KING SAUD UNIVERSITY DEPARTMENT OF MATHEMATICS
TIME: 3H, FULL MARKS: 40, SII /22/08/1438, MATH 204

**Question 1.** [4,5] a) Show that $\mu(x, y) = x^{-2}y^{-3}$ is an integrating factor for the differential equation

$$(x^3y - y^2)dx - (x^4 + xy)dy = 0, \quad x > 0, \ y > 0.$$

b) A mass of radio active material was left in a lab. After 1 year the mass decreased by 4% and after 10 years it was found that 80 gram of the material was left. Find the initial mass and the half life of this material.

**Question 2.** a) [4,3]. Solve the initial value problem

$$\frac{dy}{dx} = \frac{y^3 + 2x^2y}{xy^2 + x^3}, \quad y(1) = 1.$$

b) Find a linear differential equation that has the general solution

$$y = c_1e^x + c_2xe^x + c_3\cos 2x + c_4\sin 2x$$

**Question 3.** a) [4,4]. Use the undetermined coefficients method to solve the second order differential equation

$$y'' - 2y' + 2y = 4x - \cos x$$

b) Solve the differential equation

$$x^2y'' - 2y = 2\ln x, \quad x > 0.$$

**Question 4** [5,5]. a) Find the first six terms in a power series expansion about the ordinary point $x_0 = 0$ for a general solution to the equation

$$y'' = xy.$$

b) Use the method of elimination to solve the system of differential equations

$$\begin{cases} x' = -3x + 2y \\ y' = -3x + 4y \end{cases}$$

**Question 5.** [6]. Let $f : \mathbb{R} \to \mathbb{R}$ be the $2\pi-$periodic function defined by

$$f(x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$$

Sketch the graph of $f$ over $(-3\pi, 3\pi)$. Find the Fourier series of the function $f$ and deduce the value of the series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

1.

## Answer Sheet

Q1 a) Multiply the DE by $\mu(x,y) = x^{-2}y^{-3}$, we get

$$\underbrace{(xy^{-2} - x^{-2}y^{-1})}_{M}\,dx - \underbrace{(x^{2}y^{-3} + x^{-1}y^{-2})}_{N}\,dy = 0$$

$\frac{\partial M}{\partial y} = -2xy^{-3} + x^{-2}y^{-2}$, $\frac{\partial N}{\partial x} = -2xy^{-3} + x^{-2}y^{-2}$ (1)

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (The DE is exact) $\Rightarrow \exists F(x,y)$

$$\begin{cases} \frac{\partial F}{\partial x} = xy^{-2} - x^{-2}y^{-1} \longrightarrow (1) \\ \frac{\partial F}{\partial y} = -x^{2}y^{-3} - x^{-1}y^{-2} \longrightarrow (2) \end{cases}$$ (1)

From (1), $F(x,y) = \frac{x^2}{2}y^{-2} + x^{-1}y^{-1} + \alpha(y) \longrightarrow (3)$ (1)

From (3) $\frac{\partial F}{\partial y} = -x^{2}y^{-3} - x^{-1}y^{-2} + \alpha'(y) \longrightarrow (4)$

From (2) and (4), we have $\alpha'(y) = 0 \Rightarrow \alpha(y) = C_1$ (1)

Hence $\frac{x^2}{2}y^{-2} + x^{-1}y^{-1} = C$ is the sol of the DE.

b) Let $A(t)$ be the mass present at time $t$.
and $A_0$ is the initial mass.

$A(1) = A_0 - \frac{4}{100}A_0 = 0.96A_0 \longrightarrow ①$ (1)

$A(10) = 80 \longrightarrow ②$ (1)

Since $A(t) = A_0 e^{kt}$

$① \Rightarrow 0.96A_0 = A_0 e^{k} \Rightarrow k = \ln 0.96 \simeq -0.041$ (1)

Thus $A = A_0 e^{-0.041t}$

$② \Rightarrow 80 = A_0 e^{-0.41} \Rightarrow A_0 = 80e^{0.41} \simeq 120.55.$ (1)

Half life: $\frac{1}{2} = e^{-0.041t} \Rightarrow t = \frac{\ln 0.5}{-0.041} \simeq 16.91$ years.

(1)

$\underline{Q_2}$ a): $y' = \frac{y^3+2x^2y}{xy^2+x^3}$ the numerator and the denomentor have the Same degree, so the DE is homogeneous.

$$\frac{dy}{dx} = y' = \frac{\left(\frac{y}{x}\right)^3+2\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^2+1}, \text{ Let } u = \frac{y}{x} \Rightarrow y' = xu'+u. \quad 1$$

Hence $xu'+u = \frac{u^3+2u}{u^2+1} \Rightarrow x\frac{du}{dx} = \frac{u}{u^2+1} \Rightarrow \frac{(u^2+1)}{u}du = \frac{dx}{x}$

$$\Rightarrow \int\left(u+\frac{1}{u}\right)du = \ln|x|+C \Rightarrow \frac{u^2}{2}+\ln|u| = \ln|x|+C \quad 1$$

$$\Rightarrow \frac{y^2}{2x^2}+\ln\left|\frac{y}{x}\right|-\ln|x| = C \Rightarrow \frac{y^2}{2x^2}+\ln\left(\frac{|y|}{x^2}\right) = C \quad 1$$

$$\Rightarrow \operatorname{Exp}\left\{\frac{y^2}{2x^2}\right\}\cdot\frac{|y|}{x^2} = C^* \Rightarrow \exp\left\{\frac{y^2}{2x^2}\right\}\frac{y}{x^2} = \pm C^* = C_1$$

Since $y(1)=1$, $C_1 = \sqrt{e}$

Hence $\exp\left\{\frac{y^2}{2x^2}\right\}\frac{y}{x^2} = \sqrt{e}$. $\quad 1$

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$Q_2$. b) $m_1 = 1$, $m_2 = 1$, $m_3 = 2i$, $m_4 = -2i$. $\quad 1$

The characteristic equation is $(m-1)^2(m-2i)(m+2i) = 0$

After multiplication, we get: $m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$ $\quad 1$

Thus the DE is $y^{(4)} - 2y^{(3)} + 5y'' - 8y' + 4y = 0$ $\quad 1$

$\underline{Q_3}$ a) $y'' - 2y' + 2y = 4x - \cos x$

$$y_g = y_{gh} + y_p$$

$y_{gh} = ?$ Ch Eq: $m^2 - 2m + 2 = 0 \Rightarrow m_1 = 1 + i,\ m_2 = 1 - i$

$$y_{gh} = e^x \left( C_1 \cos x + C_2 \sin x \right).$$ 1

$$y_p = Ax + B + C\cos x + D \sin x$$ 1

$$y_p' = A - C \sin x + D \cos x$$

$$y_p'' = -C \cos x - D \sin x$$

By substitution in the DE:

$$-C\cos x - D\sin x - 2A + 2C\sin x - 2D\cos x + 2Ax + 2B + 2C\cos x + 2D\sin x = 4x - \cos x$$

$$\Leftrightarrow \cos x (C - 2D) + \sin x (D + 2C) - 2A + 2B + 2Ax = 4x - \cos x$$

Equating, we get $A = 2,\ B = 2,\ C = -\frac{1}{5},\ D = \frac{2}{5}$

Hence $y_g = e^x \left( C_1 \cos x + C_2 \sin x \right) + 2x + 2 - \frac{1}{5}\cos x + \frac{2}{5}\sin x$

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$\underline{Q_3}$ b) : $x^2y'' - 2y = 2\ln x, \quad x>0$

$$y_g = y_h + y_p$$

$x^2y''-2y=0$ : the Ch Eq : $m^2-m-2=0 \Rightarrow m_1=-1, m_2=2$

$$\Rightarrow y_h = C_1 x^{-1} + C_2 x^2,$$ (1)

$y_p = C_1(x)\,x^{-1} + C_2(x)\,x^2$, where

$$\begin{cases} C_1'(x)\,x^{-1} + C_2'(x)\,x^2 = 0 \\ -C_1'(x)\,x^{-2} + 2C_2'(x)\cdot x = \dfrac{2\ln x}{x^2} \end{cases}$$

$$W = \begin{vmatrix} x^{-1} & x^2 \\ -x^{-2} & 2x \end{vmatrix} = 3,$$

$$C_1'(x) = \begin{vmatrix} 0 & x^2 \\ \frac{2\ln x}{x^2} & 2x \end{vmatrix} / 3 = -\frac{2\ln x}{3},$$ (1)

$$\Rightarrow C_1(x) = -\frac{2}{3}(x\ln x - x),$$

$$C_2'(x) = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{2\ln x}{x^2} \end{vmatrix} / 3 = \frac{2}{3}\frac{\ln x}{x^3}$$ (1)

$$\Rightarrow C_2(x) = \frac{2}{3}\int \frac{\ln x}{x^3}\,dx = -\frac{1}{3x^2}\ln x - \frac{1}{6x^2},$$

Hence $y_g = C_1x^{-1} + C_2x^2 - \frac{2}{3}(x\ln x - x)x^{-1} - \left(\frac{1}{3x^2}\ln x + \frac{1}{6x^2}\right)x^2$

$$\boxed{y_g = C_1x^{-1} + C_2x^2 + \frac{1}{2} - \ln x}$$ (1)

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$\underline{Q_4}$ a) $y'' - xy = 0, \quad y = \sum_{n=0}^{\infty} a_n x^n,\ y' = \sum_1^{\infty} n a_n x^{n-1},\ y'' = \sum_2^{\infty} n(n-1)a_n x^{n-2}$

Hence: $\sum_2^{\infty} n(n-1)a_n x^{n-2} - \sum_0^{\infty} a_n x^{n+1} = 0$

$$\Longleftrightarrow \sum_0^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_1^{\infty} a_{n-1}x^n = 0$$ (1)

$$\Longleftrightarrow 2a_2 + \sum_1^{\infty}\left[(n+2)(n+1)a_{n+2} - a_{n-1}\right]x^n = 0$$

$$\Rightarrow a_2 = 0, \qquad a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)},\ n \geq 1$$ (2)

$\underline{n=1}$ : $a_3 = \frac{a_0}{6}$

$\underline{n=2}$: $a_4 = \frac{a_1}{12}$, $\underline{n=3}$ ; $a_5 = \frac{a_2}{20} = 0$, $\underline{n=4}$ $a_6 = \frac{a_0}{180}$

Hence $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$ 2

$$= a_0 \left[ 1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots \right] + a_1 \left[ x + \frac{x^4}{12} + \cdots \right]$$

$\underline{Q_4}$ b) operator form $\begin{cases} (D+3)[x] - 2y = 0 \longrightarrow (1) \\ 3x + (D-4)[y] = 0 \longrightarrow (2) \end{cases}$ 1

To eliminate for example $x$, we apply $(D+3)$ to $(2)$

and multiply (1) by $-3$, we obtain:

$$(D^2 - D - 6)[y] = 0 \Rightarrow y'' - y' - 6y = 0 \quad 1$$

$$m^2 - m - 6 = 0 \Rightarrow m_1 = 3, \quad m_2 = -2$$

$$y(t) = c_1 e^{3t} + c_2 e^{-2t} \quad 1 \; ; \quad y' = 3c_1 e^{3t} - 2c_2 e^{-2t}$$

From (2), we have $x(t) = \frac{-y'}{3} + \frac{4y}{3}$

$$= -\frac{1}{3}\left(3c_1 e^{3t} - 2c_2 e^{-2t}\right) + \frac{4}{3}\left(c_1 e^{3t} + c_2 e^{-2t}\right)$$

$$\Rightarrow x(t) = \frac{c_1}{3} e^{3t} + 2c_2 e^{-2t} \quad 2$$

Q5: Since f, and f' are piecewise continuous on $(-\pi, \pi)$ and since $f(x+2\pi) = f(x)$, then f has the Fourier series 1

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} (x+\pi)\,dx = \frac{1}{2\pi}(x+\pi)^2 \Big|_{-\pi}^{0} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} (x+\pi)\cos nx\,dx = \frac{1}{\pi}\left[(x+\pi)\frac{\sin nx}{n}\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\sin nx}{n}\,dx\right]$$

$$= -\frac{1}{\pi n}\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0} = \frac{1}{\pi n^2}\left(1-(-1)^n\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (x+\pi)\sin nx\,dx = \frac{1}{\pi}\left[-(x+\pi)\frac{\cos nx}{n}\Big|_{-\pi}^{0} + \int_{-\pi}^{0}\frac{\cos nx}{n}\,dx\right]$$

$$= -\frac{1}{n} - \frac{\sin nx}{1^2}\Big|_{-\pi}^{0} = -\frac{1}{n}$$

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Hence
$$f(x) \simeq \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)}{(2n+1)^2} - \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

Let $x = 0$ 1

$$f(0) = \frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

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