definition 1.17 We call a $\mathcal{P}$–valued exchangeable Markov process $\Pi := (\Pi(t))_{t \in \mathbb{R}_0^+ \cup \{\infty\}}$, continuous in probability, a *homogenous $\mathcal{P}$–fragmentation process* if

(i) $\Pi(0) = (\mathbb{N}, \emptyset, \ldots)$.

(ii) For any $t, u \in \mathbb{R}_0^+$, conditional on $\Pi(t) = (\pi_n)_{n \in \mathbb{N}}$ the random variable $\Pi(t+u)$ has the same distribution as the random variable obtained by taking the components of $\tilde{\Pi}^{(n)}(u)|_{\pi_n}$ for all $n \in \mathbb{N}$, where the $\tilde{\Pi}^{(n)}$ are i.i.d. copies of Π, and ordering the resulting sequence such that it is an element of $\mathcal{P}$, i.e.

$$\Pi(t+u) \stackrel{\mathrm{d}}{=} \left(\tilde{\Pi}_k^{(n)}(u) \Big|_{\pi_n} \right)_{k,n \in \mathbb{N}}^{\uparrow}.$$

In particular, a $\mathcal{P}$–fragmentation process starts with the trivial partition of $\mathbb{N}$, that is it starts with exactly one non–empty block that contains all natural numbers. As in the case of mass fragmentation processes, we call Property (ii) *fragmentation property.* The continuity in probability in Definition 1.17 is meant with respect to the metric $d_{\mathcal{P}}$.

For any $\pi \in \mathcal{P}$ let $\mathbb{P}_\pi$ denote the probability under which Π is conditioned to start with the partition π, that is

$$\mathbb{P}_\pi(\Pi(0) = \pi) = 1,$$

and $\mathbb{E}_\pi$ denotes the expectation under $\mathbb{P}_\pi$.

The following proposition ensures the existence of a càdlàg version of Π.

Proposition 1.18 (Proposition 1 in [Ber01]) *Fragmentation processes are Feller processes, i.e. their underlying semigroup has the Feller property.*

Proof We need to show that the map

$$\mathcal{P} \to \mathbb{R},\ \pi \mapsto \mathbb{E}_\pi(f(\Pi(t))) \tag{1.8}$$

is continuous for any continuous function $f : \mathcal{P} \to \mathbb{R}$ and $t \in \mathbb{R}_0^+$ as well as

$$\lim_{t \downarrow 0} \mathbb{E}_\pi(f(\Pi(t))) = f(\pi) \tag{1.9}$$

for each $\pi \in \mathcal{P}$.

Let $f : \mathcal{P} \to \mathbb{R}$ be continuous. For any $n \in \mathbb{N}$ and $\pi \in \mathcal{P}$ set

$$\pi^{(n)} := \pi|_{[n]} \cup \{\mathbb{N} \setminus [n]\} \in \mathcal{P}$$

and note that

$$d_{\mathcal{P}}(\pi, \pi^{(n)}) \leq 2^{-n}. \tag{1.10}$$

By means of the Heine–Cantor theorem and the compactness of $\mathcal{P}$ we have that f is uniformly continuous, and in view of (1.10) we thus infer that there exists for any $\epsilon > 0$ some $n_\epsilon \in \mathbb{N}$ such that

$$|f(\pi) - f(\pi^{(n)})| \leq \epsilon \tag{1.11}$$

holds for all $n \geq n_\epsilon$ and every $\pi \in \mathcal{P}$. Moreover, in view of the fragmentation property we have that

$$\mathbb{P}_\pi(\Pi^{(n)}(t) \in \cdot) = \mathbb{P}_{\pi'}(\Pi^{(n)}(t) \in \cdot)$$

for any $n \in \mathbb{N}$ and all $\pi, \pi' \in \mathcal{P}$ with $d_{\mathcal{P}}(\pi, \pi') \leq 2^{-n}$. Consequently, by means of (1.11) and the triangle inequality we deduce that

$$\begin{aligned}
&|\mathbb{E}_\pi(f(\Pi(t))) - \mathbb{E}_{\pi'}(f(\Pi(t)))| \\
&\leq \left|\mathbb{E}_\pi\left(f(\Pi(t)) - f(\Pi^{(n)}(t))\right)\right| + \left|\mathbb{E}_\pi\left(f(\Pi^{(n)}(t))\right) - \mathbb{E}_{\pi'}\left(f(\Pi^{(n)}(t))\right)\right| \\
&\qquad + \left|\mathbb{E}_{\pi'}\left(f(\Pi^{(n)}(t)) - f(\Pi(t))\right)\right| \\
&\leq 2\epsilon
\end{aligned}$$

for any $\epsilon > 0$, $n \geq n_\epsilon$ and $\pi, \pi' \in \mathcal{P}$ with $d_{\mathcal{P}}(\pi, \pi') \leq 2^{-n}$, which proves (1.8).

By resorting to the continuity in probability of Π and recalling that f is bounded (as $\mathcal{P}$ is compact) and continuous we conclude that also (1.9) holds, which completes the proof. □

Remark 1.19 By Kinney's regularity theorem, see [Kal01, Theorem 17.15], Feller processes have a version which is almost surely right–continuous with limits from the left. We implicitly always assume that we are dealing with such a version when considering fragmentation processes. Consequently, in view of [Chu82, Theorem 1 in Section 2.3] or [Kal01, Theorem 17.17], homogenous fragmentation processes satisfy the strong Markov property, which in this setting shall be referred to as *strong fragmentation property.* Note that here we have used that the state space $(\mathbb{P}, d_{\mathbb{P}})$ of the fragmentation process is in particular a locally compact Polish space, so that the above–mentioned results in [Chu82] and [Kal01] are applicable. See also Section III.2 of [RY99] in this regard. ◊

Similarly to the characterisation of subordinators in terms of the characteristic pair, cf. Definition **??**, by the Lévy–Khintchine formula, the following theorem provides us with a characterisation of homogenous fragmentation processes in terms of a jump measure and a continuous drift.

Theorem 1.20 (Lemma 1 in [Ber01]) *The distribution of any homogenous $\mathbb{P}$–fragmentation process is determined by*

- *a Lévy measure ν on $\mathcal{S}$,*
- *a constant $c \in \mathbb{R}_0^+$.*

Proof Let Π be some homogenous $\mathbb{P}$–fragmentation process. For any $n \in \mathbb{N}$ let $\Pi^{(n)}$ be the Markov chain (in continuous time) obtained by restricting Π to $[n]$. The fragmentation property entails that

$$\tau_n := \inf\{t \in \mathbb{R}_0^+ : \Pi^{(n)}(t) \in \mathcal{P}_{[n]}^*\} \tag{1.12}$$

is exponentially distributed with some parameter $q_n \in \mathbb{R}_0^+$. Further, denote by $\rho_n(\cdot) := \mathbb{P}(\Pi^{(n)}(\tau_n) \in \cdot)$ the distribution of $\Pi^{(n)}(\tau_n)$. By means of the strong fragmentation property we infer that ρ_n and q_n determine the jump–distribution of $\Pi^{(n)}$ for every $n \in \mathbb{N}$. Fix some $n \in \mathbb{N}$ and observe that $\Pi^{(n+1)}(\tau_{n+1}) = ([n], \{n+1\}, \emptyset, \ldots)$ if and only if $\tau_{n+1} < \tau_n$. Therefore, the restriction of $\Pi^{(n+1)}(\tau_{n+1})$ to $[n]$ is non–trivial if and only if $\tau_{n+1} = \tau_n$, and hence the jump rate of $\Pi^{(n+1)}$ on $\{\pi \in \mathcal{P}_{n+1} : \pi|_{[n]} \neq ([n], \emptyset, \ldots)\}$ equals q_n. Moreover, we have

$$\mathbb{P}\left(\Pi^{(n)}(\tau_{n+1}) \in \cdot \,\middle|\, \Pi^{(n)}(\tau_{n+1}) \neq ([n], \emptyset, \ldots)\right) = \rho_n(\cdot).$$

Consequently, the image measure of $q_{n+1}\rho_{n+1}$ under the map

$$\{\pi \in \mathcal{P}_{n+1} : \pi|_{[n]} \neq ([n], \emptyset, \ldots)\} \to \mathcal{P}_{[n]}^*, \quad \pi \mapsto \pi|_{[n]}$$

equals $q_n\rho_n$. Thus, according to Kolmogorov's extension theorem, there exists a unique measure μ on $\mathcal{P}$, with $\mu((\mathbb{N}, \emptyset, \ldots)) = 0$, such that for every $n \in \mathbb{N}$ the image measure of $\mu|_{\mathcal{P}_n^*}$ under the map $\mathcal{P}_n^* \to \mathcal{P}_{[n]}^*$, $\pi \mapsto \pi|_{[n]}$, coincides with $q_n\rho_n$. Note that μ is exchangeable, because for any $n \in \mathbb{N}$ the process $\Pi^{(n)}$ being exchangeable implies that the measure ρ_n is exchangeable. Moreover, $\mu(\mathcal{P}_2^*) < \infty$, and hence μ is an exchangeable partition measure. According to Theorem 1.12 the measure μ is thus characterised by a Lévy measure ν on $\mathcal{S}$ and a constant $c \in \mathbb{R}_0^+$. Since, for each $n \in \mathbb{N}$ we have that $q_n p_n$ determines the distribution of $\Pi^{(n)}$, we conclude that μ determines the distribution of Π, which completes the proof. □

We call the exchangeable partition measure μ obtained in the proof of Theorem 1.20 *characteristic measure* of Π. Notice that for any $n \in \mathbb{N}$ this measure satisfies

$$\mu(\pi \in \mathcal{P} : \pi|_{[n]} \in \cdot) = q_n\rho_n(\cdot) = \frac{1}{\mathbb{E}(\tau_n)}\mathbb{P}\left(\Pi^{(n)}(\tau_n) \in \cdot\right)$$

on $\mathcal{P}_{[n]}^*$, where τ_n is given by (1.12).

1.5.2 Dislocation

Here we deal with a pure jump process. In the spirit of the Lévy-Itô decomposition for subordinators we have the following representation of homogenous $\mathcal{P}$–fragmentation process via Poisson point processes.

Corollary 1.21 *Any homogenous fragmentation process without erosion, i.e. $c = 0$, is characterised by a Poisson point process.*

Proof Let Π be some homogenous fragmentation process and let the Lévy measure ν be given by Theorem 1.20. The proof of Theorem 1.20 shows that Π is characterised by a Poisson point process. More precisely, there exists a $\mathcal{P} \times \mathbb{N}$–valued Poisson point process $(\pi(t), k(t))_{t\in\mathbb{R}_0^+}$ with characteristic measure $\mu_\nu \otimes \sharp$ such that Π changes state at all times $t \in \mathbb{R}_0^+$ for which an atom $(\pi(t), k(t))$ occurs in $(\mathcal{P} \setminus (\mathbb{N}, \emptyset, \ldots)) \times \mathbb{N}$. At such a time $t \in \mathbb{R}_0^+$ the sequence $\Pi(t)$ is obtained from $\Pi(t-)$ by replacing its $k(t)$–th term, $\Pi_{k(t)}(t-) \subseteq \mathbb{N}$, with the restricted partition $\pi(t)|_{\Pi_{k(t)}(t-)}$ and reordering the terms such that the resulting partition of $\mathbb{N}$ is an element of $\mathcal{P}$. $\square$

The following result is in some sense a converse of Theorem 1.20 and Corollary 1.21 in that it shows that for any Lévy measure ν there exists a fragmentation process Π with characteristic measure μ_ν. Let us first introduce the notion of composition for partitions. To this end, let $\Gamma = (\Gamma_n)_{n\in\mathbb{N}} \in \mathcal{P}_E$ for some $E \subseteq \mathbb{N}$ and let $\pi \in \mathcal{P}$. Then we define the *composition* of Γ and π by

$$\pi \overset{k}{\circ} \Gamma := \left(\bigcup_{n\in\mathbb{N}\setminus\{k\}} \{\Gamma_n\} \cup \pi|_{\Gamma_k} \right)^{\uparrow} \in \mathcal{P}_E\,.$$

Theorem 1.22 (Theorem 1 in [Ber01]) *Let μ be a dislocation measure. Then there exists some homogenous fragmentation process with characteristic measure μ.*

Proof The proof is divided into two parts. In the first part we construct a $\mathbb{P}$–valued stochastic process via a Poisson point process with characteristic measure μ. In the second part we show that this process is indeed a fragmentation process.

Part I Let $(\pi(t), k(t))_{t\in\mathbb{R}_0^+}$ be a Poisson point process on $\mathcal{P} \otimes \mathbb{N}$ with characteristic measure $\mu \otimes \sharp$. Note that here we made use of the σ–finiteness of μ. Further, let $n \in \mathbb{N}$ and set $\Pi^{(n)}(0) := ([n], \emptyset, \ldots)$. Then we define a $\mathbb{P}_{[n]}$–valued process $\Pi^{(n)} := (\Pi^{(n)}(t))_{t\in\mathbb{R}_0^+}$ inductively by

$$\Pi^{(n)}(t) := \begin{cases} \pi(t) \overset{k(t)}{\circ} \Pi^{(n)}(t-), & (\pi(t), k(t)) \in \mathbb{P}_n^* \times [n] \\ \Pi^{(n)}(t-), & \text{otherwise,} \end{cases}$$

which is possible since $(\mu\otimes\sharp)(\mathbb{P}_n^*\times[n])<\infty$. Note that $\Pi^{(n)}$ has only finitely many jumps in any finite time–interval and is piecewise constant. Moreover, since the trivial partition of $[n+1]$ restricted to $[n]$ is the trivial partition of $[n]$ and because

$$\pi \overset{k}{\circ} \Gamma\Big|_{[n]} = \pi \overset{k}{\circ} \Gamma|_{[n]}$$

for any $\Gamma, \pi \in \mathcal{P}$ and $k \in \mathbb{N}$, we infer that the family $(\Pi^{(n)})_{n\in\mathbb{N}}$ is consistent, and thus the compatibility lemma for partitions (cf. Lemma 2.5 in [Ber06]) yields the existence of a unique $\mathbb{P}$–valued process Π such that the restriction of Π to $[n]$ coincides with $\Pi^{(n)}$ for every $n \in \mathbb{N}$. Notice that

$$\Pi(t) = \begin{cases} \pi(t) \overset{k(t)}{\circ} \Pi(t-), & \pi(t) \in \mathbb{P}^* \\ \Pi(t-), & \text{otherwise.} \end{cases}$$

Moreover, since any $\Pi^{(n)}$ is by construction càdlàg (with respect to some appropriate metric on $\mathcal{P}_{[n]}$), also Π has càdlàg paths.

<u>Part II</u> Let Π be constructed as in Part I. It follows from the construction via Poisson point processes that each $\Pi^{(n)}$, and thus Π, is continuous in probability and satisfies the fragmentation property. Moreover, it is clear that Π starts from the trivial partition. It remains to show that Π is exchangeable. To this end, consider the permutation σ on $\mathbb{N}$ given by

$$\sigma(n) := \begin{cases} 2, & n = 1 \\ 1, & n = 2 \\ n, & n \geq 3. \end{cases}$$

In addition, set

$$\tau := \inf\{t \in \mathbb{R}_0^+ : \Pi(t) \in \mathbb{P}_2^*\} = \inf\{t \in \mathbb{R}_0^+ : \pi(t) \in \mathbb{P}_2^*, k(t) = 1\}$$

and consider the point process $(\tilde{\pi}(t), \tilde{k}(t))_{t\in\mathbb{R}_0^+}$ defined by

$$\tilde{\pi}(t) : \begin{cases} \pi(t), & t \neq \tau \\ \sigma(\pi(t)), & t = \tau \end{cases} \qquad \text{and} \qquad \tilde{k}(t) := \begin{cases} k(t), & k(t) \geq 3 \text{ or } t \leq \tau \\ 2, & k(t) = 1 \text{ and } t > \tau \\ 1, & k(t) = 2 \text{ and } t > \tau. \end{cases}$$

Since μ is exchangeable, we have that $\tilde{\pi}(t)$ has the same distribution as $\tilde{\pi}(t)$, and by the construction via Poisson point processes we thus conclude that $(\tilde{\pi}(t), \tilde{k}(t))_{t\in\mathbb{R}_0^+}$ has the same distribution as $(\pi(t), k(t))_{t\in\mathbb{R}_0^+}$. Set $\tilde{\Pi}(t) := \sigma(\Pi(t))$. By construction of $(\tilde{\pi}(t), \tilde{k}(t))_{t\in\mathbb{R}_0^+}$ we then have that

$$\tilde{\Pi}(t) = \begin{cases} \tilde{\pi}(t) \overset{\tilde{k}(t)}{\circ} \tilde{\Pi}(t-), & \pi(t) \in \mathbb{P}^* \\ \tilde{\Pi}(t-), & \text{otherwise.} \end{cases}$$

Therefore, $\tilde{\Pi}$ equals Π in distribution. The same argument also works if σ permutes n with $n+1$, rather than 1 with 2, for any $n \in \mathbb{N}$. Consequently, Π is exchangeable and by Theorem 1.12 the proof is thus complete. $\square$

Definition 1.23 We call a fragmentation process with Lévy measure ν *conservative* if $\nu(\sum_{n\in\mathbb{N}} s_n < 1) = 0$ and *dissipative* otherwise. The constant $c \in \mathbb{R}_0^+$ in (ii) of Theorem 1.20 is the *rate of erosion.* Erosion means a continuous loss of mass, thus adding a continuous drift to the jumps of the fragmentation process.

Note that the stick–breaking process in Figure 1.1 is an example of a conservative homogenous mass fragmentation process without erosion, whereas the process considered in Section 1.2.2 is a dissipative homogenous mass fragmentation with erosion. Moreover, in both cases the dislocation measure is a probability measure, in particular it is finite.

Remark 1.24 The mathematical approach to tackle problems involving fragmentation processes partly depends on whether the dislocation measure is finite or infinite. If ν is finite, then a block of size x remains unchanged for an exponential period of time with parameter $\nu(\mathcal{S}) \in \mathbb{R}^+$ and this situations is sometimes referred to as the *finite activity case.* In this respect, note that even though a homogenous fragmentation process with finite ν may still have infinitely many jumps in any finite time interval, because infinitely many blocks may be present at any time and each block fragments with the same rate, each single block has only finitely many jumps up to any $t \in \mathbb{R}_0^+$ in this setting. Therefore, in this "finite activity" situation the notion of first jump of a block makes sense, but in general it is not possible to use the notion of n–th jump for any $n > 1$. By taking the negative logarithm of fragment sizes a fragmentation process with finite dislocation measure is closely related to continuous–time branching random walks and Crump–Mode–Jagers processes. If on the other hand $\nu(\mathcal{S}) = \infty$, then the jump times are dense in $\mathbb{R}_0^+$ and there is a countably infinite number of dislocations over any finite time horizon. Note that the denseness of the jump times does in particular imply that there is no first dislocation of the process and the infimum over all jump times is 0, although there is no dislocation at time 0. Fragmentation processes with an infinite dislocation measure are more interesting, both from a theoretical point of view and for applications as for instance in the mining industry. Moreover, in comparison to fragmentation processes with finite dislocation measure those processes are also mathematically more challenging. $\Diamond$

1.6 Erosion

In this section we shall briefly explain the role of the erosion coefficient. Let us start by considering a pure erosion process, i.e. without any dislocations. To this end, let $(e_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. Exp(1)–distributed random variables and for any $t \in \mathbb{R}_0^+$ set

$$N(t) := \{n \in \mathbb{N} : e_n \leq t\}.$$

For every $c, t \in \mathbb{R}_0^+$ we then define the random partition Π^c by

$$\Pi^c := \left\{ \bigcup_{n\in N(ct)} \{n\} \cup \mathbb{N} \setminus N(ct) \right\}^{\uparrow}.$$

Lemma 1.25 (Lemma 3.7 in [Ber06]) *Let $c \in \mathbb{R}_0^+$. Then the process $\Pi^c := (\Pi^c(t))_{t\in\mathbb{R}_0^+}$ is a fragmentation process with Lévy measure $\nu \equiv 0$ and erosion coefficient c. Moreover, we have*

$$|\Pi^c(t)|^{\downarrow} = (e^{-ct}, 0, \ldots).$$

Proof Since the sequence $(e_n)_{n\in\mathbb{N}}$ is i.i.d. , we infer that $\Pi^c(t)$, $t \in \mathbb{R}_0^+$, is a random exchangeable partition, and clearly $\Pi^c(0) = (\mathbb{N}, \emptyset, \ldots)$. In view of the memorylessness of the exponential distribution it follows that Π^c has the Markov property. Moreover, note that by definition Π^c contains at each time exactly one nonempty block that is not a singleton. Hence, in this case the Markov property coincides with the fragmentation property. The continuity in probability of Π^c follows from $\mathbb{P}(e_n = t) = 0$ for every $n \in \mathbb{N}$ and all $t \in \mathbb{R}_0^+$. Hence, Π^c is a fragmentation process. In order to describe the characteristic measure μ, consider a partition $\pi^{n,k} \in \mathcal{P}_{[n]}^*$, $k \in \mathbb{N}$, given by

$$\pi^{n,k} = ([n] \setminus \{k\}, \{k\}, \emptyset, \ldots)^{\uparrow}. \tag{1.13}$$

Since $e_k \sim$Exp(1), $k \in \mathbb{N}$, we then have by L'Hôpital's rule that

$$\lim_{t\downarrow 0} \frac{\mathbb{P}(\Pi^c(t)|_{[n]} = \pi^{n,k})}{t} = \lim_{t\downarrow 0} \frac{\mathbb{P}(e_k \leq ct)}{t} = \lim_{t\downarrow 0} \frac{1 - e^{-ct}}{t} = c.$$

[Note that $\lim_{t\downarrow 0} t^{-1}\mathbb{P}(e_i, e_j \leq ct) = 0$ for all $i \neq j \in \mathbb{N}$.] Moreover, for any $\pi \in \mathcal{P}_{[n]}^*$ not of the form (1.13) we have that

$$\lim_{t\downarrow 0} \frac{\mathbb{P}(\Pi^c(t)|_{[n]} = \pi)}{t} = 0.$$

Therefore, the jump rate of $\Pi^c|_{[n]}$ equals $\mu_c(\pi \in \mathcal{P} : \exists\, k \in [n] : \pi|_{[n]} = \pi^{n,k})$, and consequently the jump rate of Π^c is given by $\mu_c(\mathcal{P})$. This shows that the characteristic measure of Π^c is μ_c.

In order to compute the asymptotic frequencies, we resort to Kolmogorov's SLLN and obtain that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{\{e_k > t\}} = \mathbb{P}(e_1 > t) = e^{-ct}.$$

Hence $|\mathbb{N} \setminus N(ct)| = e^{-ct}$, and thus $|\Pi^c(t)|^{\downarrow} = (e^{-ct}, 0, \ldots)$. □

The following proposition, whose proof we omit here, combines the two phenomena of erosion and dislocation.

Proposition 1.26 (Proposition 3.4 in [Ber06]) *Let $c \geq 0$ and let ν be some Lévy measure on $\mathcal{S}$. Further, let $\Pi^\nu := (\Pi^\nu(t))_{t\in\mathbb{R}_0^+}$ be a fragmentation process with characteristic measure μ_ν and consider $\Pi^c := (\Pi^c(t))_{t\in\mathbb{R}_0^+}$, where $(e_n)_{n\in\mathbb{N}}$ is independent of Π^ν. Further, let $\Pi(t)$, $t \in \mathbb{R}_0^+$, be the unique partition such that any $n \in N(ct)$ is a singleton in $\Pi(t)$ and $\Pi(t)|_{\mathbb{N}\setminus N(ct)} = \Pi^\nu(t)|_{\mathbb{N}\setminus N(ct)}$. Then $\Pi := (\Pi(t))_{t\in\mathbb{R}_0^+}$ is a fragmentation process with characteristic measure $\mu = \mu_\nu + \mu_c$ and asymptotic frequencies given by*

$$|\Pi(t)|^{\downarrow} = e^{-ct} |\Pi^\nu(t)|^{\downarrow}$$

$\mathbb{P}$–a.s. for any $t \in \mathbb{R}_0^+$.

Proof See Proposition 3.4 in [Ber06]. □

In view of Corollary 1.21 the previous proposition say that any homogenous fragmentation process with Lévy measuere ν and erosion coefficient $c \in \mathbb{R}_0^+$ is characterised by a Poisson point process on $\mathcal{P} \otimes \mathbb{N}$ with characteristic measure $\mu_\nu \otimes \sharp$ and with drift c

1.6.1 Interval fragmentation processes

A third kind of fragmentation processes that appears in the literature are so–called interval fragmentations. This kind of fragmentation processes was introduced by Bertoin [Ber02b] and was also considered by Basdevant [Bas06]. Our definition follows the lines of [Ber02b].

In order to define interval fragmentations we first need to define some more notation. The state space of the process will be the usual topology $\mathcal{T}_{(0,1)}$ on $(0, 1)$. That is, $\mathcal{T}_{(0,1)}$ is the topology consisting of all unions of open intervals in $(0, 1)$. Further, for any $U \in \mathcal{T}_{(0,1)}$ define a function $\chi_U : [0, 1] \to [0, 1]$ by

$$\chi_U(x) = \inf_{y\in U^{\complement}} |x - y|$$

for every $x \in [0,1]$, where $U^{\complement} := [0,1] \setminus U$. We endow $\mathcal{T}_{(0,1)}$ with the metric $\rho_{\mathcal{T}_{(0,1)}}$ defined by

$$\rho_{\mathcal{T}_{(0,1)}}(U,V) = \sup_{x\in[0,1]} |\chi_U(x) - \chi_V(x)|$$

for any $U, V \in \mathcal{T}_{(0,1)}$. Observe that $(\mathcal{T}_{(0,1)}, \rho_{\mathcal{T}_{(0,1)}})$ is a compact metric space. Further, note that for all $U, V \in \mathcal{T}_{(0,1)}$ the distance $\rho_{\mathcal{T}_{(0,1)}}(U,V)$ coincides with the Hausdorff distance between $U^{\complement}$ and $V^{\complement}$. Moreover, consider two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in $[0,1]$ as well as $a, b \in [0,1]$. Then

$$\lim_{n\to\infty} \rho_{\mathcal{T}_{(0,1)}}((a_n,b_n),(a,b)) = 0 \Longleftrightarrow \lim_{n\to\infty} \max\{|a-a_n|, |b-b_n|\} = 0$$

and

$$\lim_{n\to\infty} \rho_{\mathcal{T}_{(0,1)}}((a_n,b_n),\emptyset) = 0 \Longleftrightarrow \lim_{n\to\infty} |a_n - b_n| = 0.$$

For further information regarding the metric space $(\mathcal{T}_{(0,1)}, \rho_{\mathcal{T}_{(0,1)}})$ we refer the reader to Section 2 in [Ber02b]. Let $\mathcal{T}_{0,1}$ be the topology induced by $\rho_{\mathcal{T}_{(0,1)}}$. We consider the measurable space $(\mathcal{T}_{(0,1)}, \mathscr{B}_{0,1})$, where $\mathscr{B}_{0,1}$ denotes the Borel–σ–algebra generated by $\mathcal{T}_{0,1}$, that is $\mathscr{B}_{0,1} := \sigma(\mathcal{T}_{0,1})$. Let $(p_t)_{t\in\mathbb{R}_0^+}$ be a set of probability measures on $(\mathcal{T}_{(0,1)}, \mathscr{B}_{0,1})$ such that the mapping $t \mapsto p_t$ is continuous. Further, let $\alpha \in \mathbb{R}$ and let $a, b \in [0,1]$ with $a < b$. We denote by $(\mathcal{T}_{(a,b)}, \mathcal{T}_{a,b})$ the topological subspace of $(\mathcal{T}_{(0,1)}, \mathcal{T}_{0,1})$ and we set $\mathscr{B}_{a,b} := \sigma(\mathcal{T}_{a,b})$. In addition, consider the map $g_{a,b} : \mathcal{T}_{(0,1)} \to \mathcal{T}_{(a,b)}$ by

$$g_{a,b}(U) = \{a + x(b-a) : x \in U\}.$$

for each $U \in \mathcal{T}_{(0,1)}$.

For any $t \in \mathbb{R}_0^+$ let us define a Markov kernel $p_t^{(\alpha)} : \mathcal{T}_{(0,1)} \times \mathscr{B}_{0,1} \to [0,1]$ as follows:

Definition 1.27 Set $p_t^{(\alpha)}(\emptyset, \cdot) := \delta_\emptyset$, where $\delta_\emptyset$ denotes the Dirac point mass at $\emptyset$. For any non–empty interval $(a,b) \in \mathcal{T}_{(0,1)}$ set

$$p_t^{(\alpha)}((a,b), A) := p_s(g_{a,b}^{-1}(A))$$

for every $t \in \mathbb{R}_0^+$ $A \in \mathscr{B}_{a,b}$, where $s := t(b-a)^\alpha$. Note that $g_{a,b}^{-1}(U)$ denotes the preimage of U under the function $g_{a,b}$. For any $A \in \mathscr{B}_{0,1} \setminus \mathscr{B}_{a,b}$ we set $p_t^{(\alpha)}((a,b),A) := 0$. Now let $U \in \mathcal{T}_{(0,1)}$ and consider two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in $[0,1]$ such that $U = \bigcup_{n\in\mathbb{N}}(a_n,b_n)$. Further, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables X_n with distribution $p_t^{(\alpha)}((a_n,b_n),\cdot)$. Then define $p_t^{(\alpha)}(U,\cdot)$ to be the distribution of $\bigcup_{n\in\mathbb{N}} X_n$.

We can now define interval fragmentation processes.

Definition 1.28 We call a $\mathcal{T}_{(0,1)}$–valued Markov process $\mathfrak{I} := (\mathfrak{I}(t))_{t\in\mathbb{R}_0^+}$, continuous in probability, a *self–similar (standard) interval fragmentation process* with index $\alpha \in \mathbb{R}$ if

(i) $\mathfrak{I}(0) = (0,1)$.

(ii) $\mathfrak{I}(t) \subseteq \mathfrak{I}(s)$ for all $s,t \in \mathbb{R}_0^+$ with $s \leq t$.

(iii) Denote the distribution of $\mathfrak{I}(t)$, $t \in \mathbb{R}_0^+$, by p_t. Then the transition semigroup of $\mathfrak{I}$ is determined by the Markov kernels $(p_t^{(\alpha)})_{t\in\mathbb{R}_0^+}$ provided by Definition 1.27.

If $\alpha = 0$ then the process is called *homogenous*.

Let us mention that the continuity in probability in Definition 1.28 is meant with respect to the metric $\rho_{\mathcal{T}_{(0,1)}}$. We further remark that similarly to the case of $\mathbb{P}$–fragmentation processes, see Theorem 1.22, also homogenous interval fragmentation processes without erosion can be constructed via Poisson point processes.

1.7 Bijections between different classes of fragmentation processes

According to Proposition 2.6 in [Ber02a] the $\mathcal{S}$–valued process consisting of the reordered sequences of the asymptotic frequencies of a self–similar $\mathcal{P}$–fragmentation process with index $\alpha \in \mathbb{R}$ and $\mathbb{P}$–dislocation measure μ_ν constitutes a self–similar mass fragmentation process with index α and $\mathcal{S}$–dislocation measure ν. Moreover, in [Ber02a, Proposition 2.6] Berestycki also shows that the converse holds in the sense that for any self–similar mass fragmentation process λ with index $\alpha \in \mathbb{R}$ and $\mathcal{S}$–dislocation measure ν there exists some self–similar $\mathcal{P}$–fragmentation process with index α and $\mathcal{P}$–dislocation measure μ_ν, whose asymptotic frequencies form a process having the same distribution as λ. That is, there exists a bijection between mass fragmentation processes and $\mathcal{P}$–fragmentation processes. Moreover, Section 3.2 in [Ber02b] shows that there is also a bijection between interval fragmentation processes and $\mathcal{P}$–fragmentation processes. Consequently, we have the following theorem:

Theorem 1.29 ([Ber02a], [Ber02b]) *The three classes of fragmentations that we introduced in the previous sections are mutually in a one–to–one correspondence with each other.*

Therefore, without loss of generality we can always choose the representation that is most useful in a specific situation. In this regard, we remark that

Figure 1.2 is an illustration of any kind of fragmentation processes as it is just concerned with the sizes of the blocks which always constitute a mass fragmentation process. Note that by size of a block we mean the asymptotic frequency for $\mathcal{P}$–fragmentation processes and the lengths of the interval components of open sets for interval fragmentations.

Throughout these notes we consider a homogenous standard $\mathcal{P}$–fragmentation process $\Pi = (\Pi(t))_{t\in\mathbb{R}_0^+}$ with Lévy measure ν and erosion coefficient $c \in \mathbb{R}_0^+$. In addition, let $\lambda = (\lambda(t))_{t\in\mathbb{R}_0^+} := (|\Pi(t)|^{\downarrow})_{t\in\mathbb{R}_0^+}$ and $\mathfrak{I} := (\mathfrak{I}(t))_{t\in\mathbb{R}_0^+}$ be the corresponding mass fragmentation process and interval fragmentation process respectively, given by the aforementioned bijections, see Theorem 1.29, between these classes of fragmentation processes.

CHAPTER 2

PROPERTIES OF FRAGMENTATION PROCESSES

In this chapter we provide some useful properties of fragmentation processes and consider some related concepts that are often used in the literature on fragmentations.

2.1 Connection between fragmentation processes and subordinators

2.1.1 Subordinators

It is well known that the distribution of a subordinator $(X_t)_{t\in\mathbb{R}_0^+}$ is determined by its *Laplace exponent* Φ, defined via the *Laplace transform*

$$\mathbb{E}\left(e^{-qX_t}\right) = e^{-t\Phi(q)}$$

for any $q > 0$ and $t \in \mathbb{R}_0^+$. Moreover, the Laplace exponent Φ is be characterised by the *Lévy–Khintchine formula*:

Lemma 2.1 (Theorem 1.2 in [Ber99]) *The Laplace exponent Φ of a subordinator X is given by*

$$\Phi(q) = \mathrm{k} + dq + \int_{(0,\infty)} (1 - e^{-qx})L(dx),$$

where $\mathrm{k} \in \mathbb{R}_0^+$ is the killing rate, $d \in \mathbb{R}_0^+$ specifies the drift of X and the Lévy measure L, which determines the jumps of X, is a measure on $(0,\infty)$ that satisfies $\int_{(0,\infty)}(1 \wedge x)L(dx) < \infty$.

2.1.2 Subordinators associated with fragmentations

This section is devoted to a specific subordinator that appears in the context of fragmentation processes. This subordinator plays a crucial role in this lecture.

Further, recall the concept of asymptotic frequencies for partitions that we introduced in Defintion 1.13 In this spirit we define for every $\Pi_1(t)$ the upper and lower asymptotic frequencies as follows:

$$|\Pi_1(t)|^* := \limsup_{k\to\infty} \frac{\mathrm{card}(\Pi_1(t) \cap \{1, \dots, k\})}{k}$$

$$|\Pi_1(t)|_* := \liminf_{k\to\infty} \frac{\mathrm{card}(\Pi_1(t) \cap \{1, \dots, k\})}{k}.$$

Note that in view of Theorem 1.14 the exchangeability of Π implies that for any $t \in \mathbb{R}_0^+$ the asymptotic frequency $|\Pi_1(t)| = |\Pi_1(t)|^* = |\Pi_1(t)|_*$ exists $\mathbb{P}$–almost surely. However, we now aim at studying the process $(|\Pi_1(t)|)_{t\in\mathbb{R}_0^+}$ and for this we need that the asymptotic frequencies $|\Pi_1(t)|$ exist $\mathbb{P}$–a.s. simultaneously for all $t \in \mathbb{R}_0^+$. On this note, let us remark that we clearly have the existence of $(|\Pi_1(t)|)_{t\in\mathbb{Q}}$, a fact we shall make use of below.

The goal of this section is to establish a connection between fragmentation processes and subordinators. More specifically, the main result in this regard is the following theorem.

Theorem 2.2 *The block Π_1 possesses asymptotic frequencies $\mathbb{P}$–a.s. simultaneously for all times $t \in \mathbb{R}_0^+$. Moreover, the process $(-\ln(|\Pi_1(t)|))_{t\in\mathbb{R}_0^+}$ is a possibly killed subordinator (with respect to the filtration $\mathscr{F}$) with drift $d = c$, killing rate*

$$\mathrm{k} = c + \int_{\mathcal{S}} \left(1 - \sum_{n\in\mathbb{N}} s_n\right) \nu(d\mathbf{s})$$

and Lévy measure

$$L(dx) = e^{-x} \sum_{n\in\mathbb{N}} \nu(-\ln(s_n \in dx) \qquad \forall\, x > 0.$$

We now establish several lemmas that provide us with the proof of Theorem 2.2. The first auxiliary lemma is concerned with the moments of $|\Pi_1|$.

Lemma 2.3 (Lemma 3 in [Ber01]) *Let $k \in \mathbb{N}$. Then we have*

$$\mathbb{E}\left(|\Pi_1(t)|^k\right) = \exp\left(-t\left(c(k+1) + \int_{\mathcal{S}} \left(1 - \sum_{n\in\mathbb{N}} s_n^{k+1}\right) \nu(d\mathbf{s})\right)\right).$$

for any $t \in \mathbb{R}_0^+$.

Proof Since

$$|\Pi_1(t)| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{i \in \Pi_1(t)\}},$$

we have

$$\begin{aligned}|\Pi_1(t)|^k &= \lim_{n\to\infty} \frac{1}{n^k} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \mathbb{1}_{\{i_1 \in \Pi_1(t)\}} \cdots \mathbb{1}_{\{i_k \in \Pi_1(t)\}} \\ &= \lim_{n\to\infty} \frac{1}{n^k} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \mathbb{1}_{\{\forall j=1\ldots,k:\, i_j \in \Pi_1(t)\}}. \qquad (2.1)\end{aligned}$$

Observe that for all but $\sum_{l=1}^{k-1} l = k/2$–many summands in the right–hand side of (2.1) the entries of the vector $(i_j)_{j=1\ldots,k}$ are pairwise different, and by exchangeability we thus have that for these $\sum_{l=1}^{k-1} l = k/2$–many summands

$$\mathbb{P}(\forall j = 1\ldots,k : i_j \in \Pi_1(t)) = \mathbb{P}(\mathbb{P}_k^*)$$

holds true. Consequently, since $k/2n^k \to 0$ as $n \to \infty$, we infer from (2.1) in conjunction with the DCT that

$$\mathbb{E}\left(|\Pi_1(t)|^k\right) = \mathbb{P}(\mathbb{P}_k^*). \qquad (2.2)$$

In view of Corollary 1.21 let $(\pi(t), k(t))_{t\in\mathbb{R}_0^+}$ be the Poisson point process that characterises Π. In addition, consider the Poisson point process $(\pi^{(1)}(t))_{t\in\mathbb{R}_0^+}$ (with intensity μ) given by

$$\pi^{(1)}(t) = \begin{cases} \pi(t), & k(t) = 1 \\ (\mathbb{N}, \emptyset, \ldots), & k(t) > 1. \end{cases}$$

Then the process $(\Pi_1(t))_{t\in\mathbb{R}_0^+}$ is determined by $(\pi^{(1)}(t))_{t\in\mathbb{R}_0^+}$. In particular,

$$[k] \subseteq \Pi_1(t) \iff \forall j \in \{2,\ldots,k\}\, \forall s \in [0,t] : 1 \overset{\Delta_s^{(1)}}{\sim} j. \qquad (2.3)$$

Consider the event

$$E_{1,k} := \{\exists j \in \{2,\ldots,k\} : 1 \not\sim j\}.$$

According to Proposition 2 in Section 0.5 of [Ber96] we have that

$$\tau_{1,k} := \inf\left\{t \in \mathbb{R}_0^+ : \Delta_t^{(1)} \in \{1 \not\sim j\} \text{ for some } j \in \{2,\ldots,k\}\right\} \overset{\mathrm{d}}{=} e^{\mu(E_{1,k})},$$

where $\{1 \not\sim j\} := \{\pi \in \mathcal{P} : 1 \overset{\pi}{\not\sim} j\}$. Hence, (2.2) and (2.3) result in

$$\mathbb{E}\left(|\Pi_1(t)|^k\right) = \mathbb{P}(\mathbb{P}_k^*) = \mathbb{P}\left([k] \subseteq \Pi_1(t)\right) = \mathbb{P}(\tau_{1,k} > t) = e^{-t\mu(E_{1,k})}. \qquad (2.4)$$

Moreover, Theorem 1.12 yields that

$$\mu(E_{1,k}) = \mu_c(E_{1,k}) + \mu_\nu(E_{1,k}) = c(k+1) + \int_{\mathcal{S}} \left(1 - \sum_{n\in\mathbb{N}} s_n^{k+1}\right) \nu(\mathrm{d}\mathbf{s}).$$

Plugging this into (2.4) completes the proof. □

The following lemma establishes the connection between fragmentations and subordinators. In the light of the boundedness and monotonicity of the map $t \mapsto |\Pi_1(t)|$ let us define a process $\xi := (\xi(t))_{t\in\mathbb{R}_0^+}$ by

$$\xi(t) := \lim_{\mathbb{Q}\ni u\downarrow t} (-\ln(|\Pi_1(u)|))$$

for all $t \in \mathbb{R}_0^+$.

Lemma 2.4 (Lemma 4 in [Ber01]) *The process ξ is a subordinator with respect to $\mathscr{F}$.*

Proof Since the map $t \mapsto |\Pi_1(t)|$ is càdlàg and nonincreasing, we have that ξ is a nondecreasing càdlàg process. Moreover, it follows from Lemma 2.3 and $|\Pi_1| \leq 1$ that $|\Pi_1(t)| \to 1$ in $\mathcal{L}^1(\mathbb{P})$ as $t \downarrow 0$. Since, by boundedness and monotonicity, $|\Pi_1(t)|$ converges $\mathbb{P}$–a.s. as $t \downarrow 0$ along $t \in \mathbb{Q}$, we thus infer that

$$\lim_{t\downarrow 0} |\Pi_1(t)|^* = 1 = \lim_{t\downarrow 0} |\Pi_1(t)|_* \tag{2.5}$$

$\mathbb{P}$–a.s., and consequently $|\Pi_1(t)| \uparrow 1$ as $t \downarrow 0$ along $t \in \mathbb{Q}$. Consequently, $\xi(0) = 0$ $\mathbb{P}$–almost surely.

It remains to show that ξ has stationary and independent increments. To this end, recall that the fragmentation property yields that

$$\Pi_1(t+u) = (\tilde{\Pi}|_{\Pi_1(t)})_1,$$

where the random partition $\tilde{\Pi}$ is independent of $\mathscr{F}_t$ and satisfies $\tilde{\Pi} \overset{(d)}{=} \Pi(u)$. Therefore, we have $|\Pi_1(t+u)| = |\Pi_1(t)||\tilde{\Pi}_1|$ and thus

$$-\ln(|\Pi_1(t+u)|) = -\ln|\Pi_1(t)| + (-\ln(|\tilde{\Pi}_1|))$$

$\mathbb{P}$–a.s. for all $t, u \in \mathbb{R}_0^+$, and hence we conclude that ξ has stationary and independent increments with respect to $\mathscr{F}$. □

Lemma 2.5 (Lemma 5 in [Ber01]) *The block Π possesses asymptotic frequencies $|\Pi_1(t)|$ $\mathbb{P}$–a.s. simultaneously for all $t \in \mathbb{R}_0^+$ and we have*

$$|\Pi_1(t)| = e^{-\xi(t)}.$$

Proof Resorting to the monotonicity of Π we obtain that

$$e^{-\xi(t)} = \lim_{\mathbb{Q}\ni u\downarrow t} |\Pi_1(u)| \leq |\Pi_1(t)|_*$$

and

$$|\Pi_1(t)|^* \leq \lim_{\mathbb{Q}\ni u\downarrow t-} |\Pi_1(u)| = e^{-\xi(t-)}.$$

Notice that $|\Pi_1(t)|_* = e^{-\xi(t)} = |\Pi_1(t)|^*$ for any time $t \in \mathbb{R}_0^+$ at which ξ is continuous. Hence, for each such time we have

$$|\Pi_1(t)| = e^{-\xi(t)}. \tag{2.6}$$

Le us now deal with the jump times of ξ. For this purpose fix some $\epsilon > 0$ and $k \in \mathbb{N}$ and consider

$$\tau_{\epsilon,k} := \inf\{t \in \mathbb{R}_0^+ : t \text{ is the } k\text{–th jump time of } \xi \text{ of size at least } \epsilon\}.$$

Since $\tau_{\epsilon,k}$ is an $\mathscr{F}$–stopping time, we infer from the strong fragmentation property that

$$\begin{aligned} &|\Pi_1(\tau_{\epsilon,k})|_*|\tilde{\Pi}_1(\tau_{\epsilon,k})|_* \leq |\Pi_1(\tau_{\epsilon,k}+t)|_* \leq |\Pi_1(\tau_{\epsilon,k})|_*|\tilde{\Pi}_1(\tau_{\epsilon,k})|^* \\ &|\Pi_1(\tau_{\epsilon,k})|^*|\tilde{\Pi}_1(\tau_{\epsilon,k})|_* \leq |\Pi_1(\tau_{\epsilon,k}+t)|^* \leq |\Pi_1(\tau_{\epsilon,k})|^*|\tilde{\Pi}_1(\tau_{\epsilon,k})|^* \end{aligned} \tag{2.7}$$

on $\{\tau_{\epsilon,k} < \infty\}$, where $|\tilde{\Pi}_1(t)|^*$ and $|\tilde{\Pi}_1(t)|_*$ are the upper and lower asymptotic frequencies of $\tilde{\Pi}(t)$ and $\tilde{\Pi}$ is an independent fragmentation process with the same distribution as Π. In view of (2.5) we deduce from (2.7) that

$$\lim_{t\downarrow 0} |\Pi_1(\tau_{\epsilon,k}+t)|_* = |\Pi_1(\tau_{\epsilon,k})|_* \quad \text{as well as} \quad \lim_{t\downarrow 0} |\Pi_1(\tau_{\epsilon,k}+t)|^* = |\Pi_1(\tau_{\epsilon,k})|^*$$

$\mathbb{P}$–a.s. on $\{\tau_{\epsilon,k} < \infty\}$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+$ with $t_n \downarrow 0$ as $n \to \infty$ and such that ξ is continuous at $\tau_{\epsilon,k} + t_n$ for each $n \in \mathbb{N}$. Recalling (2.6) we thus have

$$\begin{aligned} |\Pi_1(\tau_{\epsilon,k})|_* &= \lim_{t\downarrow 0} |\Pi_1(\tau_{\epsilon,k}+t)|_* \\ &= \lim_{n\to\infty} |\Pi_1(\tau_{\epsilon,k}+t_n)| \\ &= \lim_{n\to\infty} e^{-\xi(\tau_{\epsilon,k}+t_n)} \\ &= e^{-\xi(\tau_{\epsilon,k})} \end{aligned}$$

$\mathbb{P}$–a.s. on $\{\tau_{\epsilon,k} < \infty\}$, where the final equality follows from the right–continuity of ξ. Analogously, we obtain that $|\Pi_1(\tau_{\epsilon,k})|^* = e^{-\xi(\tau_{\epsilon,k})}$ $\mathbb{P}$–a.s. on $\{\tau_{\epsilon,k} < \infty\}$, which results in

$$|\Pi_1(\tau_{1/n,k})| = e^{-\xi(\tau_{1/n,k})}$$

$\mathbb{P}$–a.s. on $\{\tau_{1/n,k} < \infty\}$, simultaneously for all $k, n \in \mathbb{N}$. Note that this covers all the jumps of π_1. Since $\tau_{1/n,k} < \infty$ $\mathbb{P}$–a.s., the proof is complete. □

Let us now tackle the proof of Theorem 2.2.

Proof of Theorem 2.2 According to Lemma 2.4 and Lemma 2.5 we have that ξ is a subordinator satisfying $\xi(t) = -\ln(|\Pi_1(t)|)$ for all $t \in \mathbb{R}_0^+$ $\mathbb{P}$–almost surely. It remains to establish the Laplace exponent of ξ. For this purpose, observe first that Lemma 2.3 implies sthat

$$\begin{aligned}\mathbb{E}\left(e^{-p\xi(t)}\right) &= \mathbb{E}\left(\lambda_1^p(t)\right) \\ &= \exp\left\{-t\left(c(p+1) + \int_{\mathcal{S}}\left(1 - \sum_{n\in\mathbb{N}} s_n^{p+1}\right)\nu(\mathrm{d}\mathbf{s})\right)\right\}\end{aligned}$$

for all $p \in \mathbb{N}$. We thus obtain the representation of the Lévy–Khintchine formula, cf. Theorem 2.1, by setting

$$\mathrm{k} := c + \int_{\mathcal{S}}\left(1 - \sum_{n\in\mathbb{N}} s_i\right)\nu(\mathrm{d}\mathbf{s})$$

as well as

$$L(\mathrm{d}x) = e^{-x}\sum_{n\in\mathbb{N}}\nu(-\ln(s_n) \in \mathrm{d}x) \qquad \forall\, x > 0$$

and using the substitiution $y := e^{-x}$. Observe that L satisfies the integrability criterion $\int_{(0,\infty)}(1 \wedge x)L(\mathrm{d}x) < \infty$ and hence L is a Lévy measure on $(0, \infty)$. Consequently, we have

$$\mathbb{E}\left(e^{-p\xi(t)}\right) = e^{-t\Phi(n)} \tag{2.8}$$

for all $p \in \mathbb{N}$, where

$$\Phi(p) = \mathrm{k} + \mathrm{d}p + \int_{(0,\infty)}\left(1 - e^{-px}\right)L(\mathrm{d}x) \tag{2.9}$$

for any $p > 0$. Since by the Stone–Weierstrass theorem linear combinations of maps $x \mapsto e^{-nx}$, $n \in \mathbb{N}_0$, are dense in the space of continuous functions on $\mathbb{R}_0^+$ with a finite limit at ∞, we thus conclude that (2.8) determines the distribution of $\xi(t)$. Moreover, the Lévy–Khintchine formula implies that Φ as defined in (2.9) is the Laplace exponent of a subordinator. Hence, we infer that (2.8) holds true for all $p \in \mathbb{R}_0^+$, which completes the proof. □

Remark 2.6 Let $t > 0$ be such that $\pi^{(1)}(t) \notin (\mathbb{N}, \emptyset, \ldots)$. Then $|\Pi_1|$ has a downwards jump at t such that

$$|\Pi_1(t)| = |\pi^{(1)}(t)|\Pi_1(t-)|.$$

Consequently, the subordinator ξ jumps at time t with

$$\xi(t) = \xi(t-) - \ln(|\pi^{(1)}(t)|).$$

Therefore, the rate of partitions for which ξ has jumps of size greater than $\alpha \in \mathbb{R}^+ \cup \{\infty\}$ equals the rate for which the $\pi^{(1)}$ has asymptotic frequency less than $e^{-\alpha}$. Note that this rate can be computed by resorting to Theorem 1.12 and that for $\alpha = \infty$ this gives the killing rate

$$\mathrm{k} = c + \int_{\mathcal{S}} \left(1 - \sum_{n\in\mathbb{N}} s_i\right) \nu(\mathrm{d}\mathbf{s}).$$

On this note, let us mention that c gives the rate at which a chosen number becomes isolated from its block into a singleton and that the above integral is the rate at which a fixed number fragments from its block as a result of a sudden dislocation. In this regard, observe that for any $\mathbf{s} \in \mathcal{S}$ the set of singletons in an exchangeable random partition with distribution $\varrho_{\mathbf{s}}$ has asymptotic frequency $1 - \sum_{n\in\mathbb{N}} s_i$ $\mathbb{P}$–almost surely. $\Diamond$

From now on we make the assumption $c = 0$, i.e. we consider pure jump processes. In addition, we assume that the Lévy measure satisfies

$$\nu(s_2 \neq 0) > 0. \tag{2.10}$$

Note that (2.10) yields that at any time $t > 0$ there exist with positive probability at least two blocks with positive asymptotic frequency.

Set

$$\underline{p} := \inf\left\{p \in \mathbb{R} : \int_{\mathcal{S}} \left|1 - \sum_{n\in\mathbb{N}} s_n^{1+p}\right| \nu(\mathrm{d}\mathbf{s}) < \infty\right\} \in (-1, 0].$$

In addition, define a function $\Phi : (\underline{p}, \infty) \to \mathbb{R}$ by

$$\Phi(p) = \int_{\mathcal{S}} \left(1 - \sum_{n\in\mathbb{N}} s_n^{1+p}\right) \nu(\mathrm{d}\mathbf{s})$$

for every $p \in (\underline{p}, \infty)$ and note that on $\mathbb{R}_0^+$ this coincides with the Laplace exponent of ξ. Hence, it is well known, see e.g. Chapter III in [Ber96], that Φ is monotonically increasing and concave. Moreover, If $\underline{p} = 0$ in the conservative case, then we set $\Phi(\underline{p}) := 0$. A typical graph of Φ is depicted in Figure 2.1. Note that this graph corresponds to the dissipative case. In the conservative case we always have that Φ passes through the origin, that is $\Phi(0) = 0$. Notice further that the following three different possibilities for the behaviour of Φ at $\underline{p}$ can occur:

- $\Phi(\underline{p}) > -\infty$ and $\Phi'(\underline{p}+) < \infty$,
- $\Phi(\underline{p}) > -\infty$ and $\Phi'(\underline{p}+) = \infty$,
- $\Phi(\underline{p}) = -\infty$ and $\Phi'(\underline{p}+) = \infty$.

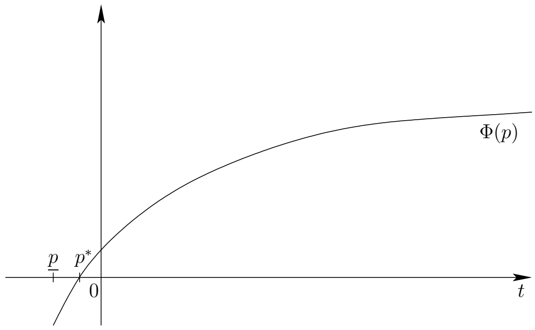


Figure 2.1: *Graph of the Laplace exponent* Φ *in the dissipative case with* $\underline{p} < 0$, $\Phi(\underline{p}) > -\infty$ *and* $\Phi'(\underline{p}) < \infty$. *Note that in this illustration there exists a* $p^* \in (\underline{p}, 0)$ *with* $\Phi(p^*) = 0$.

The illustration in Figure 2.1 depicts the first case.

Besides $\underline{p}$ two other constants related to Φ will play a crucial role. The first one is given by the following definition and the second one by the subsequent lemma.

Definition 2.7 If there exists a $p^* \in [\underline{p}, 0]$ satisfying $\Phi(p^*) = 0$, then we call p^* *Malthusian parameter.*

For the remainder of the course, unless otherwise specified, we assume that a Malthusian parameter p^* exists. More precisely, we shall work under the following hypothesis:

Hypothesis 2.1 There exists a $p^* \in (\underline{p}, 0]$ such that $\Phi(p^*) = 0$.

In what follows, we always assume that Hypotheses 2.1 holds.

If ν is conservative, that is if $\nu\left(\sum_{n\in\mathbb{N}} s_n < 1\right) = 0$, then $\Phi(0) = 0$, and thus $p^* = 0$ in that case. Moreover, observe that Hypothesis 2.1 implies that $\underline{p} < 0$ and thus $\Phi'(0+) < \infty$.

Lemma 2.8 (Lemma 1 in [Ber03]) *There exists a (unique)* $\bar{p} \in \mathbb{R}^+$ *such that the mapping* $f : (\underline{p}, \infty) \to \mathbb{R}$, *given by* $f(p) = {}^{\Phi(p)}/_{(1+p)}$, *is increasing on* $(\underline{p}, \bar{p})$ *and decreasing on* $(\bar{p}, \infty)$. *Moreover,* $\bar{p}$ *is the unique solution to*

$$(1 + p)\Phi'(p) = \Phi(p) \tag{2.11}$$

on $(\underline{p}, \infty)$, *where* Φ' *denotes the derivative of* Φ, *and the unique maximum of* f *is thus given by*

$$f(\bar{p}) = \frac{\Phi(\bar{p})}{1 + \bar{p}} = \Phi'(\bar{p}).$$

Proof Let us first mention that the map $p \mapsto (1+p)\Phi'(p) - \Phi(p)$ is decreasing on $(\underline{p}, \infty)$, since

$$\frac{\mathrm{d}}{\mathrm{d}p}((1+p)\Phi'(p) - \Phi(p)) = \Phi'(p) + (1+p)\Phi''(p) - \Phi'(p)$$
$$= (1+p)\Phi''(p) < 0, \tag{2.12}$$

where the negativity follows from Φ being concave. Applying integration by parts to the Lévy–Khintchine formula yields that

$$\lim_{p \to \infty} \frac{\Phi(p)}{p} = d = 0, \tag{2.13}$$

since we have set $d = c = 0$. Observe that $\Phi(p^*) = 0$ implies that

$$\Phi(\underline{p}) \leq 0. \tag{2.14}$$

Moreover, since according to (2.12) the map f' given by

$$f'(p) := \frac{\mathrm{d}}{\mathrm{d}p} f(p) = \frac{\Phi'(p)(1+p) - \Phi(p)}{(1+p)^2}$$

is decreasing and positive for $p = 0$, we infer from (2.13) and (2.14) that f attains its maximum at a unique point $\bar{p} \in (\underline{p}, \infty)$. Clearly, $f'(\bar{p}) = 0$ and f' is positive on $(\underline{p}, \bar{p})$ and negative on $(\bar{p}, \infty)$. Moreover, in the light of f having a unique local extremum it follows from

$$0 = f'(\bar{p}) = \frac{\Phi'(\bar{p})(1+\bar{p}) - \Phi(\bar{p})}{(1+barp)^2}$$

that $\bar{p}$ is the unique solution to (2.11). □

Notice that it follows from the above lemma that $p \geq \bar{p}$ if and only if $(1+p)\Phi'(p) \leq \Phi(p)$. Since $\Phi'(p) > 0$ for all $p \in (\underline{p}, \infty)$, we therefore have $p^* < \bar{p}$.

2.2 Many–to–one identities

In this section we develop a result that enables us to reduce the study of many fragments to that of a single block, viz the block containing 1. For this reason this kind of result may be referred to as *many–to–one identity*. Such an identity first appeared in the literature on branching processes, see e.g. [BD75], [HW96] and [Har00]. For a version of a many–to–one identity in the context of fragmentation chains we refer to Lemma 5.1 in [HK08].

Recall that $B_n(t)$, $t \in \mathbb{R}_0^+$, denotes the block in $\Pi(t)$ which contains the element $n \in \mathbb{N}$. The many–to–one identity in our setting reads as follows:

Lemma 2.9 *We have*

$$\mathbb{E}\left(\sum_{n\in\mathbb{N}} |B_n(t)| f(\{|B_n(s)| : s \le t\}) \mathbb{1}_{\{n=\min(B_n(t))\}}\right) = \mathbb{E}\left(f(\{|B_1(s)| : s \le t\})\right)$$

for every $t, u \in \mathbb{R}_0^+ \cup \{\infty\}$ *and* $f : \mathrm{RCLL}([0,t],[0,1]) \to \mathbb{R}$.

Note that the indicator function that appears on the left–hand side above is needed in order to avoid counting a block multiple times. Using the indicator function ensures that to each block corresponds exactly one summand, namely the one associated with the least element of that block.

Proof Recall that in Section 1.7 we mentioned that for the $\mathbb{P}$–fragmentation process Π there exists a corresponding interval fragmentation $\mathfrak{I}$. Hence, for any $y \in (0,1)$ and $s \in \mathbb{R}_0^+$ let $\mathfrak{I}_y(s)$ be the interval at time s in the chosen interval representation of $\Pi(s)$ that contains y, where we adopt $\mathfrak{I}_y(s) := \emptyset$ if y does not belong to any interval of the interval representation under consideration. We denote by $|\mathfrak{I}_y(s)|$ the length of the interval $\mathfrak{I}_y(s)$ and set $|\emptyset| := 0$. Further, fix $t \in \mathbb{R}_0^+$ and let $f : \mathrm{RCLL}([0,t],[0,1]) \to \mathbb{R}$. Then we have

$$\begin{aligned}
&\mathbb{E}\left(\sum_{n\in\mathbb{N}} |B_n(t)| f(\{|B_n(s)| : s \le t\}) \mathbb{1}_{\{n=\min(B_n(t))\}}\right) \\
&= \mathbb{E}\left(\int_{(0,1)} f(\{|\mathfrak{I}_y(s)| : s \le t\})\, \mathrm{d}y\right) \\
&= \mathbb{E}\left(f(\{|\mathfrak{I}_U(s)| : s \le t\})\right),
\end{aligned}$$

where $U : \Omega \to (0,1)$ is a uniformly distributed random variable that is independent of Π. By means of the exchangeability of Π the random variable $|\mathfrak{I}_U(t)|$ has the same distribution under $\mathbb{P}$ as $|B_1(t)|$ and thus we have proven the assertion. □

2.3 The intrinsic additive martingale for fragmentation processes

Set $\xi_n(t) := -\ln(|B_n(t)|)$. By means of the exchangeability of Π it follows from Theorem 2.2 that ξ_n is a possibly killed subordinator. Furthermore, recall the filtrations $\mathscr{F}$ (generated by Π) as well as $\mathscr{G}$ (generated by Π_1).

Let us start by considering the process $(e^{\Phi(p)t}|B_n(t)|^p \mathbb{1}_{\{|B_n(t)>0|\}})_{t\in\mathbb{R}_0^+}$ for $n \in \mathbb{N}$. Recall that

$$e^{\Phi(p)t}|B_n(t)|^p \mathbb{1}_{\{|B_n(t)>0|\}} = e^{\Phi(p)t - p\xi_n(t)}$$

for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$. This process with $n = 1$ was considered for instance in [BR03] and there it was used that it is a martingale with respect to the filtration $\mathscr{F}$. Let us briefly show that for any $n \in \mathbb{N}$ this process is indeed an $\mathscr{F}$–martingale. To this end, let $s, t \in \mathbb{R}_0^+$ and observe that the independent and identically distributed increments of the subordinator ξ_n yield that

$$\begin{aligned}\mathbb{E}\left(e^{\Phi(p)(t+s)-p\xi_n(t+s)}\middle|\mathscr{F}_t\right) &= e^{\Phi(p)t-p\xi_n(t)}e^{\Phi(p)s}\mathbb{E}\left(e^{-p\xi_n(s)}\right)\\ &= e^{\Phi(p)t-p\xi_n(t)},\end{aligned}$$

where the final equality follows from Φ being the Laplace exponent of the (killed) subordinator ξ_n.

Later on we shall make use of another $\mathscr{F}$–martingale that in contrast to the above process is also $\mathscr{G}$–adapted. This martingale is given by the following lemma:

Lemma 2.10 *The stochastic process $M(p) := (M_t(p))_{t\in\mathbb{R}_0^+}$, defined by*

$$M_t(p) := e^{\Phi(p)t}\sum_{n\in\mathbb{N}}|B_n(t)|^{1+p} = e^{\Phi(p)t}\sum_{n\in\mathbb{N}}\lambda_n^{1+p}(t)$$

for all $t \in \mathbb{R}_0^+$ and $p \in (\underline{p}, \infty)$, is an $\mathscr{F}$–martingale.

Proof According to the many–to–one identity in Lemma 2.9, with f given by $f(x) = x^p$ for all $x \in [0, 1]$, we have

$$\mathbb{E}\left(\sum_{n\in\mathbb{N}}|B_n(t)|^{1+p}\right) = \mathbb{E}\left(|B_1(t)|^p\right) = e^{-\Phi(p)t},$$

where the final equality results from $(e^{t\Phi(t)}|B_1(t)|^p\mathbb{1}_{\{t<\zeta\}})_{t\in\mathbb{R}_0^+}$ being a unit–mean martingale as mentioned above. Hence, we deduce from the fragmentation property that

$$\begin{aligned}\mathbb{E}(M_{t+s}(p)|\mathscr{F}_t) &= \mathbb{E}\left(e^{\Phi(p)(t+s)}\sum_{n\in\mathbb{N}}|B_n(t+s)|^{1+p}\middle|\mathscr{F}_t\right)\\ &= e^{\Phi(p)t}\sum_{n\in\mathbb{N}}|B_n(t)|^{1+p}\mathbb{E}\left(e^{\Phi(p)s}\sum_{k\in\mathbb{N}}|B_k(s)|^{1+p}\right)\\ &= e^{\Phi(p)t}\sum_{n\in\mathbb{N}}|B_n(t)|^{1+p}\\ &= M_t(p),\end{aligned}$$

which shows that $M(p)$ is a martingale. □

Remark 2.11 Let us mention that the martingale $M(p)$ appears frequently in the literature on fragmentation processes, see for example [Ber03], [BR03], [BM05], [HKK10] as well as [BHK10], and is often called *intrinsic additive martingale*. Moreover, similar additive martingales are also considered in the literature on branching processes, see for instance [Ner81], [Kyp04] and [BHK10]. In fact, $M(p)$ is the analogue of Biggins' classical additive martingale for branching random walks, see e.g. [Big92]. $\Diamond$

By the martingale convergence theorem the nonnegative martingale $M(p)$ has a $\mathbb{P}$–a.s. limit $M_\infty(p)$ for every $p > \underline{p}$. The following lemma establishes the almost sure positivity of $M_\infty(p)$ for $p \in (\underline{p}, \bar{p})$.

Lemma 2.12 *Let $p \in (\underline{p}, \bar{p})$. Then we have $M_\infty(p) > 0$ $\mathbb{P}$–almost surely.*

Proof Let $t \in \mathbb{R}^+$. Resorting to the fragmentation property of Π, we infer that

$$\mathbb{P}\left(M_\infty(p) = 0 | \mathscr{F}_t\right) = \prod_{n\in\mathbb{N}} \mathbb{P}_{\lambda_n(t)}(M_\infty(p) = 0)$$

$\mathbb{P}$–almost surely. Taking expectations we thus deduce that

$$\mathbb{P}\left(M_\infty(p) = 0\right) = \mathbb{E}\left(\prod_{n\in\mathbb{N}} \mathbb{P}_{\lambda_n(t)}(M_\infty(p) = 0)\right) \tag{2.15}$$

The homogeneity of Π yields that

$$\mathbb{P}_x(M_\infty(p) = 0) = \mathbb{P}(xM_\infty(p) = 0) = \mathbb{P}(M_\infty(p) = 0)$$

for all $x > 0$. Note that $\mathbb{P}_0(M_\infty(p) = 0) = 1$. Hence, (2.15) results in

$$\mathbb{P}\left(M_\infty(p) = 0\right) = \mathbb{E}\left(\mathbb{P}(M_\infty(p) = 0)^{\mathrm{card}(\{n\in\mathbb{N}:\lambda_n(t)>0\})}\right),$$

that is

$$\mathbb{E}\left(\mathbb{P}\left(M_\infty(p) = 0\right) - \mathbb{P}(M_\infty(p) = 0)^{\mathrm{card}(\{n\in\mathbb{N}:\lambda_n(t)>0\})}\right) = 0. \tag{2.16}$$

According to (2.10) we have $\mathrm{card}(\{n \in \mathbb{N} : \lambda_n(t) > 0\}) > 0$ $\mathbb{P}$–a.s., and thus

$$\mathbb{P}\left(M_\infty(p) = 0\right) - \mathbb{P}(M_\infty(p) = 0)^{\mathrm{card}(\{n\in\mathbb{N}:\lambda_n(t)>0\})} \geq 0$$

$\mathbb{P}$–almost surely. Hence, we infer from (2.16) that

$$\mathbb{P}\left(M_\infty(p) = 0\right) = \mathbb{P}(M_\infty(p) = 0)^{\mathrm{card}(\{n\in\mathbb{N}:\lambda_n(t)>0\})}$$

$\mathbb{P}$–almost surely. Since $\mathrm{card}(\{n \in \mathbb{N} : \lambda_n(t) > 0\}) > 1$ with positive probability, this implies that

$$\mathbb{P}(M_\infty(p) = 0) \in \{0, 1\}.$$

Since $M(p)$ is uniformly integrable, cf. Theorem 2 in [Ber03], this results in $\mathbb{P}(M_\infty(p) = 0) = 0$, because

$$\mathbb{E}(M_\infty(p)) = \mathbb{E}(M_0(p)) = 1 > 0.$$

$\square$

2.4 Speed of the largest particle

In this chapter we study the asymptotic speed of the largest fragment. The main tool in this regard turns out to be the additive martingale that we considered in Section 2.3. The main result of this section is the following theorem:

Theorem 2.13 *We have that*

$$\lim_{t\to\infty}\frac{\ln(\lambda_1(t))}{t} = -\Phi'(\bar{p}).$$

Theorem 2.13 was established [Ber06, Corollary 1.4] for the conservative case, but the proof we present here in the dissipative case is based on the same arguments.

Proof We have

$$e^{\Phi(\bar{p})t}\lambda_1^{1+\bar{p}}(t) \le e^{\Phi(\bar{p})t}\sum_{n\in\mathbb{N}}\lambda_n^{1+\bar{p}} = M_t(\bar{p}). \tag{2.17}$$

Recalling that the martingale limit $\lim_{t\to\infty} M_t(\bar{p})$ is well defined $\mathbb{P}$–a.s, we deduce from (2.17) by taking the logarithm and taking the limit superior as $t\to\infty$ that

$$\limsup_{t\to\infty}\frac{1}{t}\ln(\lambda_1(t)) \le -\frac{\Phi(\bar{p})}{1+\bar{p}} = -\Phi'(\bar{p}) \tag{2.18}$$

$\mathbb{P}$–a.s., where the final equality follows from Lemma 2.8.

In order to show the converse inequality let $p\in(\underline{p},\bar{p})$ as well as $\epsilon\in(0,p-\underline{p})$ and observe that

$$\begin{aligned}
M_t(p) &= e^{\Phi(p)t}\sum_{n\in\mathbb{N}}\lambda_n^{1+p}(t)\\
&\le e^{(\Phi(p)-\Phi(p-\epsilon))t}\lambda_1^{\epsilon}(t)e^{\Phi(p-\epsilon)t}\sum_{n\in\mathbb{N}}\lambda_n^{1+p-\epsilon}(t)\\
&= e^{(\Phi(p)-\Phi(p-\epsilon))t}\lambda_1^{\epsilon}(t)M_t(p-\epsilon).
\end{aligned} \tag{2.19}$$

According to Lemma 2.12 both $\lim_{t\to\infty} M_t^x(p)$ and $\lim_{t\to\infty} M_t^x(p-\epsilon)$ are $(0,\infty)$–valued $\mathbb{P}$–almost surely. Consequently, taking the logarithm and taking the limit superior as $t\to\infty$ we thus deduce from (2.19) that

$$\limsup_{t\to\infty}\frac{1}{t}\ln(\lambda_1^x(t)) \ge -\frac{\Phi(p)-\Phi(p-\epsilon)}{\epsilon}$$

$\mathbb{P}$–almost surely. Therefore, we have

$$\limsup_{t\to\infty}\frac{1}{t}\ln(\lambda_1^x(t)) \ge \lim_{\varepsilon\downarrow 0}\frac{\Phi(p)-\Phi(p-\varepsilon)}{-\varepsilon} = -\Phi'(p) \tag{2.20}$$

$\mathbb{P}$–almost surely. Letting $p \to \bar{p}$ and resorting to the convexity of Φ, which ensures the concavity of Φ', (2.20) results in

$$\limsup_{t\to\infty} \frac{1}{t} \ln(\lambda_1^x(t)) \geq -\Phi'(\bar{p})$$

$\mathbb{P}$–almost surely. In view of (2.18), this proves the assertion. □

Remark 2.14 It is known that the asymptotic speed of a typical fragment is $\Phi'(p^*)$. Since Φ' is increasing, and hence $\Phi'(\bar{p}) > \Phi'(p^*)$, it thus follows from Theorem 2.13 that asymptotically the largest fragment is larger than a typical fragment. ◊

2.5 Spine decomposition

The spine approach that we develop in this section is a tool that was successfully used with regard to various stochastic processes that possess a branching or fragmentation structure. For a detailed introduction to the spine method in the setting of branching diffusions we refer the reader to [**?**]. In the context of fragmentation processes we refer to [BR03] and [BR05]. Let us consider the following change of measure.

Definition 2.15 (cf. Section 3.3 in [BR05]) We define for each $p \in (\underline{p}, \infty)$ a probability measure $\mathbb{P}^{(p)}$ on $\mathscr{F}_\infty := \bigcup_{t\in\mathbb{R}_0^+} \mathscr{F}_t$ by

$$\left.\frac{\mathrm{d}\mathbb{P}^{(p)}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = e^{\Phi(p)t - p\xi(t)}$$

for all $t \in \mathbb{R}_0^+$ and we denote by $\mathbb{E}^{(p)}$ the expectation under $\mathbb{P}^{(p)}$.

The change of measure in Definition 2.15 is a so–called *Esscher transform*, cf. Section 3.3 in [Kyp06]. Theorem 3.9 in [Kyp06] shows that under the measure $\mathbb{P}^{(p)}$ the process ξ is a subordinator with Laplace exponent Φ_p given by

$$\Phi_p(a) = \Phi(p+a) - \Phi(p) \tag{2.21}$$

for every $a \in \mathbb{R}_0^+$. Moreover, considering projections onto the sub–filtration $\mathscr{G}$ results in

$$\left.\frac{\mathrm{d}\mathbb{P}^{(p)}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{G}_t} = M_t(p) \tag{2.22}$$

for any $p \in (\underline{p}, \infty)$ and $t \in \mathbb{R}_0^+$. Indeed, (2.22) holds true because we have

$$\mathbb{E}\left(e^{\Phi(p)t - p\xi(t)}\middle|\, \mathscr{G}_t\right) = e^{\Phi(p)t}\mathbb{E}\left(|B_1(t)|^p \middle|\, \mathscr{G}_t\right)$$

$$= e^{\Phi(p)t}\mathbb{E}\left(\sum_{n\in\mathbb{N}}|\Pi_n(t)|^{1+p}\middle|\mathscr{G}_t\right)$$

$$= M_t(p)$$

for all $p \in (\underline{p}, \infty)$ and $t \in \mathbb{R}_0^+$, where the second equality follows from a version of the many–to–one identity in Lemma 2.9 for conditional expectations.

Remark 2.16 We remark that in view of Lemma 2.12 we have that restricted to the σ–algebra $\mathscr{G}_\infty := \bigcup_{t\in\mathbb{R}_0^+}\mathscr{G}_t$ the measures $\mathbb{P}^{(p)}$ and $\mathbb{P}$ are equivalent for any $p \in (\underline{p}, \bar{p})$. Moreover, since $M(p)$ is a uniformly integrable unit–mean martingale, we infer that $\mathbb{P}^{(p)}$ is a probability measure on $\mathscr{G}_\infty$. $\diamondsuit$

A similar change of measure has fruitfully been applied for branching processes in [LPP95] and [Lyo97]. In the light of these papers Bertoin and Rouault (cf. [BR03] and [BR05]) showed that under $\mathbb{P}^{(p)}$ the process Π has the same distribution as the decreasingly ordered asymptotic frequencies of a $\mathcal{P}$–valued fragmentation process with a distinguished nested sequence of fragments. In the literature this sequence, from which all the other fragments descend, is often called the "spine" of the process. Bertoin and Rouault call the blocks in this distinguished sequence "tagged fragment" as one can imagine giving at each time of dislocation a tag to a uniformly chosen (among all fragments that exist at that time) fragment. This motivates the following definition:

Definition 2.17 We call the stochastic process $(\Pi_1(t))_{t\in\mathbb{R}_0^+}$ the *spine* of Π and for any $t \in \mathbb{R}_0^+$ we call $\Pi_1(t) = B_1(t)$, that is the block containing the element 1 at time t, the *tagged fragment*. Further, let $\mathscr{F}^1 := (\mathscr{F}_t^1)_{t\in\mathbb{R}_0^+}$ be the filtration generated by the spine, i.e. $\mathscr{F}_t^1 = \sigma(B_1(t))$ for all $t \in \mathbb{R}_0^+$.

Note that by means of the exchangeability of Π, see Remark 1.7, we could also assume that the spine is $|B_n(t)|$ for any $n \in \mathbb{N}$.

The evolution of Π under $\mathbb{P}^{(p)}$ differs from the evolution of Π under $\mathbb{P}$ exactly at the behaviour of the spine, and all fragments that come off the spine evolve according to the behaviour of Π. More precisely, the evolution of Π under $\mathbb{P}^{(p)}$ can be described by a Poisson point process on $\mathcal{P}\times\mathbb{N}$ with the following characteristic measure:

$$(\mu_\nu^{(p)} \otimes \sharp)|_{\mathcal{P}\times\{1\}} + (\mu_\nu \otimes \sharp)|_{\mathcal{P}\times\mathbb{N}\setminus\{1\}},$$

where the measure $\mu_\nu^{(p)}$ on $\mathcal{P}$ is given by

$$\mu_\nu^{(p)}(\mathrm{d}\pi) = |\pi_1|^p\mu_\nu(\mathrm{d}\pi)$$

for all $\pi \in \mathcal{P}$. Define the operators "+" and "Σ" on $\mathcal{S}$ as the concatenations of their arguments. Further, let $(t_i)_{i\in\mathcal{I}_1}$ be the jump times of Π_1. Then we have the following *spine decomposition*:

$$|\Pi(t)| = (|\Pi_1(t)|, 0, \ldots) + \sum_{i\in\mathcal{I}_1 : t_i \le t} \sum_{j\in\mathbb{N}\setminus\{1\}} \left|\Pi^{i,j}(t - t_i)\right|$$

$\mathbb{P}^{(p)}$–a.s., where the $\Pi^{i,j}$ are independent and satisfy

$$\mathbb{P}^{(p)}\left(|\Pi^{i,j}(u)| \in \cdot \,\middle|\, \mathscr{F}^1_{t_i}\right) = \mathbb{P}\left(x|\Pi(u)| \in \cdot\right)$$

with $x = |\Pi_1(t_i-) \cap \pi_j(t_i)|$. Moreover, the behaviour of the block Π_1 under $\mathbb{P}^{(p)}$ is determined by a Poisson point process with characteristic measure $\mu^{(p)}_\nu$.

Let us now return to the martingale $M(p)$ considered in Section 2.3.

Lemma 2.18 *Let $p \in [\bar{p}, \infty)$. Then we have that $M_\infty(p) = 0$ $\mathbb{P}$–almost surely.*

Proof By concavity of Φ we have

$$\frac{\mathrm{d}}{\mathrm{d}p}\left[(1+p)\Phi'(p) - \Phi(p)\right] = \Phi''(p) < 0,$$

and hence it follows from $(1+\bar{p})\Phi'(\bar{p}) = \Phi(\bar{p})$ that

$$(1+p)\Phi'(p) \le \Phi(p). \tag{2.23}$$

Moreover,

$$M_t(p) \ge e^{t\Phi(p)}|B_1(t)|^{1+p} = e^{\Phi(p)t-(1+p)\xi(t)}. \tag{2.24}$$

Since, as mentioned above, cf. (2.21), under the tilted measure $\mathbb{P}^{(p)}$ the process ξ is a subordinator with Laplace exponent Φ_p given by

$$\Phi_p(a) = \Phi(p+a) - \Phi(p)$$

for every $a \in \mathbb{R}^+_0$, we infer that $\mathbb{E}^{(p)}(\xi(1)) = \Phi_p'(0+) = \Phi'(p)$. Therefore, the Lévy process $X_t(p) := \Phi(p)t - (1+p)\xi(t)$ has mean

$$\mathbb{E}^{(p)}(X_1(p)) = \Phi(p) - (1+p)\mathbb{E}^{(p)}\left(\xi(1)\right) = (\Phi(p) - (1+p)\Phi'(p) \ge 0,$$

where the nonnegativity follows from (2.23). According to Corollary 3.13 in [Kyp06] we thus deduce that $\limsup_{t\to\infty} X_t(p) = \infty$ $\mathbb{P}^{(p)}$–a.s. and in view of (2.24) this results in $\limsup_{t\to\infty} M_t(p) = \infty$ $\mathbb{P}^{(p)}$–almost surely. Consequently, because $M_\infty(p) < \infty$ $\mathbb{P}$–a.s., we conclude from

$$0 = \mathbb{P}^{(p)}(M_\infty(p) < \infty) = \int_{\{M_\infty(p)<\infty\}\cap\{M_\infty(p)>0\}} M_\infty(p)\, \mathrm{d}\mathbb{P}$$

that $M_\infty(p) = 0$ $\mathbb{P}$–almost surely. □

2.6 Stopped fragmentations

In the present section we consider stopping lines as a generalisation of the more common concept of stopping times. More specifically, this section is devoted to introducing fragmentation processes stopped at a specific stopping line. In the context of branching processes the concept of stopping lines was considered by various authors and in the setting of fragmentation processes it was introduced by Bertoin, cf. Definition 3.4 in [Ber06].

Recall the spine–filtration $\mathscr{F}^1$ given by $\mathscr{F}^1_t = \sigma(B_1(t))$ for all $t \in \mathbb{R}_0^+$. With this in mind we define for any $n \in \mathbb{N} \setminus \{1\}$ a filtration $\mathscr{F}^n := (\mathscr{F}^n_t)_{t\in\mathbb{R}_0^+}$ by $\mathscr{F}^n_t := \sigma(B_n(t))$ for each $t \in \mathbb{R}_0^+$.

Definition 2.19 A sequence $(L_n)_{n\in\mathbb{N}}$ of $\mathbb{R}_0^+ \cup \{\infty\}$–valued random variables is called *stopping line* if

(i) L_n is an $\mathscr{F}^n$–stopping time for every $n \in \mathbb{N}$.

(ii) $L_n = L_k$ for all $n \in \mathbb{N}$ and $k \in B_n(L_n)$.

Stopping lines were first considered in the theory of branching processes, see for example [Nev87], [Jag89] and [Cha91].

The strong fragmentation property of Π extends to the situation where the stopping times are replaced by stopping lines and is then called *extended fragmentation property*. More precisely, for any stopping line $L := (L_n)_{n\in\mathbb{N}}$ set

$$\mathcal{F}_L := \sigma(\{\Pi(L \wedge t) : t \in \mathbb{R}_0^+\}) = \sigma\left(\bigcup_{n\in\mathbb{N}} \mathscr{F}^n_{L_n}\right).$$

Note that $\Pi(L) \in \mathcal{P}$ consists of all the blocks $\{B_n(L_n)) : n \in \mathbb{N}\}$. The extended fragmentation property then says that the conditional distribution, given $\mathcal{F}_L$, of the process $(\Pi(L+t))_{t\in\mathbb{R}_0^+}$ equals $\mathbb{P}_\pi(\Pi_t \in \cdot)$, where $\pi = \Pi(L)$.

The extended fragmentation property for fragmentation processes was established by Bertoin for $\mathcal{P}$–valued fragmentations in Lemma 3.14 in [Ber06] and for interval fragmentation processes (with the appropriate changes in Definition 2.19 and with an analogous definition of the extended fragmentation property) in Theorem 1 in [Ber02b].

We are mainly interested in a specific example of a stopping line, namely in the first passage times, defined by

$$\upsilon_{\eta,k} := \inf\left\{s \in \mathbb{R}_0^+ : |B_n(s)| < e^{-t}\right\} \tag{2.25}$$

for any $\eta \in (0, 1]$, when the asymptotic frequency of the block containing $n \in \mathbb{N}$ enters the interval $(0, \eta)$. Observe that $(\upsilon_{\eta,k})_{k\in\mathbb{N}}$ does indeed define a stopping line for any $\eta \in (0,1]$. In particular,

$$B_n(\upsilon_{\eta,k}) \cap B_k(\upsilon_{\eta,l}) \in \{\emptyset, B_k(\upsilon_{\eta,k})\}$$

for all $k, l \in \mathbb{N}$.

This section is devoted to introducing fragmentation processes stopped at the stopping line $(\upsilon_{\eta,k})_{n\in\mathbb{N}}$ that was defined in (2.25).

Our approach is to first describe the evolution of $B_{\eta,k}$, $\eta \in (0,1]$, the block in the stopped process that contains $k \in \mathbb{N}$. To this end, let $k \in \mathbb{N}$ as well as $\eta \in (0,1]$ and set

$$B_{\eta,k}(s) := B_k(t \wedge \upsilon_{\eta,k})$$

for any $t \in \mathbb{R}_0^+$. The evolution $t \mapsto B_{\eta,k}(t)$ of distinct blocks is independent and happens according to the above description. Hence, at a given time $t \in \mathbb{R}_0^+$ only those blocks $B_{\eta,k}(t)$, $k \in \mathbb{N}$, still dislocate that are of size bigger than or equal to η. This procedure describes $(B_{\eta,k}(t))_{\eta\in(0,1],t\in\mathbb{R}_0^+}$ for all $\eta \in (0,1]$, $k \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$. As with the non–stopped fragmentations it will be convenient to consider the $\mathcal{S}$–valued processes of the decreasingly ordered asymptotic frequencies, and consequently we adopt

$$\lambda_{\eta,k}(t) := \Big((|B_{\eta,l}(t)| \mathbb{1}_{\{l=\min(B_{\eta,k}(t)\}})_{l\in\mathbb{N}} \Big)_k^{\downarrow}$$

for every $\eta \in (0,1]$, $k \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$. We shall be interested in this stopped process at the time at which it is stopped. In this regard, we set

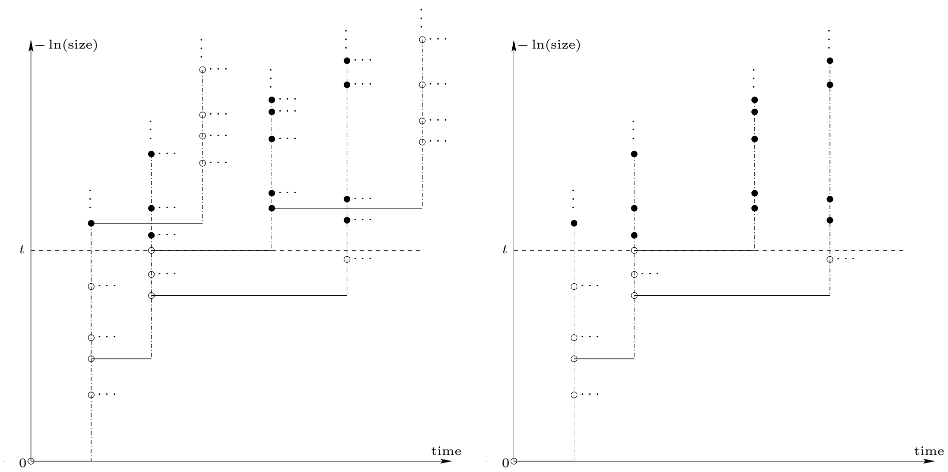
$$\lambda_{\eta,k} := \lim_{t\to\infty} \lambda_{\eta,k}(t)$$

for all $\eta \in (0,1]$ and $k \in \mathbb{N}$. Note that the above limit exists as for sufficiently large $t \in \mathbb{R}_0^+$ the map $t \mapsto |B_{\eta,l}(t)| \mathbb{1}_{\{l=\min(B_{\eta,l}(t)\}}$ is constant. Let us now define the stopped fragmentation process, see Figure 2.2.

Definition 2.20 The $\mathcal{S}$–valued stochastic process $\lambda^S := (\lambda_t^{\mathcal{S}})_{\eta\in(0,1]}$ defined by

$$\lambda_t^{\mathcal{S}} := (\lambda_{\eta,k})_{k\in\mathbb{N}}$$

for all $\eta \in (0,1]$ is called *stopped fragmentation process.*



(a) Illustration of $\upsilon_{\eta,k}$.

(b) Illustration of λ^S.

Figure 2.2: *Illustration (a) depicts the stopping line $\upsilon_{\eta,k}$ given by the first passage of the block sizes below η and (b) illustrates the stopped fragmentation process λ^S, stopped at $\upsilon_{\eta,k}$. The black dots indicate the blocks at the stopping line $\upsilon_{\eta,k}$, since their sizes are smaller than η and they result from the dislocation of blocks with size greater than or equal to η.*

CHAPTER 3

ENERGY COST OF FRAGMENTATIONS

3.1 Introduction

The present chapter is based upon [BM05] and deals with one of the most advanced applications of fragmentations processes that is known in the literature. More specifically, we shall be concerned with a theoretical limit theorem for the cost it takes in the mining industry to crush rocks of minerals. The purpose of this chapter is to show how the techniques and properties derived in the previous two chapters can be used to tackle applied problems in the real world.

Clearly, one should be interested in minimising the energy that is required to crush blocks of minerals as this contributes a significant amount to the worldwide problem of overly excessive energy usage (and sadly also energy wasting). As Bertoin and Martínez point out in [BM05] the crushing processes currently used are not optimal and the best way to minimise the energy cost is by choosing mesh size of the crushers in an optimal way. The goal of this chapter is therefore to compute the energy cost as a function of the mesh size. For more information related to the connection of fragmentation and issues regarding the mining industry we refer the interested reader to Sections 1 and 4 of [BM05]. We would also like to point out that the material presented here is only a strict subset of the results obtained in [BM05].

In order to make the connections with results on branching processes we shall need that the $X \log X$ condition is satisfied, thus throughout this chapter we assume that the followinh hypothesis holds:

Hypothesis 3.1 We have

$$\int_{\mathcal{S}} \sum_{n\in\mathbb{N}} s_n^{1+p^*} \ln^+ \left(\sum_{k\in\mathbb{N}} s_k^{1+p^*} \right) \nu(\mathrm{d}\mathbf{s}) < \infty,$$

where $\ln^+$ denotes the positive part of the ln–function. Observe that in the conservative case Hypothesis 3.1 is always satisfied.

In what follows, we assume that Hypotheses 2.1 and 3.1 hold.

Throughout this chapter we consider a mass fragmentation process $\lambda = (\lambda_t)_{t\in\mathbb{R}_0^+}$ whose underlying Poisson point process we denote by $(k(t), \Delta(t))_{t\in\mathbb{R}_0^+}$.

Let us now state the main result of the present chapter. To this end we define the energy that is needed for crushing blocks of unit size to blocks that have size less than $\eta \in (0, 1)$ by

$$\mathcal{E}(\eta) = \sum_{i\in\mathcal{I}} \mathbb{1}_{\{\lambda_{k(t_i)}(t_i-)\geq\eta\}} \lambda_{k(t_i)}^{1+p}(t_i-)\varphi(\Delta(t_i)),$$

where $(t_i)_{i\in\mathcal{I}}$ are the jump times of λ and $\varphi : \mathcal{S} \to \mathbb{R}_0^+$ is a measurable function satisfying $\varphi((1, 0, \ldots)) = 0$ that we call *cost function*.

The main result of this chapter then reads as follows.

Theorem 3.1 (Theorem 2 in [BM05]) *Let Hypotheses 2.1–3.1 be satisfied. In addition, assume that*

$$\int_{\mathcal{S}} \varphi(\mathbf{s})\nu(d\mathbf{s}) < \infty \tag{3.1}$$

and let $p \in (\underline{p}, p^*)$. *Then*

$$\eta^{p^*-p}\mathcal{E}(\eta) \to \frac{M_\infty(p^*)}{(p^*-p)\Phi'(p^*)} \int_{\mathcal{S}} \varphi(\mathbf{s})\nu(d\mathbf{s})$$

in $\mathscr{L}^1(\mathbb{P})$ *as* $\eta \downarrow 0$.

Remark 3.2 We remark that $\Phi(\eta)$, $\eta > 0$, can be interpreted as the energy cost of the process to fragment certain items (e.g. blocks of metallic ore in the mining industry) until the sizes of all the fragments are less than η (cf. Section 2 in [BM05]).

3.2 The case of finite activity

3.2.1 Main result

As mentioned in Section 1.2 there is a close relationship between general branching processes and fragmentation processes and in this section we shall make use of this connection in order to prove Theorem 3.1 in the setting of fragmentations with a finite dislocation measure. Later on this result will be used to establish the proof in the general case of an arbitrary dislocation measure. In this section we are concerned with a finite dislocation measure ν and w.l.o.g. we thus assume that ν is a probability measure, i.e. $\nu(\mathcal{S}) = 1$. Note that any block $B_n(t)$ is the result of finitely many dislocation events that affected the block containing $n \in \mathbb{N}$ at times $s \leq t \in \mathbb{R}_0^+$. In view of the genealogical structure that the fragmentation chain possesses this motivates using the Ulam–Harris notation for labelling the blocks as follows. Denote by $\emptyset$ the root, i.e. the single initial block, and consider the genealogical tree

$$\mathscr{I} := \{\emptyset\} \cup \bigcup_{n \in \mathbb{N}} \mathbb{N}^n.$$

For every $n \in \mathbb{N}$ we denote by (n) the n^{th} fragment resulting at the dislocation of $\emptyset$. Then any $u = (u_1, \ldots, u_n) \in \mathscr{I}$, $n \geq 2$, can inductively be interpreted as the u_n^{th} largest block resulting from the fragmentation of $(u_1, \ldots, u_{n-1}) \in \mathscr{I}$, and u is in the n^{th} step of fragmentation of $\emptyset$. If two or more blocks in the n^{th} generation have the same size, then the order among these blocks is random. We adopt $\hat{u} = \emptyset$ for any $u \in \mathbb{N}$, and we denote by $u(n)$, $n \in \mathbb{N}$, the n^{th} largest block resulting from the fragmentation of $u \in \mathscr{I}$. To every $u \in \mathscr{I}$ we assign a random variable $s_u \in [0,1]$, which can be interpreted as the mass or size of the block u, and we set $s_n := s_{(n)}$. Note that for any $u \in \mathscr{I}$ with $s_u > 0$, the sequence $({}^{s_{u(n)}}/_{s_u})_{n \in \mathbb{N}}$ has distribution ν and in particular that ν does not depend on the size s_u. If $s_u = 0$, then we adopt $s_{u(n)} = 0$ for each $n \in \mathbb{N}$.

Let $\mathcal{L}$ is be t8he space of possible life careers ω. For any $\omega \in \mathcal{L}$ denote by ω_u an independent copy of ω that is associated with a block $u \in \mathscr{I}$. Let $\tilde{\varphi} : \mathcal{L} \to \mathbb{R}_0^+$ be a measurable function that plays the role of the cost function.

Proposition 3.3 (Theorem 1 in [BM05]) *Assume that* $\nu(\mathcal{S}) = 1$ *and that Hypotheses 2.1–3.1 are satisfied. In addition, let* ω *be such that* $\mathbb{E}(\tilde{\varphi}(\omega)) < \infty$ *and let* $p \in (\underline{p}, p^*)$. *Then*

$$\eta^{p^*-p}\mathcal{E}(\eta) \to \frac{M_\infty(p^*)}{(p^*-p)\Phi'(p^*)} \int_{\mathcal{S}} \varphi(\mathbf{s})\nu(d\mathbf{s})$$

in $\mathscr{L}^1(\mathbb{P})$ *as* $\eta \downarrow 0$.

Recall that (3.1) says that $\int_{\mathcal{S}}(\mathbf{s})\nu(\mathrm{d}\mathbf{s}) < \infty$. Moreover, observe that in the discrete setting

$$\mathcal{E}(\eta) = \sum_{u \in \mathscr{I}} s_u^{1+p} \mathbb{1}_{\{s_u \geq \eta\}} \varphi(\omega_u)$$

for all $\eta > 0$.

3.2.2 Crump–Mode–Jagers processes

Our aim is to express the homogeneous fragmentation chain as a Crump–Mode–Jagers process (see e.g. Chapter 6 in [Jag75] and Section 1 in [Ner81]). For this purpose, we adopt $\sigma_u := -\ln s_u$ for each $u \in \mathscr{I}$. Then the genealogical tree $\mathscr{I}$ gives rise to a C–M–J process, where the reproduction rate is given by the kernel $\xi : \Omega \times \mathscr{B}(\mathbb{R}^+) \to \bar{\mathbb{N}}$, that is defined by

$$\xi(\mathrm{d}t) := \sharp(\{n \in \mathbb{N} : \sigma_n \in \mathrm{d}t\})$$

for all $t \in \mathbb{R}_0^+$. Further, denote by μ_{p^*} the measure with μ–density $e^{-(1+p^*)(\cdot)}$, where μ is defined by

$$\forall\, t \in \mathbb{R}_0^+ : \mu(\mathrm{d}t) = \mathbb{E}(\xi(\mathrm{d}t)) = \int_{\mathcal{S}} \sum_{n \in \mathbb{N}} \mathbb{1}_{\{-\ln(s_n) \in \mathrm{d}t\}} \nu(\mathrm{d}\mathbf{s}),$$

and set

$$m(p^*) := \int_{\mathbb{R}_0^+} t \mu_{p^*}(\mathrm{d}t).$$

Then we have

$$m(p^*) = \int_{\mathbb{R}_0^+} t e^{-(1+p^*)t} \mu(\mathrm{d}t) = \int_{\mathcal{S}} \sum_{n \in \mathbb{N}} s_n^{1+p} \ln(s_n^{-1}) \nu(\mathrm{d}\mathbf{s}) = \Phi'(p^*). \qquad (3.2)$$

Let us consider so–called *random characteristics* with life careers (see Section 7 in [Jag89]), i.e. measurable, separable, nonnegative stochastic processes $\phi : \mathbb{R} \times \mathcal{L} \times \Omega$ with $\phi(x, \cdot, \cdot) = 0$ for all $x < 0$. For any such characteristic ϕ and $t \in \mathbb{R}_0^+$ we define $Z_t^\phi := \sum_{u \in \mathscr{I}} \phi_u(t - \sigma_u, \omega_u)$. The process $(Z_t^\phi)_{t \in \mathbb{R}_0^+}$ is called *Crump–Mode–Jagers process counted with characteristic* ϕ, cf. Section 6.9 in [Jag75].

The result on C–M–J processes that we shall makes use of is the following result, due to Jagers.

Theorem 3.4 (Theorem 7.3 in [Jag89]) *Let ϕ be a characteristic such that $\mathbb{P}$–a.s. the function $\phi(\cdot, \omega)$ is continuous Lebesgue–almost everywhere, and further assume that*

$$\xi(\mathbb{R}_0^+) < \infty \qquad \mathbb{P}\text{–}a.s. \qquad (3.3)$$

as well as

(i)

$$\forall\, \omega \in \mathcal{L} : \quad \int_{\mathbb{R}_0^+} e^{-(1+p^*)t} \mathbb{E}(\phi(t), \omega)\, dt < \infty$$

(ii)

$$\forall \omega \in \mathcal{L}: \quad \lim_{t\to\infty} e^{-(1+p^*)t}\mathbb{E}(\phi(t),\omega) = 0$$

(iii)

$$\mathbb{E}\left(\int_{\mathbb{R}_0^+} e^{-(1+p^*)t}\xi(dt)\ln^+ \int_{\mathbb{R}_0^+} e^{-(1+p^*)t}\xi(dt)\right) < \infty.$$

Then

$$e^{-(1+p^*)t}Z_t^\phi \to \frac{M_\infty}{m(p^*)}\int_0^\infty e^{-(1+p^*)s}\mathbb{E}(\phi(s),\omega)ds$$

in $\mathscr{L}^1$ *as* $t\to\infty$ *for any* $\omega\in\mathcal{L}$.

Since Theorem 3.4 is not a result on fragmentation processes and doesn't have a short proof, we omit presenting its proof here and rather refer to the proof of Theorem 7.3 in [Jag89].

Remark 3.5 The assumption (3.3), which is needed for C–M–J processes, is not necessary in the fragmentation setting, as for any $s\in\mathbb{R}_0^+$ the stopped fragmentation, stopped at the stopping line $(\upsilon_{s,n})_{n\in\mathbb{N}}$, contains only finitely many blocks of size greater than e^{-t}, $t>s$. This fact can replace assumption (3.3) in the proof of [Jag89, Theorem 7.3]. This assumption is used in the first formula on page 208 in the proof of [Jag89, Theorem 7.2]. The proof of [Jag89, Theorem 7.3] is based on this result and doesn't make use of assumption (3.3) anywhere else. ◊

3.2.3 Proof in the fragmentation setting

The goal of this section is to prove Proposition 3.3. The idea of the proof (which was presented in [BM05]) is to apply Theorem 3.4.

Proof of Proposition 3.3 Let $p\in(\underline{p},p^*)$ and consider the random characteristic ϕ given by

$$\phi(t) = \mathbb{1}_{\{t\geq 0\}}e^{(1+p)t}\varphi(\omega).$$

For any $u\in\mathscr{I}$ let ϕ_u be an independent copy of ϕ. Then we have

$$e^{-(1+p^*)\ln(t)}Z^\phi_{\ln(t)} = t^{-(1+p^*)}\sum_{u\in\mathscr{I}}\phi_u(\ln(t)-\sigma_u) = t^{p-p^*}\mathcal{E}(t^{-1}). \tag{3.4}$$

We aim at resorting to Theorem 3.4. To this end, let us check that the assumptions of Theorem 3.4 are satisfied in our situation. The continuity of the exponential function implies that $\mathbb{E}\circ\phi$ is continuous on $\mathbb{R}\setminus\{0\}$, and moreover we infer from (3.1) that

$$\int_{\mathbb{R}_0^+} e^{-(1+p^*)t}\mathbb{E}(\phi(t,\omega)\mathrm{d}t = \mathbb{E}(\varphi(\omega))\int_{\mathbb{R}_0^+} e^{-(1+p^*)t}\mathrm{d}t < \infty.$$

Furthermore, we have that

$$e^{-(1+p^*)t}\mathbb{E}(\phi(t),\omega) = \mathbb{E}(\varphi(\omega))e^{-(p^*-p)t} \to 0$$

as $t \to \infty$. Hence, since Hypothesis 3.1 implies that condition (iii) in Theorem 3.4 is satisfied, it follows from Theorem 3.4, bearing in mind Remark 3.5, and (3.4) that

$$t^{p-p^*}\mathcal{E}(t^{-1}) \to \frac{M_\infty}{m(p^*)}\int_{\mathbb{R}_0^+} e^{-(1+p^*)s}\mathbb{E}(\phi(s,\omega))\mathrm{d}s = \frac{M_\infty}{m(p^*)(p^*-p)}\mathbb{E}(\varphi(\omega))$$

in $\mathscr{L}^1$ as $t \to \infty$. With $\eta = t^{-1}$ this proves the assertion, because according to (3.2) we have that $m(p^*) = \Phi'(p^*)$. □

3.3 The general case

In this section we extend Proposition 3.3 to fragmentation processes with an infinite dislocation measure. The technique we use is a discretisation method that allows us to reduce the considerations to the finite activity case considered in Section 3.2. In this section we shall make use of the properties of fragmentations that we were concerned with in the previous two chapters. The proof of Theorem 3.1 is based on the following lemmas.

Lemma 3.6 (Bertoin, cf. Lemma 1 in [BM05]) *We have*

$$\mathbb{E}\left(\sum_{n\in\mathbb{N}}\lambda_n^{1+p^*}(t)\ln\left(\frac{1}{\lambda_n(t)}\right)\right) = t\Phi'(p^*)$$

for all $t > 0$.

Proof Since, by definition of the Laplace exponent, $\Phi = t^{-1}\ln(\mathbb{E}(e^{-p\xi(t)})$, we infer that

$$t\Phi'(p^*) = \frac{\mathbb{E}(\xi(t)e^{-p^*\xi(t)})}{\mathbb{E}\left(e^{-p^*\xi(t)}\right)} = \mathbb{E}(\xi(t)e^{-p^*\xi(t)}),$$

as $(e^{-p^*\xi(t)})_{t\in\mathbb{R}_0^+}$ being a unit–mean martingale implies that $\mathbb{E}(e^{-p^*\xi(t)}) = 1$. Hence, an application of the many–to–one identity in Lemma 2.9 yields that

$$\mathbb{E}\left(\sum_{n\in\mathbb{N}}\lambda_n^{1+p^*}(t)\ln\left(\frac{1}{\lambda_n(t)}\right)\right) = \mathbb{E}\left(\xi(t)e^{-p^*\xi(t)}\right) = t\Phi'(p^*).$$

□

The key lemma reads as follows:

Lemma 3.7 *Let $p \in (\underline{p}, p^*)$. Then there exists some $c \in \mathbb{R}_0^+$ such that*

$$\mathbb{E}\left(\sum_{k\in\mathbb{N}} \lambda_{\eta,k}^{1+p}\right) \leq c\eta^{p-p^*}.$$

Proof Recall that for any $y \in (0,1)$ and $s \in \mathbb{R}_0^+$ we defined by $\mathfrak{I}_y(s)$ the interval at time s in the chosen interval representation of $\Pi(s)$ that contains y and that $|\mathfrak{I}_y(s)|$ denotes the length of this interval. In addition, set

$$\tau_\eta(y) := \inf\{t \in \mathbb{R}_0^+ : |\mathfrak{I}_y(t)| < \eta\}.$$

Then we have

$$\begin{aligned}
\mathbb{E}\left(\sum_{k\in\mathbb{N}} \lambda_{\eta,k}^{1+p}\right) &= \mathbb{E}\left(|\mathfrak{I}_U(\tau_\eta(U))|^p\right) \\
&= \mathbb{E}\left(\int_{(0,1)} |\mathfrak{I}_y(\tau_\eta(y))|^p \,\mathrm{d}y\right) \\
&= \mathbb{E}\left(e^{-p\xi(\upsilon_{\eta,1})}\right),
\end{aligned}$$

where $U : \Omega \to (0,1)$ is a uniformly distributed random variable that is independent of Π. Further, recall the change of measure of Definition 2.15 with p^*, i.e.

$$\left.\frac{\mathrm{d}\mathbb{P}^{(p^*)}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = e^{-p^*\xi(t)},$$

as well as the following expression

$$\forall\, a \in \mathbb{R}_0^+ : \Phi_{p^*}(a) = \Phi(p^* + a) \tag{3.5}$$

of the Laplace exponent of ξ under $\mathbb{P}^{(p^*)}$ in terms of that under $\mathbb{P}$, cf. (2.21). Consequently,

$$\mathbb{E}\left(\sum_{k\in\mathbb{N}} \lambda_{\eta,k}^{1+p}\right) = \mathbb{E}^{(p^*)}\left(e^{(p^*-p)\xi(\upsilon_{\eta,1})}\right). \tag{3.6}$$

According to Lemma 1.10 of [Ber99] (see also Thm. 5.6 in [Kyp06]) we have

$$\mathbb{P}^{(p^*)}\left(\xi(\upsilon_{\eta,1}) \in \mathrm{d}z\right) = \int_{[0,-\ln(\eta)]} \mathbb{1}_{\{z>-\ln(\eta)\}} \Pi^{(p^*)}(\mathrm{d}z - y)\mathrm{d}U^{(p^*)}(y),$$

where $\Pi^{(p^*)}$ and $U^{(p^*)}$ are the Lévy measure resp. the renewal function of Π under $\mathbb{P}^{(p^*)}$. Recall that the *renewal function* is defined by

$$U^{(p^*)}(x) = \mathbb{E}^{(p^*)}\left(\int_{(0,\infty)} \mathbb{1}_{\{\xi(t)\leq x\}}\mathrm{d}t\right).$$

Observe that

$$\eta^{p^*-p} e^{(p^*-p)z} \leq e^{(p^*-p)x}$$

for $y < -\ln(\eta)$ and $x = z - y$. Therefore, we infer that

$$\begin{aligned}
&\eta^{p^*-p}\mathbb{E}^{(p^*)}\left(e^{(p^*-p)\xi(\upsilon_{\eta,1})}\mathbb{1}_{\{\xi(\upsilon_{\eta,1})\geq 1-\ln(\eta)\}}\right)\\
&= \eta^{p^*-p}\int_{(-\ln(\eta),\infty)} e^{(p^*-p)z}\mathbb{1}_{\{z\geq 1-\ln(\eta)\}}\mathbb{P}^{p^*)}\left(\xi(\upsilon_{\eta,1}\in \mathrm{d}z)\right)\\
&= \eta^{p^*-p}\int_{z\in(-\ln(\eta),\infty)}\int_{y\in(0,-\ln(\eta]} e^{(p^*-p)z}\mathbb{1}_{\{z\geq 1-\ln(\eta)\}}\Pi^{(p^*)}(\mathrm{d}(z-y))\ \mathrm{d}U^{(p^*)}(y)\\
&\leq \int_{[1,\infty)} e^{(p^*-p)x}(U^{(p^*)}(-\ln(\eta)) - U^{(p^*)}(-\ln(\eta)-x)^+)\Pi^{(p^*)}(\mathrm{d}x)\\
&\leq \int_{[1,\infty)} e^{(p^*-p)x}U^{(p^*)}(x)\Pi^{(p^*)}(\mathrm{d}x),
\end{aligned}$$

where we used the substitution $x := z-y$ and where the last estimate follows from the subadditivity of the renewal function (see p. 10 in [Ber99]). In view of (3.6) and since

$$\eta^{p^*-p}\mathbb{E}^{(p^*)}\left(e^{(p^*-p)\xi(\upsilon_{\eta,1})}\mathbb{1}_{\{\xi(\upsilon_{\eta,1})<1-\ln(\eta)\}}\right) < e^{p^*-p}\eta^{p^*-p}\eta^{-(p^*-p)} = e^{p^*-p},$$

the assertion is proven once we have shown that

$$\int_{[1,\infty)} e^{(p^*-p)x}U^{(p^*)}(x)\Pi^{(p^*)}(\mathrm{d}x) < \infty. \tag{3.7}$$

In order to prove (3.7) recall that $\mathbb{E}^{(p)}(\xi(1)) = \Phi'_{p^*}(0+) = \Phi'(p^*) < \infty$. Hence, the elementary renewal theorem yields that

$$\lim_{x\to\infty}\frac{U^{(p^*)}(x)}{x} = \frac{1}{\Phi'(p^*)}. \tag{3.8}$$

Notice that (3.5) implies that the subordinator ξ is not killed $\mathbb{P}^{p^*}$–almost surely. Moreover, in the light of (3.5) the Lévy-Khintchine formula implies that

$$\int_{(1,\infty)}(1-e^{-qx})\Pi^{(p^*)}(\mathrm{d}x) > \int_{(0,\infty)}(1-e^{-qx})\Pi^{(p^*)}(\mathrm{d}x) = \Phi_{p^*}(q) > -\infty$$

for all $q \in (\underline{p}-p^*, 0)$. As $\Pi([1,\infty)) < \infty$, this yields $\int_{[1,\infty)} e^{-qx}\Pi^{(p^*)}(\mathrm{d}x) < \infty$ and hence

$$\int_{[1,\infty)} xe^{-\tilde{q}x}\Pi^{(p^*)}(\mathrm{d}x) < \infty.$$

for all $\tilde{q} \in (q, 0)$. Since $p \in (\underline{p}, p^*)$, we can choose $\tilde{q} > \underline{p} - p^*$ such that $-\tilde{q} > p^* - p$, which by means of (3.8) results in (3.7) and thus completes the proof. □

Our aim is to use a discretisation method in order to extend Proposition 3.3 to the situation of an infinite dislocation measure. To this end, consider the time–discrete skeleton $(\Pi_{n\delta})_{n\in\mathbb{N}}$, where $\delta > 0$. The discretisation method forces us to consider the life careers of the fragments, and fot this purpose we introduce the life career of $\Pi_i(n\delta)$, $i \in \mathbb{N}$, $n \in \mathbb{N}_0$, by

$$\omega_i^{n\delta} := (k_i^{n\delta}(t), \Delta_i^{n\delta}(t))_{t\in(0,\delta]},$$

where $k_i^{n\delta}(t) := k(n\delta + t)$ and

$$\Delta_i^{n\delta}(t) := \begin{cases} \Delta(n\delta + t), & \Pi_{k(n\delta+t)}(n\delta + t-) \subseteq \Pi_i(n\delta) \\ (1, 0, \ldots), & \text{otherwise.} \end{cases}$$

That is, $\omega_i^{n\delta}$ denotes the Poisson point process that determines the evolution of the block $\Pi_i(n\delta)$ during the time interval $(n\delta, (n+1)\delta)$. Recall the cost function $\varphi : \mathcal{S} \to \mathbb{R}$. To deal with the time–discrete skeleton, we shall use the function φ_δ defined by

$$\varphi_\delta(\omega_i^{n\delta}) = \sum_{t\in(0,\delta]} \left(\lambda_{k_i^{n\delta}(t)}(n\delta + t-)\right)^{1+p} \varphi(\Delta_i^{n\delta}(t)).$$

Note that in the definition of φ_δ there does not appear the indicator function that is present in φ and that this definition implies in particular that

$$\varphi_\delta(\omega_1^0) = \sum_{t\in(0,\delta]} \left(\lambda_{k(t)}(t-)\right)^{1+p} \varphi(\Delta(t)).$$

Lemma 3.8 (Bertoin, cf. Lemma 2 in [BM05]) *Let $p > p_c$. Then we have*

$$\forall\, \delta > 0 : \mathbb{E}(\varphi_\delta(\omega_1^0)) = \frac{1 - e^{-\Phi(p)\delta}}{\Phi(p)} \int_{\mathcal{S}_d^\downarrow} \varphi(\mathbf{s})\nu(d\mathbf{s}).$$

Proof By the compensation formula for Poisson point processes we have

$$\mathbb{E}\left(\sum_{t\in(0,\delta]} \lambda_{l(t)}^{1+p}(t-)\varphi(\Delta(t))\right) = \mathbb{E}\left(\int_0^\delta \sum_{n\in\mathbb{N}} \lambda_n^{1+p}(t)\mathrm{d}t\right) \int_{\mathcal{S}_\mathrm{d}^\downarrow} \varphi(\mathbf{s})\nu(\mathrm{d}\mathbf{s}).$$

Since $\mathbb{E}\left(e^{\Phi(p)t} \sum_{n\in\mathbb{N}} \lambda_n^{1+p}(t)\right) = 1$, we infer from Tonelli's theorem that

$$\mathbb{E}\left(\sum_{n\in\mathbb{N}} \lambda_n^{1+p}(t)\right) = e^{-\Phi(p)t},$$

which proves the assertion. □

Proof of Theorem 3.1 Let

$$\mathcal{E}_\delta(\eta) := \sum_{u\in\mathscr{I}} s_u^{1+p}\mathbb{1}_{\{s_u\geq\eta\}}\varphi_\delta(\omega_u)$$

denote the fragmentation energy for $(\Pi_{n\delta})_{n\in\mathbb{N}}$ and the cost function φ_δ. Observe that here the blocks may be crushed unnecessarily small as the stopping can only happen at the discrete time set $(n\delta)_{n\in\mathbb{N}}$. Therefore, $\mathcal{E}(\eta) \leq \mathcal{E}_\delta(\eta)$ for every $\delta > 0$.

Let us now estimate the overcost $\mathcal{E}_\delta(\eta) - \mathcal{E}(\eta)$ which is due to the discretisation. To this end, notice that

$$\mathcal{E}_\delta(\eta) - \mathcal{E}(\eta) \leq \sum_{k\in\mathbb{N}} \mathcal{E}(\lambda_{\eta,k},\delta), \tag{3.9}$$

where $\mathcal{E}(x,\delta)$, $x \in (0,1]$, denotes the energy cost for an initial block of size x and an interval of duration δ in the fragmentation process. By means of Lemma 3.7 and Lemma 3.8 there exists some $c \in \mathbb{R}_0^+$ such thst that

$$\begin{aligned}\mathbb{E}\left(\sum_{k\in\mathbb{N}} \mathcal{E}(\lambda_{\eta,k},\delta)\right) &= \mathbb{E}\left(\sum_{k\in\mathbb{N}} \lambda_{\eta,k}^{1+p}\right)\mathbb{E}\left(\varphi\left(\omega_1^0\right)\right)\\ &\leq \eta^{p-p^*}\frac{c\left(1-e^{-\delta\Phi(p)}\right)}{\Phi(p)}\int_{\mathcal{S}}\varphi(\mathbf{s})\nu(\mathrm{d}\mathbf{s})\end{aligned}$$

for every $\eta \in (0,1)$, where for the fist equality we also used the homogeneity of the cost function. In view of (3.9) this estimate results in

$$\eta^{p^*-p}\mathbb{E}\left(\mathcal{E}_\delta(\eta) - \mathcal{E}(\eta)\right) \leq \frac{c\left(1-e^{-\delta\Phi(p)}\right)}{\Phi(p)}\int_{\mathcal{S}}\varphi(\mathbf{s})\nu(\mathrm{d}\mathbf{s}) \to 0 \tag{3.10}$$

uniformly in η as $\delta \downarrow 0$. Moreover, the hypotheses of Proposition 3.3 are satisfied, and hence that result is applicable. Consequently, by an application of the triangle inequality and recalling (3.2) we deduce from Proposition 3.3, Lemma 3.6 as well as Lemma 3.8 and (3.10) that

$$\begin{aligned}&\mathbb{E}\left(\left|\eta^{p^*-p}\mathcal{E}(\eta) - \frac{M_\infty(p^*)}{(p^*-p)\Phi'(p^*)}\int_{\mathcal{S}}\varphi(\mathbf{s})\nu(\mathrm{d}\mathbf{s})\right|\right)\\ &\leq \eta^{p^*-p}\mathbb{E}\left(\mathcal{E}(\eta) - \mathcal{E}_\delta(\eta)\right)\\ &+\mathbb{E}\left(\left|\eta^{p^*-p}\mathcal{E}_\delta(\eta) - \frac{M_\infty(p^*)}{(p^*-p)\Phi'(p^*)}\frac{1-e^{-\delta\Phi(p)}}{\delta\Phi(p)}\int_{\mathcal{S}}\varphi(\mathbf{s})\nu(\mathrm{d}\mathbf{s})\right|\right)\\ &+\mathbb{E}\left(\left|\frac{M_\infty(p^*)}{(p^*-p)\Phi'(p^*)}\frac{1-e^{-\delta\Phi(p)}}{\delta\Phi(p)} - \frac{M_\infty(p^*)}{(p^*-p)\Phi'(p^*)}\right|\int_{\mathcal{S}}\varphi(\mathbf{s})\nu(\mathrm{d}\mathbf{s})\right),\end{aligned}$$

since

$$\mathbb{E}\left(\left|\eta^{p^*-p}\mathcal{E}_\delta(\eta) - \frac{M_\infty(p^*)}{(p^*-p)\Phi'(p^*)}\frac{1-e^{-\delta\Phi(p)}}{\delta\Phi(p)}\int_{\mathcal{S}}\varphi(\mathbf{s})\nu(\mathrm{d}\mathbf{s})\right|\right) \to 0$$

as $\eta \downarrow 0$ and

$$\eta^{p^*-p}\mathbb{E}\left(\mathcal{E}_\delta(\eta) - \mathcal{E}(\eta)\right) \to 0 \qquad \text{as well as} \qquad \frac{1-e^{-\delta\Phi(p)}}{\delta\Phi(p)} \to 1$$

uniformly in η as $\delta \downarrow 0$, we thus conclude that

$$\eta^{p^*-p}\mathcal{E}(\eta) \to \frac{M_\infty(p^*)}{(p^*-p)\Phi'(p^*)}\int_{\mathcal{S}}\varphi(\mathbf{s})\nu(\mathrm{d}\mathbf{s})$$

in $\mathscr{L}^1(\mathbb{P})$ as $\eta \downarrow 0$. □

Remark 3.9 According to Theorem 3.1 the expectation of the energy cost $\mathcal{E}(0+)$ to reduce some block to dust is ∞ for $p \in (\underline{p}, p^*]$. However, for $p \in (p^*, \infty)$ this expected cost is finite. To see that $\mathcal{E}(0+) \in \mathscr{L}^1(\mathbb{P})$ if $p \in [p^*, \infty)$, let us again consider the finite activity skeleton $(\Pi_{n\delta})_{n\in\mathbb{N}}$ and the corresponding cost function $\mathcal{E}_\delta(\eta)$. Then the fragmentation property yields that

$$\begin{aligned}\mathbb{E}(\mathcal{E}_\delta(0+)) = \sum_{u\in\mathscr{I}} s_u^{1+p}\mathbb{1}_{\{s_u\geq 0+\}}\varphi_\delta(\omega_u) &= \sum_{n\in\mathbb{N}_0}\sum_{u\in\mathbb{N}^n}\mathbb{E}(s_u^{1+p}\varphi_\delta(\omega_u)) \\ &= \mathbb{E}(\varphi_\delta(\omega_u))\sum_{n\in\mathbb{N}_0}\sum_{u\in\mathbb{N}^n}\mathbb{E}(s_u^{1+p}).\end{aligned}$$

Further, observe that the the fragmentation property entails that

$$\sum_{u\in\mathbb{N}^n}\mathbb{E}(s_u^{1+p}) = \left(\int_{\mathcal{S}}\sum_{k\in\mathbb{N}} s_k^{1+p}\nu(\mathrm{d}\mathbf{s})\right)^n.$$

holds for any $n \in \mathbb{N}$. In view the definition of p^* it follows from $p < p^*$ that $\int_{\mathcal{S}}\sum_{k\in\mathbb{N}} s_k^{1+p}\nu(\mathrm{d}\mathbf{s}) < 1$, and hence the geometric series

$$\sum_{n\in\mathbb{N}_0}\left(\int_{\mathcal{S}}\sum_{k\in\mathbb{N}} s_k^{1+p}\nu(\mathrm{d}\mathbf{s})\right)^n$$

converges, i.e. $\mathbb{E}(\mathcal{E}_\delta(0+)) < \infty$. Recalling that $\mathcal{E}(\eta) \leq \mathcal{E}_\delta(\eta)$ for every $\eta > 0$ therefore proves that $\mathbb{E}(\mathcal{E}(0+)) < \infty$. ◊

BIBLIOGRAPHY

[Ald85] D. J. Aldous. *Exchangeability and related topics.* Ecole d'été de Probabilités de Saint-Flour XIII, Lecture Notes in Mathematics, Springer, **1117**, 1985

[Bas06] A.-L. Basdevant. Fragmentation of ordered partitions and intervals, *Elect. J. Probab.*, **11**, pp. 394–417, 2006

[Ber02a] J. Berestycki. Ranked fragmentations, *ESAIM Probab. Statist.*, **6**, pp. 157–175, 2002

[BHK10] J. Berestycki, S. C. Harris, A. E. Kyprianou. Travelling waves and homogeneous fragmentation, to appear in *Ann. Appl. Probab.*, 2010

[Ber96] J. Bertoin. *Lévy Processes*, Cambridge University Press, 1996

[Ber99] J. Bertoin. *Subordinators: examples and applications.* Ecole d'été de Probabilités de Saint-Flour XXVII, Lecture Notes in Mathematics, Springer, **1717**, pp. 1–91, 1999

[Ber01] J. Bertoin. Homogeneous fragmentation processes, *Probab. Theory Related Fields* **121**, pp. 301–318, 2001

[Ber02b] J. Bertoin. Self–similar fragmentations, *Ann. Inst. H. Poincaré Probab. Statist.*, **38**, pp. 319–340, 2002

[Ber03] J. Bertoin. The asymptotic behavior of fragmentation processes, *J. Europ. Math. Soc.*, **5**, pp. 395–416, 2003

[Ber06] J. Bertoin. *Random fragmentation and coagulation Processes*, Cambridge University Press, 2006

[BM05] J. Bertoin, S. Martínez. Fragmentation energy, *Adv. Appl. Prob.*, **37**, pp. 553–570, 2005

[BR03] J. Bertoin, A. Rouault. Additive martingales and probability tilting for homogeneous fragmentations, preprint, 2003

[BR05] J. Bertoin, A. Rouault. Discritization methods for homogeneous fragmentations, *J. London Math. Soc.*, **72**, pp. 91–109, 2005

[Big92] J. D. Biggins. Uniform convergence of martingales in the branching random walk, *Ann. Probab.*, **20** (1), pp. 137–151, 1992

[BD75] N. Bingham, R. A. Doney. Asymptotic properties of supercritical branching processes. II: Crump–Mode and Jirana processes, *Adv. Appl. Probab.*, **7**, pp. 66–82, 1975

[Cha91] B. Chauvin. Product martingales and stopping lines for branching Brownian motion, *Ann. Probab.*, **19**, pp. 1195–1205, 1991

[Chu82] K. L. Chung. *Lectures from Markov processes to Brownian motion*, Springer, 1982

[FKM10] J. Fontbona, N. Krell, S. Martínez. Energy efficiency of consecutive fragmentation processes, *J. Appl. Probab.*, **47** (2), pp. 543-561, 2010

[Haa03] B. Haas. Loss of mass in deterministic and random fragmentations, *Stoch. Process. Appl.*, **106**, pp. 245-277, 2003

[Har00] S. C. Harris. Convergence of a Gibbs-Boltzmann random measure for a typed branching diffusion, *Séminaire de Probabilités* **XXXIV**, pp. 239–256, 2000

[HW96] S. C. Harris, D. Williams. Large-deviations and martingales for a typed branching diffusion : I, *Astérisque* **236**, pp. 133–154, 1996

[HKK10] S. C. Harris, R. Knobloch, A. E. Kyprianou. Strong law of large numbers for fragmentation processes, *Ann. Inst. H. Poincaré Probab. Statist.*, **46** (1), pp. 119–134, 2010

[HK08] M. Hoffmann, N. Krell. Statistical analysis of self–similar conservative fragmentation chains, to appear in *Bernoulli*, 2008

[Jag75] P. Jagers. *Branching processes with biological applications*, Wiley, 1975

[Jag89] P. Jagers. General branching processes as Markov fields, *Stoch. Process. Appl.*, **32**, pp. 183–212, 1989

[Kal01] O. Kallenberg. *Foundations of modern probability*, second edition, Springer, 2001

[Kin82] J. F. C. Kingman. The coalescent, *Stoch. Process. Appl.*, **13**, pp. 235–248, 1982

[Kyp04] A. E. Kyprianou. Travelling wave solutions to the K–P–P equation: alternatives to Simon Harris' probabilistic analysis, *Ann. Inst. H. Poincaré Probab. Statist.*, **40**, pp. 53–72, 2004

[Kyp06] A. E. Kyprianou. *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer, 2006

[KP10] A. E. Kyprianou, J. C. Pardo. An optimal stopping problem, submitted, 2010

[Lyo97] R. Lyons. A simple path to Biggin's martingale convergence for branching random walk, In: *Classical and modern branching processes*, K. Athreya and P. Jagers (editors), IMA Vol. Math. Appl., Springer, pp. 217–221, 1997

[LPP95] R. Lyons, R. Pemantle, Y. Peres. Conceptual proofs of L log L criteria for mean behaviour of branching processes, *Ann. Probab.*, **23**, pp. 1125–1138, 1995

[Ner81] O. Nerman. On the convergence of supercritical general (C–M–J) branching processes. *Z. Wahrsch. Verw. Gebiete*, **57** (3), pp. 365–395, 1981

[Nev87] J. Neveu. Multiplicative martingales for spatial branching processes. *Seminar on Stochastic Processes* Progress in Probability and Statistics, Birkhäuser, **15**, pp. 223–242, 1987

[Pit99] J. Pitman. Coalescent with multiple collisions, *Ann. Probab.*, **27**, pp. 1870–1902, 1999

[RY99] D. Revuz, M. Yor. *Continuous martingales and Brownian motion*, third edition, Springer, 1999

[Sch00] J. Schweinsberg. Coalescent with simultaneous multiple collisions, *Electr. J. Probab.*, **5** (12), pp. 1–50, 2000