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Optimal and self-tuning optimal control of a periodic-review hybrid production inventory system☆

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Abstract

In this paper, a periodic-review dynamic production inventory system for a single reusable product is investigated. There are two stocks, one for the serviceable items and one for the remanufactured ones. We assume that the items in either stock may be subject to deterioration. Items deterioration is of great importance to inventory theory. An optimal control is derived in the case where the deterioration parameters are known and a self-tuning optimal control strategy is applied in the case where the deterioration parameters are unknown. In particular, the recursive least-squares (RLS) method is used to identify the deterioration parameters. Simulations are conducted to illustrate the results obtained.

Keywords: Hybrid; Periodic-review; Deterioration; Optimal control; Self-tuning; Lagrange technique; Recursive least-squares

1. Introduction

On the one hand, the environment has become one of the most important issues of our time, and no doubt it will continue to be so well into the future. Most studies of recycling and remanufacturing focus on environmental benefits because the major drivers – such as preventing the pollution associated with producing and refining virgin materials, reducing the amount of trees that are cut down, and decreasing the amount of material that is landfilled – of recycling and remanufacturing programs have traditionally been environmental in nature.

On the other hand, Operations Research (OR) has become one of the most important disciplines applying advanced analytical methods to help make better decisions, and no doubt it will also continue to be so well into the future. One of the environmental problems tackled by OR researchers concerns the important issue of remanufacturing, which is the process of restoring or upgrading used products to a like-new condition. Indeed, even though the focus of past studies has been almost solely on environmental benefits, recycling and remanufacturing are also important from an economic perspective for many reasons. For example,

1. they directly employ individuals and supply valuable materials and products to downstream industries, thus increasing the number of jobs;

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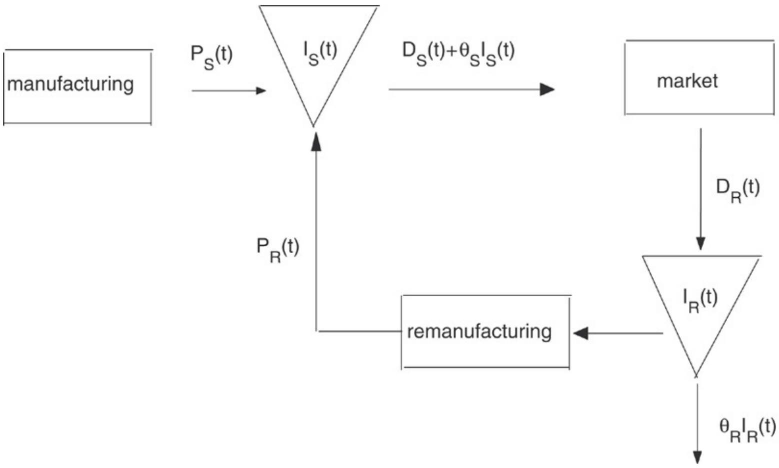

Fig. 1. Product recovery system.

2. they are high-growth industries since the amount of materials recycled or remanufactured has grown dramatically over the years;
3. they allow savings in material costs and lower energy and waste disposal costs;
4. they improve the competitive position of some domestic industries in the international marketplace by lowering costs.

Examples of industries that greatly benefited from recycling and remanufacturing are the steel industry, the aluminium industry, and the paper industry.

The model we are considering was introduced by Hedjar et al. [7]. A graphical illustration of this simple manufacturing/remanufacturing system is given in Fig. 1.

A number of authors have studied such hybrid production inventory systems [1,3–5,7]. The recent paper of Hedjar et al. [7] contains a comprehensive survey of the literature on the topic and, to avoid unnecessary repetition, we refrain from reproducing it here.

In this paper we examine a product recovery system and assume that there is no difference between newly produced and recycled items, i.e., we apply the as-good-as-new principle, so that the customer makes no distinction between manufactured and remanufactured products. Demand is satisfied either from production or remanufacturing of returned products. We therefore deal with a deterministic model with dynamic demand and return. We also assume that the item in either stock (serviceable and returned) may be subject to deterioration. Items deterioration is of great importance in inventory theory, as shown by the survey of Goyal and Giri [6]. Examples of deteriorating items include blood, photographic films, certain pharmaceuticals, and food stuffs.

Hedjar et al. [7] applied a receding horizon control strategy to the model. Receding horizon control, also known as model predictive control, is a discrete-time technique in which the control action is obtained by repeatedly solving online open loop optimization problems at each time step. In this paper, we are interested in the self-tuning optimal control of this same model. That is, we study how to use control and identification methods for controlling the system when its parameters are not all *a priori* known to the designer. This is a situation virtually always met in practice.

In the next section we introduce the notation and formally describe the system. We are assuming that the firm of interest to us adopts a periodic-review (instead of a continuous-review) policy. Therefore, in this section, the problem, which is initially formulated in continuous time, is also adapted to fit the periodic-review policy. In Section 3 we study the optimal control of the system when all system parameters are known while in Section 4, we study the optimal control of the system when not all system parameters are known. Illustrative examples are provided in both Sections 3 and 4.

2. Model formulation and notation

We consider a production inventory system with two stocks. Items in the first stock are called serviceable. Items in the second stock are called remanufactured. We will use throughout the paper the subscript "$_S$" to indicate the quantity corresponding to the first (serviceable) stock and the subscript "$_R$" to indicate the quantity corresponding to the second (remanufactured) stock. So, for example, $I_S(t)$ represents the inventory level at time t in the first stock while $I_R(t)$ represents the inventory level at time t in the second stock.

Let H denote the length of a finite planning horizon and consider the first stock. The production of new items at rate $P_S(t)$ and the remanufacturing of returned items to like-new condition at rate $P_R(t)$ increases the inventory level. Also, demand at rate $D_S(t)$ and deterioration at rate $\theta_S > 0$ decreases the inventory level. Similarly, the inventory level in the second stock increases by the returned items at rate $D_R(t)$ and decreases due to deterioration at rate $\theta_R > 0$ and to the remanufactured items that are moved to the first stock at rate $P_R(t)$. The change in the level of inventory in both stocks is therefore given by the following state equations

$$\begin{cases} \dot{I}_S(t) = -\theta_S I_S(t) + P_S(t) + P_R(t) - D_S(t), & \forall t \in [0, H], \\ \dot{I}_R(t) = -\theta_R I_R(t) - P_r(t) + D_R(t), & \forall t \in [0, H]. \end{cases} \tag{2.1}$$

We assume that the initial stocks are known

$$I_S(0) = I_{S0}, \qquad I_R(0) = I_{R0}. \tag{2.2}$$

In the terminology of optimal control theory, $I_S(t)$ and $I_R(t)$ represent the *state variables* while the *control variables* are $P_S(t)$ and $P_R(t)$ which need to be nonnegative:

$$P_S(t) \geq 0, \qquad P_R(t) \geq 0. \tag{2.3}$$

In other words, we seek to find the optimal production and remanufacturing rates, that is the rates that minimize some performance index. Now, in order to build this performance index, we assume that the firm has set the following goals

$\hat{I}_S(t)$: inventory goal level for the first stock,
$\hat{I}_R(t)$: inventory goal level for the second stock,
$\hat{P}_S(t)$: production goal rate,
$\hat{P}_R(t)$: remanufacturing goal rate,

and penalties are incurred for each variable to deviate from its corresponding goal. Denoting these penalties by $q_i \geq 0$ and $r_i > 0$, $(i = S, R)$, the problem is to minimize

$$J = \frac{1}{2}\int_0^H \{q_S[I_S(t) - \hat{I}_S(t)]^2 + q_R[I_R(t) - \hat{I}_R(t)]^2 + r_S[P_S(t) - \hat{P}_S(t)]^2 + r_R[P_R(t) - \hat{P}_R(t)]^2\}\mathrm{d}t \tag{2.4}$$

subject to constraints (2.1)–(2.3). Since obviously it is more convenient to handle this problem using matrix notation, we let

$$I(t) = \begin{pmatrix} I_S(t) \\ I_R(t) \end{pmatrix}, \qquad \hat{I}(t) = \begin{pmatrix} \hat{I}_S(t) \\ \hat{I}_R(t) \end{pmatrix}, \qquad I_0 = \begin{pmatrix} I_{S0} \\ I_{R0} \end{pmatrix}, \qquad \Delta I(t) = I(t) - \hat{I}(t),$$

$$P(t) = \begin{pmatrix} P_S(t) \\ P_R(t) \end{pmatrix}, \qquad \hat{P}(t) = \begin{pmatrix} \hat{P}_S(t) \\ \hat{P}_R(t) \end{pmatrix}, \qquad \Delta P(t) = P(t) - \hat{P}(t),$$

$$Q = \begin{pmatrix} q_S & 0 \\ 0 & q_R \end{pmatrix}, \qquad R = \begin{pmatrix} r_S & 0 \\ 0 & r_R \end{pmatrix},$$

and

$$D(t) = \begin{pmatrix} D_S(t) \\ D_R(t) \end{pmatrix}.$$

Also for any vector X (with transpose denoted by $X^\top$) and matrix A, let

$$\|X\|_A^2 = X^\top A X.$$

Then the problem is to minimize

$$J = \frac{1}{2}\int_0^H \{\|\Delta I(t)\|_Q^2 + \|\Delta P(t)\|_R^2\}\mathrm{d}t \tag{2.5}$$

subject to:

$$\dot{I}(t) = -AI(t) + BP(t) + CD(t), \quad \forall t \in [0, H] \tag{2.6}$$

and

$$I(0) = I_0, \qquad P(t) \geq 0, \tag{2.7}$$

where

$$A = \begin{pmatrix} \theta_S & 0 \\ 0 & \theta_R \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now suppose the firm adopts a periodic-review policy. To adapt our problem to this situation, we first note that the solution to the state equation (2.5) is given by

$$I(t) = \mathrm{e}^{-At} I(0) + \int_0^t \mathrm{e}^{-A(t-u)}[BP(u) + CD(u)]\mathrm{d}u. \tag{2.8}$$

Dividing the planning horizon $[0, H]$ into $(N + 1)$ intervals $[t_k, t_{k+1}]$ with $t_k = kT$, $k = 0, 1, \ldots, N$, and $t_{N+1} = H$, and rewriting the above expression at the end points t_k yields

$$I(t_{k+1}) = \mathrm{e}^{-AT} I(t_k) + \int_{t_k}^{t_{k+1}} \mathrm{e}^{-A(t_{k+1}-t)}[BP(t) + CD(t)]\mathrm{d}t. \tag{2.9}$$

Choose T small enough so that $BP(t) + CD(t)$ is constant over the intervals $[t_k, t_{k+1}]$. Then, expression (2.9) can be written

$$I((k + 1)T) = \phi(T)I(kT) + \Gamma(T)P(kT) + \psi(T)D(kT), \quad k = 0, \ldots, N, \tag{2.10}$$

where

$$\phi(T) = \mathrm{e}^{-AT}, \qquad \Gamma(T) = \begin{pmatrix} \dfrac{1-\mathrm{e}^{-\theta_S T}}{\theta_S} & \dfrac{1-\mathrm{e}^{-\theta_S T}}{\theta_S} \\ 0 & -\dfrac{1-\mathrm{e}^{-\theta_R T}}{\theta_R} \end{pmatrix}, \quad \text{and} \quad \psi(T) = I_2 - \phi(T)A^{-1}.$$

Here I_2 is the 2×2 identity matrix. It is easy to see that

$$\phi(T) = \begin{pmatrix} \mathrm{e}^{-\theta_S T} & 0 \\ 0 & \mathrm{e}^{-\theta_R T} \end{pmatrix} \quad \text{and} \quad \psi(T) = \begin{pmatrix} -\dfrac{1-\mathrm{e}^{-\theta_S T}}{\theta_S} & 0 \\ 0 & \dfrac{1-\mathrm{e}^{-\theta_R T}}{\theta_R} \end{pmatrix}.$$

Without loss of generality, take $T = 1$, and for more simplicity in the notation, let $\phi = \phi(1)$, $\Gamma(1) = \Gamma$, and $\psi = \psi(1)$. Then, expression (2.10) becomes

$$I(k + 1) = \phi I(k) + \Gamma P(k) + \psi D(k). \tag{2.11}$$

Now introduce the shift operator Δ:

$$\Delta I(k) = I(k) - \hat{I}(k), \qquad \Delta P(k) = P(k) - \hat{P}(k).$$

Expression (2.11) becomes

$$\Delta I(k + 1) = \phi\Delta I(k) + \Gamma\Delta P(k), \tag{2.12}$$

and the optimal control problem is to minimize

$$J = \frac{1}{2}\sum_{0}^{N}\{\|\Delta I(k)\|_Q^2 + \|\Delta P(k)\|_R^2\} \tag{2.13}$$

subject to:

$$\Delta I(k+1) = \phi\Delta I(k) + \Gamma\Delta P(k), \tag{2.14}$$

and

$$I(0) = I_0, \qquad P(k) \geq 0, \quad k = 0, \ldots, N. \tag{2.15}$$

In the next section, we solve the optimal control problem, which consists of determining the optimal production and remanufacturing rates, in the case where the deterioration rates θ_S and θ_R are known. In the following section, we solve the same problem under the additional assumption that the deterioration rates are unknown.

3. Optimal control

We assume in this section that the rate parameters θ_S and θ_R are known so that the matrices ϕ and Γ are also known. In this case, the problem is nonlinear and can be solved using the Lagrangian technique.

3.1. Analytical solution

To use the Lagrangian technique, introduce the Lagrange multipliers

$$\lambda(k) = \begin{pmatrix}\lambda_S(k)\\ \lambda_R(k)\end{pmatrix},$$

and the Lagrangian function

$$L = \frac{1}{2}\sum_{0}^{N}\{\|\Delta I(k)\|_Q^2 + \|\Delta P(k)\|_R^2\} + \lambda^\top(k+1)[-\Delta I(k+1) + \phi\Delta I(k) + \Gamma\Delta P(k)]. \tag{3.1}$$

The control equation $\nabla_{\Delta P(k)} L = 0$ is equivalent to

$$R\Delta P(k) + \Gamma^\top\lambda(k+1) = 0. \tag{3.2}$$

The adjoint equation $\nabla_{\Delta I(k)} L = 0$ is equivalent to

$$Q\Delta I(k) + \phi\lambda(k+1) - \lambda(k) = 0, \tag{3.3}$$

while $\nabla_{\lambda(k+1)} L = 0$ yields the state equation

$$\Delta I(k+1) = \phi\Delta I(k) + \Gamma\Delta P(k). \tag{3.4}$$

To solve this set of equations, we use the sweep method of Bryson and Ho [2] and set

$$\lambda(k) = S(k)\Delta I(k), \tag{3.5}$$

where $S(k), k = 0, \ldots, N$ are positive matrices. Combining (3.2), (3.3) and (3.5) leads to

$$\Delta P(k) = R^{-1}\Gamma^\top\phi^{-1}[Q - S(k)]\Delta I(k). \tag{3.6}$$

Combining (3.2), (3.4) and (3.5) leads to

$$\Delta P(k) = -[R + \Gamma^\top S(k+1)\Gamma]^{-1}\Gamma^\top S(k+1)\phi\Delta I(k). \tag{3.7}$$

Equating (3.6) and (3.7) yields the following discrete Ricatti equation (DRE):

$$S(k) = \phi(\Gamma^\top)^{-1}R[R + \Gamma^\top S(k+1)\Gamma]^{-1}\Gamma^\top S(k+1)\phi + Q. \tag{3.8}$$

The recursive DRE can be solved backwards, starting from

$$S(N) = Q, \tag{3.9}$$

since $\Delta P(N)$ has no effect on the objective function and therefore $\Delta P(N) = 0$. Now, combining (3.3) and (3.5) we obtain

$$\Delta I(k+1) = S(k+1)^{-1}\phi^{-1}[S(k) - Q]\Delta I(k), \quad k = 0, \ldots, N-1, \tag{3.10}$$

which allows to compute recursively, starting from $I(0) = I_0$, the optimal trajectory. Also, combining (3.4) and (3.10), we get

$$\Delta P(k) = \Gamma^{-1}[\Delta I(k+1) - \phi\Delta I(k)], \quad k = 0, \ldots, N-1. \tag{3.11}$$

Since only nonnegative production rates are allowed, the optimal production rates are chosen equal to

$$\max\{\Gamma^{-1}[\Delta I(k+1) - \phi\Delta I(k)], 0\}, \quad k = 0, \ldots, N-1. \tag{3.12}$$

A closed form expressions for $\Delta I(k)$ can be obtained from (3.10) as:

$$\Delta I(k) = (\phi^{-1})^k \prod_{i=0}^{k}[I_2 - S(i)Q][S(k) - Q]^{-1}S(0)\Delta I(0), \tag{3.13}$$

and for $\Delta P(k)$ from (3.11) as:

$$\Delta P(k) = \Gamma^{-1}(\phi^{-1})^{k+1} \prod_{i=0}^{k}[I_2 - S(i)^{-1}Q]\Gamma R^{-1}\Gamma^{\top} S(0)\Delta I(0). \tag{3.14}$$

For expressions (3.13) and (3.14) to be fully explicit, one still has to determine $S(0)$. Note that from (3.3), one has

$$\lambda(N-i) = Q\sum_{j=0}^{i-1}\phi^j \Delta I(N-i+j), \quad i = 0, \ldots, N. \tag{3.15}$$

Since $\Delta I(N) = 0$, setting $i = N$ in (3.15), one readily has:

$$S(0)\Delta I(0) = Q\sum_{j=0}^{N-1}\phi^j \Delta I(j). \tag{3.16}$$

Although closed form expressions have been explicitly derived for $\Delta I(k)$ and $\Delta P(k)$, it is the recursive relations (3.10) and (3.11) that are used for computational purposes.

3.2. Numerical illustration

For this numerical example, we will use the same data as Minner and Kleber [8]. The demand and return rates over a planning horizon of $H = 54$ weeks (approximately 1 year) are given by the following two functions:

$$D_S(t) = c_1 + c_2\sin(t),$$
$$D_R(t) = \begin{cases} c_3 D_S(t-4), & t \geq c_4, \\ 0, & t > c_4. \end{cases}$$

The demand and return rates are sinusoidal functions of time. The term c_4 represents the delay between the demand and the corresponding return. Fig. 2 depicts the variations of the demand and return rates for the following values of the parameters: $c_1 = 1$, $c_2 = 0.5$, $c_3 = 0.4$, and $c_4 = \frac{H}{4}$.

For the simulation, the inventory goal levels and the initial inventory level were chosen to be $\hat{I}_S(t) = 15$, $\hat{I}_R(t) = 10$ and $I_{S0} = 10$, $I_{R0} = 15$, respectively. We also assumed the following deterioration rates $\theta_S = .01$ and $\theta_R = .02$ and

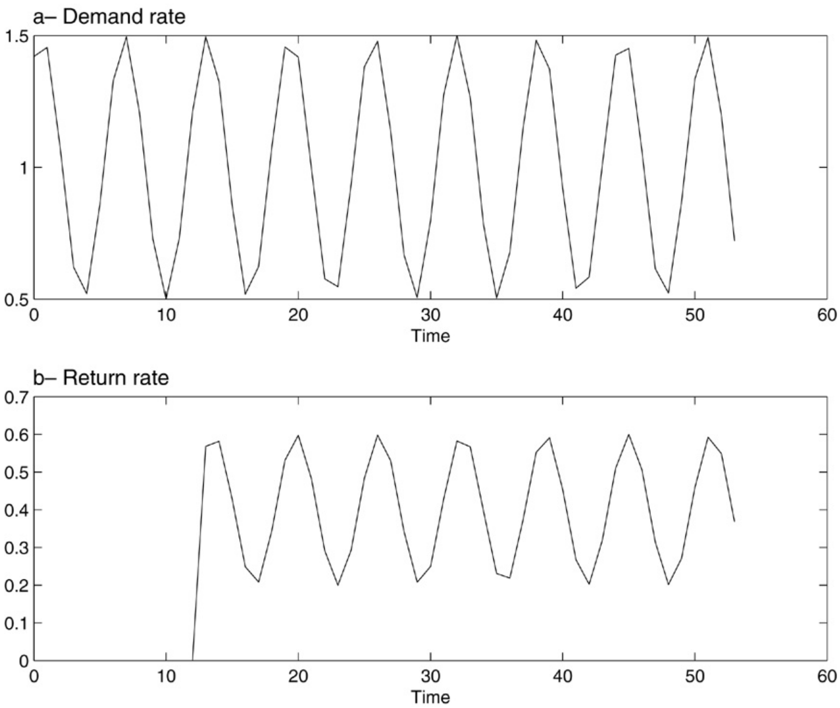

Fig. 2. Demand and return rates.

the following penalty costs $q_S = 10$, $q_R = 4$, $r_S = 12$, and $r_R = 8$. The first step is to compute the production goal rate which must satisfy the state equation (2.6), and thus

$$\hat{P}(k) = \max\{B^{-1}[A\hat{I}(k) - CD(k)], 0\}.$$

Next we successively compute the matrices S from (3.8), the optimal inventory levels I from (3.10), and the optimal production rates P from (3.11). Fig. 3 shows the optimal state variable and as can be seen, I (solid line) converges to $\hat{I}$ (dashed line).

Also, Fig. 4 shows the optimal control variable and also, as can be seen, P (solid line) converges to $\hat{P}$ (dashed line).

3.3. Extension to the infinite horizon

Assume now the planning horizon is no longer finite, i.e., ($H = \infty$). Then it is straightforward to extend the results obtained previously in this section. In this case, we look for the steady-state solution of the DRE (3.8). Denoting by S_∞ the limit of the sequence $S(k)$ when $k \to \infty$, we have from (3.8)

$$S_\infty = \phi(\Gamma^\top)^{-1}R[R + \Gamma^\top S_\infty \Gamma]^{-1}\Gamma^\top S_\infty \phi + Q. \tag{3.17}$$

The computation of the elements of the matrix S_∞ is now much simpler than before. Indeed, when H was finite, we needed to compute a whole sequence of matrices $S(k)$, while now we only need to compute the elements of the matrix S_∞, which is a simple matter since Eq. (3.17) reduces to a simple system of two linear equations. Then, from (3.10), the optimal states are now given by

$$\Delta I(k+1) = S_\infty^{-1}\phi^{-1}[S_\infty - Q]\Delta I(k), \quad k = 0, 1, \ldots \tag{3.18}$$

while the optimal controls are still given by (3.11)

$$\Delta P(k) = \Gamma^{-1}[\Delta I(k+1) - \phi\Delta I(k)], \quad k = 0, 1, \ldots. \tag{3.19}$$

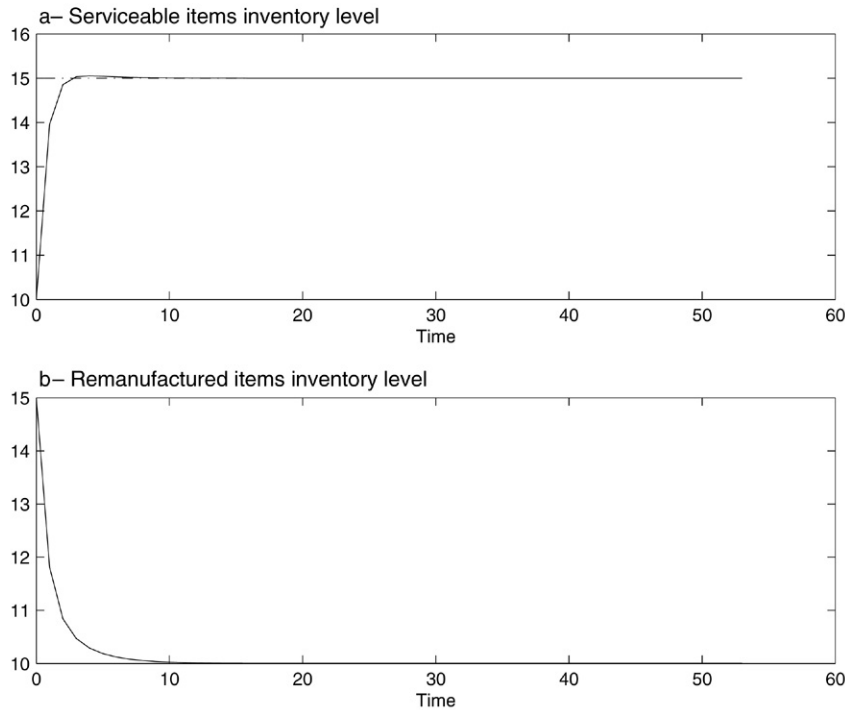

Fig. 3. Optimal inventory levels I.

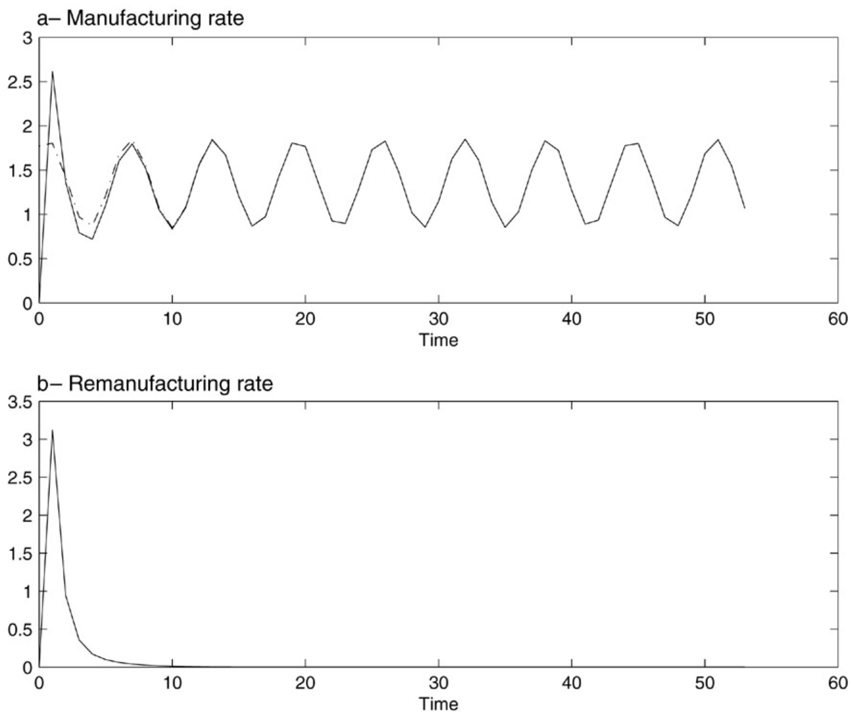

Fig. 4. Optimal production rates P.

4. Self-tuning optimal control

In this section the rate parameters θ_S and θ_R are assumed to be no longer known and therefore so are the matrices ϕ and Γ. In this case, we are faced with the problem of estimating these matrices. This is known as *parameters identification* in optimal control theory. This step is carried out below using the *recursive least-squares* (RLS) algorithm.

4.1. Analytical solution

In a first step, we will assume that all the states $I(0), \ldots, I(N)$ are known *a priori*. Then, from (2.11), we have

$$\begin{aligned} I(k+1) &= \phi I(k) + \Gamma P(k) + \psi D(k) \\ &= \phi I(k) + \Gamma[P(k) + \Gamma^{-1}\psi D(k)] \end{aligned} \tag{4.1}$$

so that by setting

$$\begin{aligned} U(k) &= P(k) + \Gamma^{-1}\psi D(k) \\ &= P(k) + F D(k) \end{aligned} \tag{4.2}$$

where

$$F = \Gamma^{-1}\psi = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

expression (4.1) can be written

$$I(k) = \phi I(k-1) + \Gamma U(k-1). \tag{4.3}$$

For convenience, let

$$a_S = \mathrm{e}^{-\theta_S}, \qquad a_R = \mathrm{e}^{-\theta_R}, \qquad b_s = \frac{1-\mathrm{e}^{-\theta_S}}{\theta_S}, \qquad b_s = \frac{1-\mathrm{e}^{-\theta_S}}{\theta_S}.$$

Also, let

$$U(k) = \begin{pmatrix} U_S(k) \\ U_R(k) \end{pmatrix}.$$

Then, (4.3) becomes

$$I(k) = M(k)W, \tag{4.4}$$

where

$$M(k) = \left(\begin{array}{cc|cc} I_S(k-1) & 0 & U_S(k-1)+U_R(k-1) & 0 \\ & & & \\ 0 & I_R(k-1) & 0 & -U_R(k-1) \end{array}\right), \tag{4.5}$$

and

$$W = \begin{pmatrix} a_S & a_R & b_S & b_R \end{pmatrix}^\top,$$

is the vector that we need to estimate. Let $e(k, \hat{W})$ be the equation error defined as

$$e(k, \hat{W}) = I(k) - M(k-1)\hat{W},$$

where $\hat{W}$ is the estimate of W. The principle of least-squares says that the estimate $\hat{W}$ is the point of the minimum of the performance measure

$$J_2(\hat{W}) = \sum_{k=1}^{N+1} e^2(k, \hat{W}) = E^\top(N, \hat{W}) E(N, \hat{W}),$$

where

$$E(N,\hat{W}) = Y(N) - X(N)\hat{W},$$
$$Y(N) = \begin{pmatrix} I^\top(1) & I^\top(2) & \cdots & I^\top(N+1) \end{pmatrix}^\top,$$
$$X(N) = \begin{pmatrix} M^\top(0) & M^\top(1) & \cdots & M^\top(N) \end{pmatrix}^\top.$$

Taking the gradient vector with respect to $\hat{W}$:

$$\nabla_{\hat{W}} J_2(\hat{W}) = 0 \Longleftrightarrow \hat{W} = [X^\top(N)X(N)]^{-1}X^\top(N)Y(N). \tag{4.6}$$

Recall that in order to derive the estimate (4.6), we have assumed that all the states $I(0),\ldots,I(N)$ were known. Obviously, this is not the case since only $I(0)$ is known at time 0. In order to estimate W, the RLS algorithm can be used. Knowing $I(0)$ and guessing initial values for W, we can use (4.6) to compute our first estimate of W. This first estimate is then used to compute $I(1)$. Then using $I(1)$, we use (4.6) to compute our second estimate of W. This second estimate is then used to compute $I(2)$, and so on. At the $(k+1)$th iteration, in order to minimize the number of operations, use is made of the computations carried out at the kth iteration. This is known as the RLS algorithm, and the full procedure is described in the next paragraph.

Recursive least-squares algorithm. Let $\hat{W}(k)$ denote the least-squares estimate based on k measurements $I(1),\ldots,I(k)$. From (4.6) we have

$$\hat{W}(k) = [X^\top(k)X(k)]^{-1}X^\top(k)Y(k).$$

Assume an additional measurement, the $(k+1)$th, is added:

$$X(k+1) = \begin{pmatrix} X(k) \\ M(k+1) \end{pmatrix}, \qquad Y(k+1) = \begin{pmatrix} Y(k) \\ I(k+1) \end{pmatrix}.$$

The estimate $\hat{W}(k+1)$ based on $(k+1)$ measurements can be written as

$$\begin{aligned} \hat{W}(k+1) &= [X^\top(k+1)X(k+1)]^{-1}X^\top(k+1)Y(k+1) \\ &= [X^\top(k)X(k) + M^\top(k+1)M(k+1)]^{-1}[X^\top(k)Y(k) + M^\top(k+1)I(k+1)]. \end{aligned} \tag{4.7}$$

We define the 4×4 matrix $R(k)$ by the recursion

$$R(k+1) = [X^\top(k+1)X(k+1)]^{-1} = G(k)R(k),$$

where

$$G(k) = [I_4 + R(k)M^\top(k+1)M(k+1)]^{-1}.$$

Substituting for $R(k+1)$ into (4.7), we obtain

$$\begin{aligned} \hat{W}(k+1) &= G(k)\hat{W}(k) + G(k)R(k)M^\top(k+1)I(k+1) \\ &= G(k)[\hat{W}(k) + R(k)M^\top(k+1)I(k+1)]. \end{aligned}$$

Note that the new estimate $\hat{W}(k+1)$ uses the previous estimate $\hat{W}(k)$ and the new measurement $I(k+1)$.

4.2. Numerical illustration

We carry out a numerical example using the same data as in Section 3.2, except that now θ_S and θ_R are unknown. To implement the RLS algorithm, we start with the initial guess $\hat{W}(0) = \begin{pmatrix} 0.5 & 0.5 & 0.4 & 0.4 \end{pmatrix}^\top$. We also take $R(0) = 100I_4$. Fig. 5 shows the successive updates of the estimates a_S and b_S while Fig. 6 shows the successive updates of the estimates a_R and b_R.

Both estimates converge quite rapidly to the exact values, represented by *. Also, Fig. 7 shows the optimal state variable and as can be seen, I (solid line) converges to $\hat{I}$ (dashed line).

Finally, Fig. 8 shows the optimal control variable and also, as can be seen, P (solid line) converges to $\hat{P}$ (dashed line).

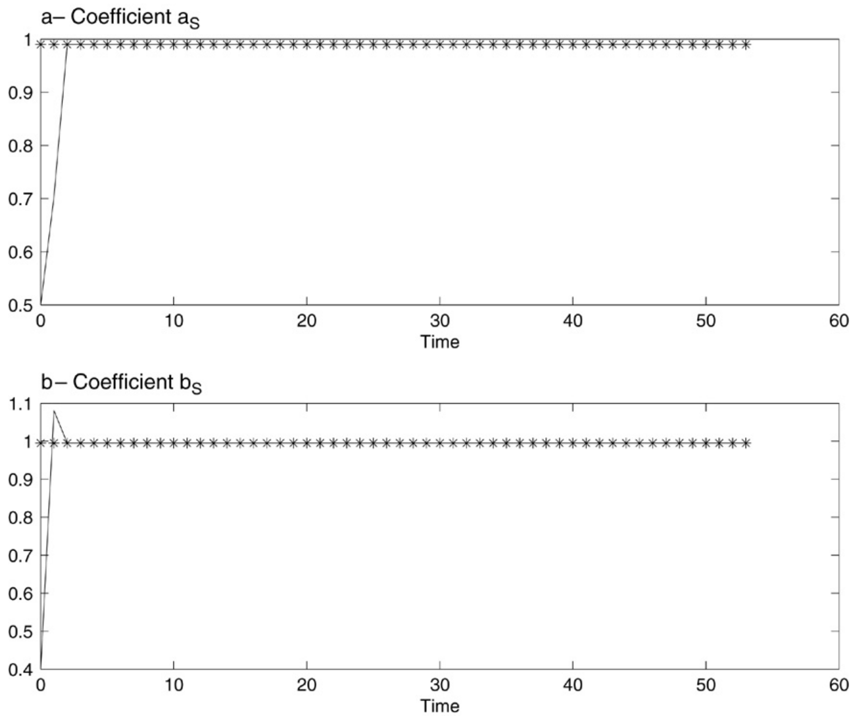

Fig. 5. Estimates of a_S and b_S.

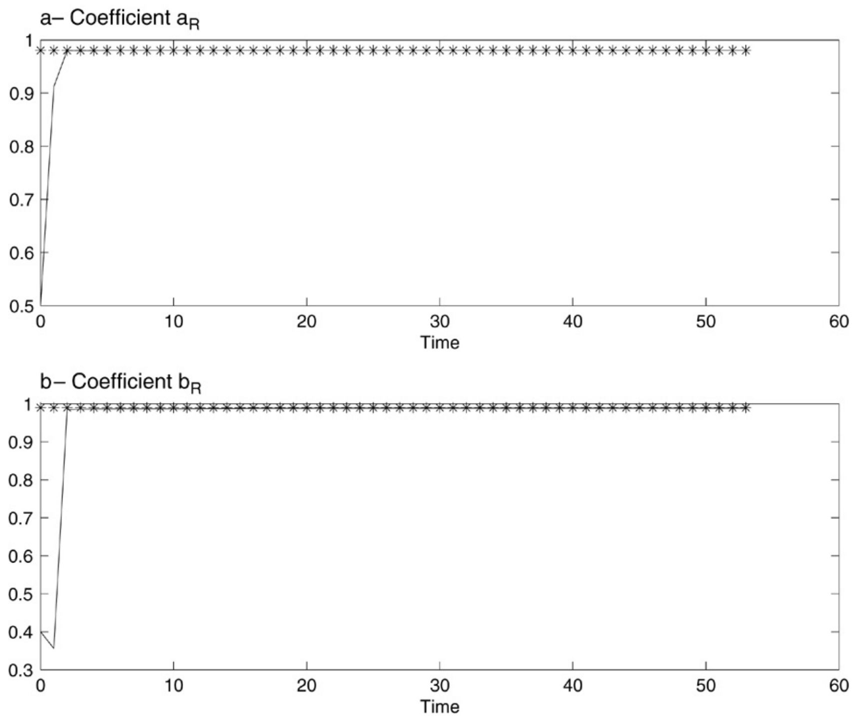

Fig. 6. Estimates of a_R and b_R.

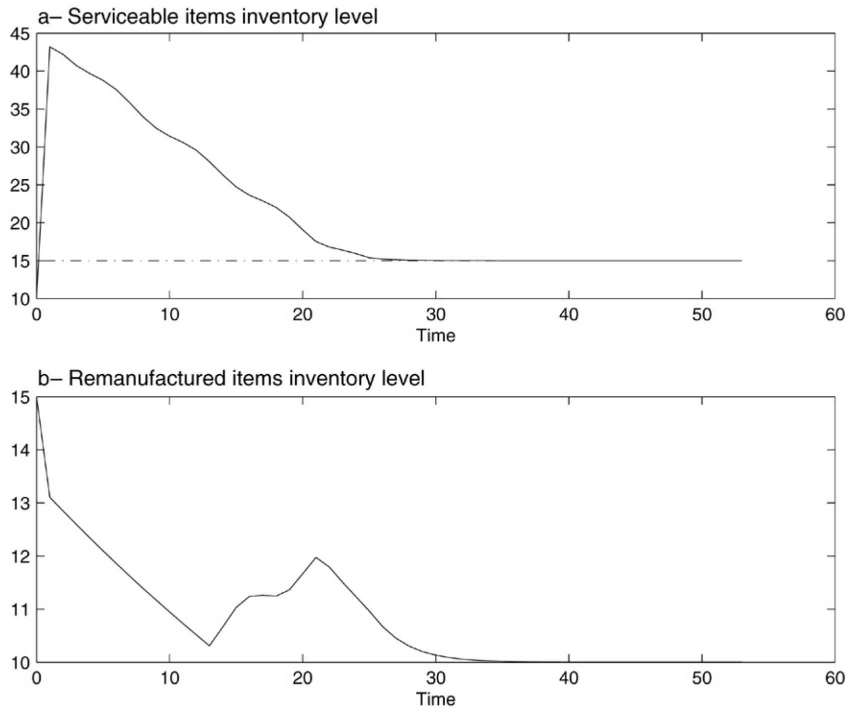

Fig. 7. Optimal inventory levels I.

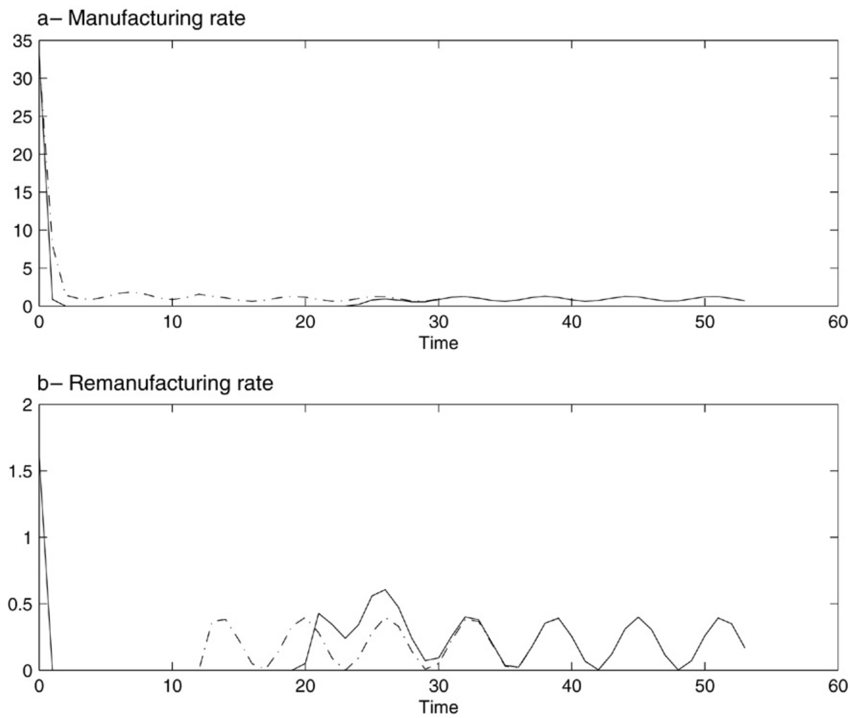

Fig. 8. Optimal production rates P.

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