College of Sciences
Department of
Physics and Astronomy

مولعلا ةيلك
ايزيفلا مسقلفلكلفلاو ء

جامعة الملك سعود
King Saud University

Final Exam

Tuesday 08/Rabeea'l Thani/1439 (26/12/2017)	**PHYS 505**	**Academic year 1438-39H**
8:00 – 11:00 am	**Advanced Quantum Mechanics**	**First Semester**

Student's Name		اسم الطالب
ID number		الرقم الجامعي
Section No.		رقم الشعبة
Classroom No.		رقم قاعة الاختبار
Teacher's Name		اسم أستاذ المقرر
Roll Number		رقم التحضير

40

Please answer all questions

1. You are given that the state $\left|j_1 = 1, j_2 = 1, J = 2, M = 0\right\rangle$ for two p electrons is given by:

$$\left|j_1 = 1, j_2 = 1, J = 2, M = -1\right\rangle = \frac{1}{\sqrt{2}}\left\{\left|j_1 = 1, m_1 = 0\right\rangle\left|j_2 = 1, m_2 = -1\right\rangle + \left|j_1 = 1, m_1 = -1\right\rangle\left|j_2 = 1, m_2 = 0\right\rangle\right\}$$

Find the state $\left|j_1 = 1, j_2 = 1, J = 2, M = -2\right\rangle$.

You are given the lowering operator for all different types of angular momentum ($K = J, j, l, s$): $K_-\left|k, m_k\right\rangle = \hbar\sqrt{k(k+1) - m_k\left(m_k - 1\right)}\left|k, m_k - 1\right\rangle$. Also for two particles the lowering operator is $K_- = K_-^1 + K_-^2$

(10 marks)

Solution:

$$l_-\left|j_1 = 1, j_2 = 1, J = 2, M = -1\right\rangle = \hbar\sqrt{J\left(J+1\right) - M\left(M-1\right)}\left|j_1 = 1, j_2 = 1, J = 2, M = -2\right\rangle =$$
$$\hbar\sqrt{2\left(2+1\right) - (-1)\left(-1-1\right)}\left|j_1 = 1, j_2 = 1, J = 2, M = -2\right\rangle = 2\hbar\left|j_1 = 1, j_2 = 1, J = 2, M = -2\right\rangle$$

(a)

$$\left(l_-^1 + l_-^2\right)\frac{1}{\sqrt{2}}\left\{\left|j_1 = 1,\ m_1 = 0\right\rangle\left|j_2 = 1,\ m_2 = -1\right\rangle + \left|j_1 = 1,\ m_1 = -1\right\rangle\left|j_2 = 1,\ m_2 = 0\right\rangle\right\} =$$

$$\frac{1}{\sqrt{2}}\left\{\left(l_-^1 + l_-^2\right)\left|j_1 = 1,\ m_1 = 0\right\rangle\left|j_2 = 1,\ m_2 = -1\right\rangle + \left(l_-^1 + l_-^2\right)\left|j_1 = 1,\ m_1 = -1\right\rangle\left|j_2 = 1,\ m_2 = 0\right\rangle\right\} =$$

$$\frac{1}{\sqrt{2}}\left\{l_-^1\left|j_1 = 1,\ m_1 = 0\right\rangle\left|j_2 = 1,\ m_2 = -1\right\rangle + l_-^2\left|j_1 = 1,\ m_1 = 0\right\rangle\left|j_2 = 1,\ m_2 = -1\right\rangle +\right.$$

$$\left. l_-^1\left|j_1 = 1,\ m_1 = -1\right\rangle\left|j_2 = 1,\ m_2 = 0\right\rangle + l_-^2\left|j_1 = 1,\ m_1 = -1\right\rangle\left|j_2 = 1,\ m_2 = 0\right\rangle\right\} =$$

(b)

$$\frac{1}{\sqrt{2}}\left\{l_-^1\left|j_1 = 1,\ m_1 = 0\right\rangle\left|j_2 = 1,\ m_2 = -1\right\rangle + \left|j_1 = 1,\ m_1 = 0\right\rangle\left(l_-^2\left|j_2 = 1,\ m_2 = -1\right\rangle\right) +\right.$$

$$\left. l_-^1\left|j_1 = 1,\ m_1 = -1\right\rangle\left|j_2 = 1,\ m_2 = 0\right\rangle + \left|j_1 = 1,\ m_1 = -1\right\rangle\left(l_-^2\left|j_2 = 1,\ m_2 = 0\right\rangle\right)\right\} =$$

$$\frac{1}{\sqrt{2}}\left\{\hbar\sqrt{1(1+1) - 0(0-1)}\left|j_1 = 1,\ m_1 = -1\right\rangle\left|j_2 = 1,\ m_2 = -1\right\rangle + 0 +\right.$$

$$\left. 0 + \hbar\sqrt{1(1+1) - 0(0-1)}\left|j_1 = 1,\ m_1 = -1\right\rangle\left|j_2 = 1,\ m_2 = -1\right\rangle\right\} =$$

$$\frac{1}{\sqrt{2}}\left\{\hbar\sqrt{2}\left|j_1 = 1,\ m_1 = -1\right\rangle\left|j_2 = 1,\ m_2 = -1\right\rangle + \hbar\sqrt{2}\left|j_1 = 1,\ m_1 = -1\right\rangle\left|j_2 = 1,\ m_2 = -1\right\rangle\right\} =$$

$$2\hbar\left|j_1 = 1,\ m_1 = -1\right\rangle\left|j_2 = 1,\ m_2 = -1\right\rangle$$

(b)

Comparing (a) and (b) we must have (a)=(b) thus,

$$2\hbar\left|j_1 = 1, j_2 = 1, J = 2,\ M = -2\right\rangle = 2\hbar\left|j_1 = 1,\ m_1 = -1\right\rangle\left|j_2 = 1,\ m_2 = -1\right\rangle \Rightarrow$$

$$\left|j_1 = 1, j_2 = 1, J = 2,\ M = -2\right\rangle = \left|j_1 = 1,\ m_1 = -1\right\rangle\left|j_2 = 1,\ m_2 = -1\right\rangle$$

2. Three particles with spin $s_1 = s_2 = s_3 = 1/2$ make up a system. A) Calculate the total spin of the system. (Hint: consider that the three particles are combined as follows: first you combine the two of them and then this system with the third one). B) If the three particles interact with the Hamiltonian $H = A[\mathbf{s}_1 \cdot \mathbf{s}_2 + \mathbf{s}_1 \cdot \mathbf{s}_3 + \mathbf{s}_2 \cdot \mathbf{s}_3]$, where A is a given constant. Find the possible energy eigenvalues. (Hint: start with the total angular momentum $\mathbf{S}^2 = (\mathbf{s_1} + \mathbf{s_2} + \mathbf{s_3})^2$. You are given that for any type of angular momentum ($K = J$, j, l, s): $K^2 |k,m\rangle = \hbar^2 k(k+1)|k,m\rangle$.

(10 marks)

Solution:

A) We combine first the two spins s_1, s_2 and we get:

$$s_{1,2} = |s_1 - s_2| \ldots |s_1 + s_2| = \left|\frac{1}{2} - \frac{1}{2}\right| \ldots \left|\frac{1}{2} + \frac{1}{2}\right| = 0,\ 1.$$

Then we combine this with the third one:

i) For $s_{1,2} = 0$ we have $S = |s_{1,2} - s_3| \ldots |s_{1,2} + s_3| = \left|0 - \frac{1}{2}\right| \ldots \left|0 + \frac{1}{2}\right| = \frac{1}{2}$.

ii) For $s_{1,2} = 1$ we have $S = |s_{1,2} - s_3| \ldots |s_{1,2} + s_3| = \left|1 - \frac{1}{2}\right| \ldots \left|1 + \frac{1}{2}\right| = \frac{1}{2},\ \frac{3}{2}$.

B) For the calculation of the energy we have that:

$$\mathbf{S}^2 = (\mathbf{s_1} + \mathbf{s_2} + \mathbf{s_3})^2 \Rightarrow \mathbf{S}^2 = \mathbf{s}_1^2 + \mathbf{s}_2^2 + \mathbf{s}_3^2 + 2\mathbf{s_1} \cdot \mathbf{s}_2 + 2\mathbf{s}_2 \cdot \mathbf{s}_3 + 2\mathbf{s_1} \cdot \mathbf{s}_3 \Rightarrow$$

$$2\mathbf{s_1} \cdot \mathbf{s}_2 + 2\mathbf{s}_2 \cdot \mathbf{s}_3 + 2\mathbf{s_1} \cdot \mathbf{s}_3 = \mathbf{S}^2 - \mathbf{s}_1^2 - \mathbf{s}_2^2 - \mathbf{s}_3^2 \Rightarrow$$

$$\mathbf{s_1} \cdot \mathbf{s}_2 + \mathbf{s}_2 \cdot \mathbf{s}_3 + \mathbf{s_1} \cdot \mathbf{s}_3 = \frac{1}{2}\left(\mathbf{S}^2 - \mathbf{s}_1^2 - \mathbf{s}_2^2 - \mathbf{s}_3^2\right)$$

Thus the energy eigenvalues are:

$$H = \frac{A\hbar^2}{2}\left[S(S+1) - s_1(s_1+1) - s_2(s_2+1) - s_3(s_3+1)\right].$$ So we have:

i) For $S = 1/2$ we have

$$H = \frac{A\hbar^2}{2}\left[\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}+1\right)\right] = \frac{A\hbar^2}{2}\left(-\frac{3}{2}\right) = -\frac{3A\hbar^2}{4}.$$

ii) For $S = 3/2$ we have

$$H = \frac{A\hbar^2}{2}\left[\frac{3}{2}\left(\frac{3}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}+1\right)\right]=$$

$$\frac{A\hbar^2}{2}\left(\frac{15}{4}-\frac{3}{4}-\frac{3}{4}-\frac{3}{4}\right)=\frac{3A\hbar^2}{4}.$$

3. We know that for a simple harmonic oscillator the unperturbed eigenfunctions and eigenenergies of the system are: $\psi_n^{(0)} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-m\omega x^2/2\hbar}$ and $E_n^{(0)} = \left(n+\frac{1}{2}\right)\hbar\omega$.

We add a small perturbation to the system given by $V(x)=\lambda^2\delta(x)$. Find the first order corrections of the energy eigenvalues.
You are given:

$$E_n^{(1)} = \left\langle \psi_n^{(0)} \middle| V\psi_n^{(o)}\right\rangle = \int_{-\infty}^{+\infty}\left(\psi_n^{(0)}\right)^* V\psi_n^{(0)}dx,\ \int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0).$$

$$H_{2n}(0) = (-1)^n\frac{(2n)!}{n!},\ H_{2n+1}(0)=0.$$

(10 marks)

Solution:

$$E_n^{(1)} = \int_{-\infty}^{+\infty}\left(\psi_n^{(0)}(x)\right)^* V\psi_n^{(0)}(x)dx = \int_{-\infty}^{+\infty}\left(\psi_n^{(0)}(x)\right)^*\lambda^2\delta(x)\psi_n^{(0)}(x)dx = \int_{-\infty}^{+\infty}\lambda^2\delta(x)\left|\psi_n^{(0)}(x)\right|^2dx =$$

$$\lambda^2\int_{-\infty}^{+\infty}\lambda^2\delta(x)\left|\psi_n^{(0)}(x)\right|^2dx = \lambda^2\left|\psi_n^{(0)}(0)\right|^2$$

But for the wave-function we have:

$$\psi_n^{(0)}(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-m\omega x^2/2\hbar} \Rightarrow \psi_n^{(0)}(0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(0) \Rightarrow$$

$$\left|\psi_n^{(0)}(0)\right|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{1}{2^n n!}\left[H_n(0)\right]^2$$

Thus:

$$E_n^{(1)} = \lambda^2\left|\psi_n^{(0)}(0)\right|^2 = \lambda^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{1}{2^n n!}\left[H_n(0)\right]^2.$$

i) If n is even (let $n= 2m$, $m =0, 1, 2,\ldots$) *the* energy shift is given by:

$$E_{2m}^{(1)} = \lambda^2 \left|\psi_n^{(0)}(0)\right|^2 = \lambda^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{1}{2^n n!}\left[H_n\left(0\right)\right]^2 =$$

$$\lambda^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{1}{2^{2m}\left(2m\right)!}\left[\left(-1\right)^m \frac{\left(2m\right)!}{m!}\right]^2 = \lambda^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{\left(2m\right)!}{2^{2m}\left(m!\right)^2}$$

ii) If n is odd the energy shift is zero.

Thus only even states experience a first order correction to the energy.

4. A) For the scattering in the spherically symmetric potential $V(\mathbf{r}) = A\delta\left(r-R\right)$, (where $A>0$ a positive constant and $\delta\left(r-R\right)$ the δ function, calculate in the Born approximation the quantities $f_B\left(\theta\right)$ and $d\sigma/d\Omega$.

(8 marks)

You are given that:

$$\int_0^\infty f\left(r\right)\delta\left(r-R\right)dr = f(R)$$

$$\frac{d\sigma}{d\Omega} = \left|f\left(\theta\right)\right|^2 \ , \ f\left(\theta\right) = -\frac{2m}{q\hbar^2}\int_0^\infty rV(r)\sin\left(qr\right)dr, \quad q = 2k\sin\left(\theta/2\right)$$

(10 marks)

Solution:

$$f\left(\theta\right) = -\frac{2m}{q\hbar^2}\int_0^\infty rV(r)\sin\left(qr\right)dr = -\frac{2m}{q\hbar^2}\int_0^\infty rA\delta\left(\mathbf{r}-\mathbf{R}\right)\sin\left(qr\right)dr =$$

$$-\frac{2mA}{q\hbar^2}\int_0^\infty r\delta\left(\mathbf{r}-\mathbf{R}\right)\sin\left(qr\right)dr = -\frac{2mA}{q\hbar^2}R\sin(qR)$$

Thus

$$\frac{d\sigma}{d\Omega} = \left|f\left(\theta\right)\right|^2 = -\left(\frac{2mA}{q\hbar^2}R\sin(qR)\right)^2 = \frac{4m^2A^2}{q^2\hbar^4}R^2\sin^2(qR) =$$

$$\frac{4m^2A^2}{4k^2\hbar^4\sin^2\left(\theta/2\right)}R^2\sin^2(2kR\sin\left(\theta/2\right))$$