Section 2.21

Formula Sheet

Geometric Series

$$1+r+r^2+\ldots+r^{n-1}=\frac{1-r^n}{1-r}=\frac{r^n-1}{r-1} \quad r\neq 1 \qquad 1+r+r^2+\ldots=\frac{1}{1-r} \quad |r|<1$$

Annuities

Level Annuities:

Immediate $a_{\overline{n}|}=\dfrac{1-v^n}{i}$ $\quad s_{\overline{n}|}=(1+i)^n\cdot a_{\overline{n}|}=\dfrac{(1+i)^n-1}{i}$ $\quad a_{\overline{n}|}=v^n\cdot s_{\overline{n}|}$

Due $\ddot{a}_{\overline{n}|}=\dfrac{1-v^n}{d}$ $\quad \ddot{s}_{\overline{n}|}=(1+i)^n\cdot \ddot{a}_{\overline{n}|}=\dfrac{(1+i)^n-1}{d}$ $\quad \ddot{a}_{\overline{n}|}=v^n\cdot \ddot{s}_{\overline{n}|}$

Payable m-thly $a_{\overline{n}|}^{(m)}=\dfrac{1-v^n}{i^{(m)}}=\dfrac{i}{i^{(m)}}\cdot a_{\overline{n}|}$

$$s_{\overline{n}|}^{(m)}=(1+i)^n\cdot a_{\overline{n}|}^{(m)}=\frac{(1+i)^n-1}{i^{(m)}}=\frac{i}{i^{(m)}}\cdot s_{\overline{n}|}$$

$$\ddot{a}_{\overline{n}|}^{(m)}=\frac{1-v^n}{d^{(m)}}=\frac{i}{d^{(m)}}\cdot a_{\overline{n}|}=\left(1+\frac{i^{(m)}}{m}\right)\cdot a_{\overline{n}|}^{(m)}$$

$$\ddot{s}_{\overline{n}|}^{(m)}=\frac{(1+i)^n-1}{d^{(m)}}=\frac{i}{d^{(m)}}\cdot s_{\overline{n}|}=\left(1+\frac{i^{(m)}}{m}\right)\cdot s_{\overline{n}|}^{(m)}$$

Continuously payable $\bar{a}_{\overline{n}|}=\dfrac{1-v^n}{\delta}=\dfrac{i}{\delta}\cdot a_{\overline{n}|}$ $\quad \bar{s}_{\overline{n}|}=\dfrac{(1+i)^n-1}{\delta}=\dfrac{i}{\delta}\cdot s_{\overline{n}|}$

Perpetuities $a_{\overline{\infty}|}=v+v^2+v^3+\ldots=\dfrac{1}{i}$ $\quad \ddot{a}_{\overline{\infty}|}=1+v+v^2+\ldots=\dfrac{1}{d}$ $\quad \bar{a}_{\overline{\infty}|}=\dfrac{1}{\delta}$

Deferred ${}_{k|}a_{\overline{n}|}=v^k\cdot a_{\overline{n}|}$ $\quad {}_{k|}a_{\overline{n}|}=a_{\overline{n+k}|}-a_{\overline{k}|}$

Relations $\ddot{a}_{\overline{n}|}=\dfrac{i}{d}\cdot a_{\overline{n}|}=(1+i)a_{\overline{n}|}$ $\quad \ddot{s}_{\overline{n}|}=\dfrac{i}{d}\cdot s_{\overline{n}|}=(1+i)s_{\overline{n}|}$

$\ddot{a}_{\overline{n}|}=a_{\overline{n-1}|}+1$ $\quad s_{\overline{n}|}=\ddot{s}_{\overline{n-1}|}+1$

$\ddot{a}_{\overline{\infty}|}=a_{\overline{\infty}|}+1=(1+i)\cdot a_{\overline{\infty}|}$

$a_{\overline{k}|}=a_{\overline{n+k}|}-{}_{k|}a_{\overline{n}|}$ $\quad a_{\overline{n}|}=a_{\overline{\infty}|}-{}_{n|}a_{\overline{\infty}|}=\dfrac{1}{i}-v^n\cdot\dfrac{1}{i}=\dfrac{1-v^n}{i}$

Arithmetic Annuities:

Increasing: Payments are 1, 2, . . ., n

$$(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i} \qquad (I\ddot{a})_{\overline{n|}} = (1+i)\cdot(Ia)_{\overline{n|}} = \frac{i}{d}\cdot(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{d}$$

$$(Is)_{\overline{n|}} = (1+i)^n\cdot(Ia)_{\overline{n|}} = \frac{\ddot{s}_{\overline{n|}} - n}{i} \qquad (I\ddot{s})_{\overline{n|}} = (1+i)^n\cdot(I\ddot{a})_{\overline{n|}} = \frac{i}{d}\cdot(Is)_{\overline{n|}} = \frac{\ddot{s}_{\overline{n|}} - n}{d}$$

$$(I\bar{a})_{\overline{n|}} = \frac{i}{\delta}\cdot(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - n\cdot v^n}{\delta} \qquad (I\bar{s})_{\overline{n|}} = (1+i)^n\cdot(I\bar{a})_{\overline{n|}} = \frac{i}{\delta}\cdot(Is)_{\overline{n|}} = \frac{\ddot{s}_{\overline{n|}} - n}{\delta}$$

$$(\bar{I}\bar{a})_{\overline{n|}} = \frac{\bar{a}_{\overline{n|}} - n\cdot v^n}{\delta} \qquad (\bar{I}\bar{s})_{\overline{n|}} = (1+i)^n\cdot(\bar{I}\bar{a})_{\overline{n|}} = \frac{\bar{s}_{\overline{n|}} - n}{\delta}$$

Increasing perpetuity $\quad (Ia)_{\overline{\infty|}} = \dfrac{1}{id} \quad (I\ddot{a})_{\overline{\infty|}} = \dfrac{1}{d^2} \quad (I\bar{a})_{\overline{\infty|}} = \dfrac{1}{\delta d} \quad (\bar{I}\bar{a})_{\overline{\infty|}} = \dfrac{1}{\delta^2}$

Decreasing: Payments are n, n-1,…, 2, 1

$$(Da)_{\overline{n|}} = \frac{n - a_{\overline{n|}}}{i} \qquad (D\ddot{a})_{\overline{n|}} = \frac{n - a_{\overline{n|}}}{d} = (1+i)(Da)_{\overline{n|}}$$

$$(Ds)_{\overline{n|}} = (1+i)^n\,(Da)_{\overline{n|}} = \frac{n\cdot(1+i)^n - s_{\overline{n|}}}{i}$$

$$(D\ddot{s})_{\overline{n|}} = (1+i)^n\,(D\ddot{a})_{\overline{n|}} = \frac{n\cdot(1+i)^n - s_{\overline{n|}}}{d} = \frac{i}{d}\cdot(Ds)_{\overline{n|}}$$

$$(D\bar{a})_{\overline{n|}} = \frac{i}{\delta}\cdot(Da)_{\overline{n|}} = \frac{n - a_{\overline{n|}}}{\delta} \qquad (D\bar{s})_{\overline{n|}} = (1+i)^n\,(D\bar{a})_{\overline{n|}} = \frac{n\cdot(1+i)^n - s_{\overline{n|}}}{\delta}$$

$$(\bar{D}\bar{a})_{\overline{n|}} = \frac{n - \bar{a}_{\overline{n|}}}{\delta} \qquad (\bar{D}\bar{s})_{\overline{n|}} = (1+i)^n\,(\bar{D}\bar{a})_{\overline{n|}} = \frac{n\cdot(1+i)^n - \bar{s}_{\overline{n|}}}{\delta}$$

***PQ* Formula** for Arithmetic Annuities-Immediate:

Payments are $P, (P+Q), (P+2Q), \ldots, (P+(n-1)Q)$

$$PV = P\cdot a_{\overline{n|}} + Q\cdot\left(\frac{a_{\overline{n|}} - nv^n}{i}\right) \qquad FV = P\cdot s_{\overline{n|}} + Q\cdot\left(\frac{s_{\overline{n|}} - n}{i}\right)$$

For perpetuities-immediate (infinite n): $\quad PV = \dfrac{P}{i} + \dfrac{Q}{i^2}$

Geometric Annuities:

Payments are 1, $(1+g)$, $(1+g)^2$, …, $(1+g)^{n-1}$ $\qquad g \neq i$

$$a^{g}_{\overline{n}|i} = \frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g} \qquad \ddot{a}^{g}_{\overline{n}|i} = (1+i)\cdot a^{g}_{\overline{n}|i} = \frac{1-\left(\frac{1+g}{1+i}\right)^n}{d-v\cdot g} = \frac{1-\left(\frac{1+g}{1+i}\right)^n}{\frac{i-g}{1+i}}$$

$$s^{g}_{\overline{n}|i} = \frac{(1+i)^n-(1+g)^n}{i-g} \qquad \ddot{s}^{g}_{\overline{n}|i} = (1+i)\cdot s^{g}_{\overline{n}|i} = \frac{(1+i)^n-(1+g)^n}{d-v\cdot g} = \frac{(1+i)^n-(1+g)^n}{\frac{i-g}{1+i}}$$

$$\bar{a}^{\bar{g}}_{\overline{n}|} = \frac{1-e^{-n\cdot(\delta-\bar{g})}}{\delta-\bar{g}} \qquad \bar{s}^{\bar{g}}_{\overline{n}|} = \bar{a}^{\bar{g}}_{\overline{n}|}\cdot e^{n\delta} = \frac{e^{n\delta}-e^{n\cdot\bar{g}}}{\delta-\bar{g}}$$

$$\ddot{a}^{g}_{\overline{n}|i} = \ddot{a}_{\overline{n}|j} \quad \text{where } 1+j = \frac{1+i}{1+g}$$

Geometric perpetuity $(i>g)$:

$$a^{g}_{\overline{\infty}|i} = \frac{1}{i-g} \qquad \ddot{a}^{g}_{\overline{\infty}|i} = (1+i)\cdot a^{g}_{\overline{\infty}|i} = \frac{1}{d-v\cdot g} = \frac{1+i}{i-g} \qquad \bar{a}^{\bar{g}}_{\overline{\infty}|i} = \frac{1}{\delta-\bar{g}}$$

Varying the Payment Frequency:

Converting from an annuity-immediate with annual payments (at end of year) to an annuity with a different timing or frequency of payments:

To convert to:	Multiply by:	Example
Annuity-due (beginning-of-year pmts)	$\frac{i}{d}$	$\ddot{a}_{\overline{n}|} = \frac{i}{d}\cdot a_{\overline{n}|}$
Continuous payments	$\frac{i}{\delta}$	$\bar{s}_{\overline{n}|} = \frac{i}{\delta}\cdot s_{\overline{n}|}$
m-thly annuity-immediate (payments at end)	$\frac{i}{i^{(m)}}$	$(Ia)^{(m)}_{\overline{\infty}|} = \frac{i}{i^{(m)}}\cdot (Ia)_{\overline{\infty}|}$
m-thly annuity-due (payments at beginning)	$\frac{i}{d^{(m)}}$	$\ddot{s}^{g\,(m)}_{\overline{n}|} = \frac{i}{d^{(m)}}\cdot s^{g}_{\overline{n}|}$

Note: In each case, we begin with a unit annuity-immediate. That is, it is an annuity-immediate that pays 1 at the end of each year (if it is a level annuity), or it pays 1 at the end of the first year (if it is a geometric annuity or an arithmetic increasing annuity) or it pays n at the end of the first year and 1 at the end of the n^th year (if it is an arithmetic decreasing annuity). And the total payments each year for the converted annuity are the same as for the original annuity (e.g., for a level unit annuity, it could be monthly payments of 1/12 each month, or it could be continuous payments at a rate of 1 per year).