coachingäctuaries

You have what it takes to pass

Exam FM

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INTEREST MEASUREMENT

Effective Rate of Interest

$$i_t = \frac{A(t) - A(t-1)}{A(t-1)}$$

Effective Rate of Discount

$$d_t = \frac{A(t) - A(t-1)}{A(t)}$$

Accumulation Function and Amount Function

$$A(t) = A(0) \cdot a(t)$$

All-in-One Relationship Formula

$$(1+i)^t = \left(1 + \frac{i^{(m)}}{m}\right)^{mt} = (1-d)^{-t}$$

$$= \left(1 - \frac{d^{(m)}}{m}\right)^{-mt} = e^{\delta t}$$

Simple Interest

$$a(t) = 1 + it$$

Variable Force of Interest

$$\delta_t = \frac{a'(t)}{a(t)}$$

Accumulate 1 from time t_1 to time t_2:

$$AV = \exp\left(\int_{t_1}^{t_2} \delta_u \, du\right)$$

Discount Factor

$$v = \frac{1}{1+i} = 1 - d$$

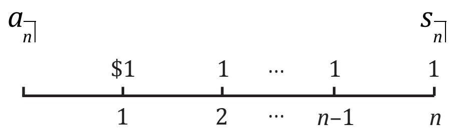
$$d = \frac{i}{1+i} = iv$$

ANNUITIES

Annuity-Immediate

$$PV = a_{\overline{n}|}$$
$$= v + v^2 + \cdots + v^n$$
$$= \frac{1 - v^n}{i}$$

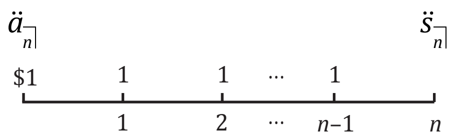
$$AV = s_{\overline{n}|}$$
$$= 1 + (1+i) + \cdots + (1+i)^{n-1}$$
$$= \frac{(1+i)^n - 1}{i}$$



Annuity-Due

$$PV = \ddot{a}_{\overline{n}|}$$
$$= 1 + v + v^2 + \cdots + v^{n-1}$$
$$= \frac{1 - v^n}{d}$$

$$AV = \ddot{s}_{\overline{n}|}$$
$$= (1+i) + (1+i)^2 + \cdots + (1+i)^n$$
$$= \frac{(1+i)^n - 1}{d}$$



Immediate vs. Due

$$\ddot{a}_{\overline{n}|} = a_{\overline{n}|}(1+i) = 1 + a_{\overline{n-1}|}$$

$$\ddot{s}_{\overline{n}|} = s_{\overline{n}|}(1+i) = s_{\overline{n+1}|} - 1$$

Deferred Annuity

m-year deferred n-year annuity-immediate:

$$PV = {}_{m|}a_{\overline{n}|} = v^m \cdot a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$$

Perpetuity

- *Perpetuity-immediate:*

  $$PV = a_{\overline{\infty}|} = v + v^2 + \cdots = \frac{1}{i}$$

- *Perpetuity-due:*

  $$PV = \ddot{a}_{\overline{\infty}|} = 1 + v + v^2 + \cdots = \frac{1}{d}$$

  $$\ddot{a}_{\overline{\infty}|} = 1 + a_{\overline{\infty}|}$$

MORE GENERAL ANNUITIES

j-effective method is used when payments are more or less frequent than the interest period.

"j-effective" Method

Convert the given interest rate to the equivalent effective interest rate for the period between each payment.

Example: To find the present value of n monthly payments given annual effective rate of i, define j as the monthly effective rate where $j = (1+i)^{1/12} - 1$. Then apply $PV = a_{\overline{n}|}$ using j.

Payments in Arithmetic Progression

- *PV of n-year annuity-immediate with payments of* $P, P+Q, P+2Q, \ldots, P+(n-1)Q$

  $$PV = Pa_{\overline{n}|} + Q\frac{a_{\overline{n}|} - nv^n}{i}$$

  Calculator-friendly version:

  $$PV = \left(P + \frac{Q}{i}\right)a_{\overline{n}|} + \left(-\frac{Qn}{i}\right)v^n$$

  $N = n, I/Y = i$ (in %),

  $PMT = P + \frac{Q}{i}, FV = -\frac{Qn}{i}$

- *PV of n-year annuity-immediate with payments of* $1, 2, 3, \ldots, n$

  Unit increasing: $(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$

  P&Q version: $P = 1, Q = 1, N = n$

- *PV of n-year annuity-immediate with payments of* $n, n-1, n-2, \ldots, 1$

  Unit decreasing: $(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$

  P&Q version: $P = n, Q = -1, N = n$

- *PV of perpetuity-immediate and perpetuity-due with payments of* $1, 2, 3, \ldots$

  $$(Ia)_{\overline{\infty}|} = \frac{1}{id} = \frac{1}{i} + \frac{1}{i^2}$$

  $$(I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2}$$

Payments in Geometric Progression

PV of an n-year annuity-immediate with payments of

$1, (1+k), (1+k)^2, \ldots, (1+k)^{n-1}$

$$PV = \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{i-k}, i \neq k$$

Level and Increasing Continuous Annuity

$$\bar{a}_{\overline{n|}} = \int_0^n v^t \, dt = \frac{1-v^n}{\delta} = \frac{i}{\delta} a_{\overline{n|}}$$

$$(\bar{I}\bar{a})_{\overline{n|}} = \int_0^n t v^t \, dt = \frac{\bar{a}_{\overline{n|}} - n v^n}{\delta}$$

YIELD RATES

Two methods for comparing investments:

- *Net Present Value (NPV):* Sum the present value of cash inflows and cash outflows. Choose investment with greatest positive NPV.
- *Internal Rate of Return (IRR):* The rate such that the present value of cash inflows is equal to the present value of cash outflows. Choose investment with greatest IRR.

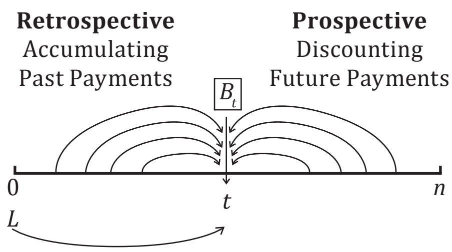
LOAN AMORTIZATION

Outstanding Balance Calculation

- *Prospective:* $B_t = Ra_{\overline{n-t|}}$,

  Present value of future level payments of R.
- *Retrospective:* $B_t = L(1+i)^t - Rs_{\overline{t|}}$

  Accumulated value of original loan amount L minus accumulated value of all past payments.



Loan Amortization

For a loan of $a_{\overline{n|}}$ repaid with n payments of 1:

	Period t
Interest (I_t)	$1 - v^{n-t+1}$
Principal repaid (P_t)	v^{n-t+1}
Total	1

General Formulas for Amortized Loan with Level/Non-Level Payments

$I_t = i \cdot B_{t-1}$

$B_t = B_{t-1}(1+i) - R_t = B_{t-1} - P_t$

$P_t = R_t - I_t$

$P_{t+k} = P_t(1+i)^k$ (only for Level Payments)

BONDS

Bond Pricing Formulas

P	Price of bond
F	Par value (face amount) of bond (not a cash flow)
r	Coupon rate per payment period
Fr	Amount of each coupon payment
C	Redemption value of bond ($F = C$ unless otherwise stated)
i	Interest rate per payment period
n	Number of coupon payments

Basic Formula

$P = Fra_{\overline{n|}i} + Cv^n$

Premium/Discount Formula:

$P = C + (Fr - Ci)a_{\overline{n|}i}$

Premium vs. Discount

	Premium	Discount
Condition	$P > C$ or $Fr > Ci$	$P < C$ or $Fr < Ci$
Amortization Process	Write-Down	Write-Up
Amount	$\lvert(Fr - Ci) \cdot v^{n-t+1}\rvert = \lvert B_{t-1} - B_t\rvert = \lvert Fr - I_t\rvert$	

General Formulas for Bond Amortization

- Book value:

  $B_t = Fra_{\overline{n-t|}i} + Cv^{n-t}$

  $= C + (Fr - Ci)a_{\overline{n-t|}i}$
- Interest earned = iB_{t-1}

Callable Bonds

Calculate the lowest price for all possible redemption dates at a certain yield rate. This is the highest price that guarantees this yield rate.

- Premium bond – call the bond on the FIRST possible date.
- Discount bond – call the bond on the LAST possible date.

STOCKS

Price of Level Dividend-Paying Stock

$$P = \frac{Fr}{i}$$

- F = par value
- r = fixed dividend rate

Price of Increasing Dividend-Paying Stock

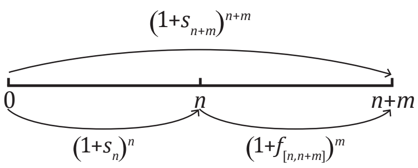
$$P = \frac{D}{i-k}$$

- D = expected first dividend
- k = growth rate

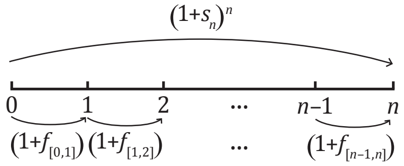
SPOT RATES AND FORWARD RATES

s_t is the t-year spot rate.
$f_{[t_1,t_2]}$ is the forward rate from time t_1 to time t_2, expressed annually.

$$(1+s_n)^n \cdot \left(1+f_{[n,n+m]}\right)^m = (1+s_{n+m})^{n+m}$$



$$(1+s_n)^n = \left(1+f_{[0,1]}\right)\cdot\left(1+f_{[1,2]}\right)\cdots\left(1+f_{[n-1,n]}\right)$$



INTEREST RATE SWAP

An agreement between two parties in which both parties agree to exchange a series of cash flows based on interest rates.

Swap Rate

The swap rate can be calculated by equating the present value of swap payments with the present value of expected variable payments.

- *If notional amount is not level:*
  Example:
  $$\frac{X_1R}{1+s_1}+\frac{X_2R}{(1+s_2)^2}+\frac{X_3R}{(1+s_3)^3} = \frac{X_1f_{[0,1]}}{1+s_1}+\frac{X_2f_{[1,2]}}{(1+s_2)^2}+\frac{X_3f_{[2,3]}}{(1+s_3)^3}$$
- *If notional amount is level:*
  Since an interest rate swap is equivalent to borrowing at a floating rate to buy a fixed-rate bond, fixed swap rate is the coupon rate on a par coupon bond.
  $$\frac{R}{1+s_1}+\frac{R}{(1+s_2)^2}+\cdots+\frac{R+1}{(1+s_n)^n}=1$$
  $$R=\frac{1-P_n}{P_1+P_2+\cdots+P_n}$$

Net Swap Payment

The difference between the fixed interest payment and variable interest payment.

Net Interest Payment

The combination of the net swap payment and the interest payment made by the borrower to the lender.

Deferred Interest Rate Swap

For an x-year deferred n-year swap with level notional amount:

$$R=\frac{P_x-P_n}{P_{x+1}+P_{x+2}+\cdots+P_n}$$

where x is the number of deferred years and n is the term of the swap.

Market Value of a Swap

- The market value of a swap at time t is the present value at time t of its expected future cash flows.
- The market value of a swap is 0 at inception.

DETERMINANTS OF INTEREST RATES

- Interest rate can be viewed as the equilibrium price of money.
- Interest rate can be decomposed into five components:
  - Real risk-free rate (r)
  - Maturity risk premium
  - Default risk premium (s)
  - Inflation premium (i_e, i_u, c, i_a)
  - Liquidity premium
- $R = r + s + i_e + i_u - c + i_a$
  - For loans with inflation protection, set $i_e = i_u = 0$.
  - For loans without inflation protection, set $i_a = c = 0$.
  - $R = r + s - c$ is the real interest rate.
  - $R = r + s + i_e + i_u$ is the nominal interest rate.
- Four theories explaining why interest rates differ by terms:
  - Market segmentation theory
  - Preferred habitat theory
  - Liquidity preference theory / Opportunity cost theory
  - Expectations theory
- Federal Reserve facilitates a country's payment operations and functions as a last resort lender to commercial banks.
- U.S. T-bills are quoted:
  $$\text{Quoted Rate} = \frac{360}{N}\times\frac{I}{C}$$
- Canadian T-bills are quoted:
  $$\text{Quoted Rate} = \frac{365}{N}\times\frac{I}{P}$$
  - N is the number of days to maturity.
  - I is the amount of interest.
  - C is the maturity value.
  - P is the price.

INTEREST MEASUREMENT OF A FUND

Dollar-weighted Interest Rate

The yield rate computation depends on the amount invested.

Method:

- Calculate interest: $I = B - A - C$
  A: Amount at the beginning of period
  B: Amount at the end of period
  C: Deposit/withdrawal
- Calculate dollar-weighted interest rate:

$$i_{DW} = \frac{I}{A + \sum C_t(1-t)}$$

Time-weighted Interest Rate

The yield rate computation depends on successive sub-intervals of the year each time a deposit or withdrawal is made.

Method:

$$1 + i_{TW} = \left(\frac{A_2}{B_1}\right) \cdot \left(\frac{A_3}{B_2}\right) \cdot \left(\frac{A_4}{B_3}\right) \cdot \ldots \cdot \left(\frac{A_n}{B_{n-1}}\right)$$

	Date 1	Date 2
Account Before CF	A_1	A_2
Cash Flow (CF)	C_1	C_2
Account After CF	$B_1 = A_1 + C_1$	$B_2 = A_2 + C_2$

DURATION AND CONVEXITY

Duration

$$MacD = -\frac{P'(\delta)}{P(\delta)} = \frac{\sum_{t=0}^{n} t \cdot v^t \cdot CF_t}{\sum_{t=0}^{n} v^t \cdot CF_t}$$

$$ModD = -\frac{P'(i)}{P(i)} = \frac{\sum_{t=0}^{n} t \cdot v^{t+1} \cdot CF_t}{\sum_{t=0}^{n} v^t \cdot CF_t}$$

$$ModD = MacD \cdot v$$

	$MacD$
n-year zero-coupon bond	n
Geometrically increasing perpetuity	$\frac{1+i}{i-k}$
n-year par bond	$\ddot{a}_{\overline{n}\vert}$

First-order Modified Approximation

$$P(i_n) \approx P(i_o) \cdot [1 - (i_n - i_o)(ModD)]$$

First-order Macaulay Approximation

$$P(i_n) \approx P(i_o) \cdot \left(\frac{1+i_o}{1+i_n}\right)^{MacD}$$

Passage of Time

Given that the future cash flows are the same at time t_1 and time t_2:

$$MacD_{t_2} = MacD_{t_1} - (t_2 - t_1)$$

$$ModD_{t_2} = ModD_{t_1} - v(t_2 - t_1)$$

Duration of a portfolio

For a portfolio of m securities where invested amount $P = P_1 + P_2 + \cdots + P_m$ at time 0:

$$MacD_P = \frac{P_1}{P} MacD_1 + \cdots + \frac{P_m}{P} MacD_m$$

Convexity

$$ModC = \frac{P''(i)}{P(i)} = \frac{\sum_{t=0}^{n} t \cdot (t+1) \cdot v^{t+2} \cdot CF_t}{\sum_{t=0}^{n} v^t \cdot CF_t}$$

$$MacC = \frac{P''(\delta)}{P(\delta)} = \frac{\sum_{t=0}^{n} t^2 \cdot v^t \cdot CF_t}{\sum_{t=0}^{n} v^t \cdot CF_t}$$

$$ModC = v^2(MacC + MacD)$$

$MacC$(n-year zero-coupon bond) $= n^2$

IMMUNIZATION

Redington and Full Immunization

Redington	*Full*
$PV_{Assets} = PV_{Liabilities}$	
$MacD_A = MacD_L$ or $P'_A = P'_L$	
$C_A > C_L$ or $P''_A > P''_L$	There has to be asset cash flows before and after each liability cash flow.
Immunizes against small changes in i	Immunizes against any changes in i

Immunization Shortcut

(*works for immunization questions that have asset cash flows before and after the liability cash flow*)

1. Identify the asset allocation at the time the liability occurs by equating face amounts (prices) and durations.

$$w = \frac{t_2 - t_L}{t_2 - t_1}$$

t_1	Shorter bond duration
t_2	Longer bond duration
t_L	Liability duration
w	Shorter bond's weight
$1 - w$	Longer bond's weight

2. Adjust for interest to the asset maturity date.

BA-II PLUS CALCULATOR GUIDE

Basic Operations

[ENTER (SET)]: Send value to a variable (option)

[↑][↓]: Navigate through variables

[2ND]: Access secondary functions (yellow)

[STO] + [0~9]: Send on-screen value into memory

[RCL] + [0~9]: Recall value from a memory

Time Value of Money (TVM)

Good for handling annuities, loans and bonds.

Note: Be careful with signs of cash flows.

[N]: Number of periods

[I/Y]: Effective interest rate per period (in %)

[PV]: Present value

[PMT]: Amount of each payment of an annuity

[FV]: Future value

[CPT] + (one of above): Solve for unknown

[2ND] + [BGN], [2ND] + [SET], [2ND] + [QUIT]: Switch between annuity immediate and annuity due

[2ND] + [P/Y]: *Please keep P/Y and C/Y as 1*

[2ND] + [CLR TVM]: Clear TVM worksheet

[2ND] + [AMORT]: Amortization (See Below)

For bonds $\left(P = Fra_{\overline{n}|i} + Cv^n\right)$:

[N] $= n$; [I/Y] $= i$; [PV] $= -P$;

[PMT] $= Fr$; [FV] $= C$.

Cash Flow Worksheet

([CF], [NPV], [IRR])

Good for non-level series of payments.

Input ([CF])

CF_0: Cash flow at time 0

C_n: n^{th} cash flow

F_n: Frequency of the cash flow

Output ([NPV], [IRR])

I: Effective interest rate (in %)

[NPV] + [CPT]: Solve for net present value

[IRR] + [CPT]: Solve for internal rate of return

Amortization Schedule

([2ND] + [AMORT])

Good for finding outstanding balance of the loan and interest/principal portion of certain payments.

Note: BA-II Plus requires computing the unknown TVM variable before entering into AMORT function.

P1: Starting period

P2: Ending period

BAL: Remaining balance of the loan after P2

PRN: Sum of the principal repaid from P1 to P2

INT: Sum of the interest paid from P1 to P2