King Saud University
Department of Mathematics
1st Semester 1433-1434 H

MATH 253-MATH 352 (Numerical Analysis)
Final Exam
Duration: 3 Hours

Student's Name	Student's ID	Lecturer's Name

Question No.	I	II	III	IV	V	VI	Total
Mark							

[**I**] Determine whether the following is **True** or **False**. **Justify** your answer.

(a) If the divided differences $f[x_0, x_1, x_2] = 5$ and $f[x_0, x_1] = 2$ are given for $x_0 = 1$, $x_1 = 2$ and $x_2 = 4$, then $f[x_1, x_2] = 12$. (   )

(b) The sequence $\mathbf{x}^{(\mathbf{k})} = \left(ke^{-k}, e^{-k}\sin k, \frac{k-1}{k+1}\right)$ converges to $(0, 0, 1)$ as $k \to \infty$. (   )

(c) $g(x) = \sqrt{\frac{x+2}{x^2+1}}$ has a fixed point at p, where p is a root of $f(x) = x^4 + x^2 - x - 2$. (   )

OVER

(d) If $|c| < 3$ for $A = \begin{bmatrix} 4 & -1 & c \\ c & 6 & 2 \\ 3 & 1 & 5 \end{bmatrix}$, then Gauss-Seidel method for solving $A\mathbf{x} = \mathbf{b}$ is convergent for any initial vector $\mathbf{x^{(0)}}_{3\times 1}$ and any $\mathbf{b}_{3\times 1}$. (   )

(e) If the Trapezoidal rule approximation of $I =: \int_0^2 f(x)dx$ is 6 and the Simpson's rule approximation of I is 7, then the Midpoint rule approximation of I is 6.5. (   )

(f) The bisection method for root-finding generates a sequence $\{p_n\}$ approximating p with rate of convergence $O\left(2^{-n}\right)$. (   )

OVER

[II] Use the data in the following table to answer all parts of this question.

x	1	2	3	4
$f(x)$	0	0.6931	1.098	1.386

(a) Approximate $f(2.5)$ using a Lagrange polynomial of degree 2.

(b) If $|f'''(\zeta)| < 2$ for $1 < \zeta < 4$, find a bound for the error of your approximation in (a).

(c) Approximate $f'(3)$ using a 3-point formula.

OVER

[III] For $f(x) = x^3 - 3x + 2$,

(a) Why does Newton's method for finding the root $p = 1$ of f converges only linearly?

(b) Use a modified Newton's method that converges quadratically to approximate the root $p = 1$ of f with accuracy 10^{-3} and $p_0 = 1.6$.

OVER

[IV] For $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix}$,

(a) Find P, L and U that satisfies $PA = LU$, where P is a permutation matrix, L and U are lower and upper triangular matrices, respectively.

(b) Can you factorize A as $A = LDL^T$, where D and L are diagonal and lower triangular matrices, respectively? Justify your answer.

OVER

[V] For the system $A\mathbf{x} = \mathbf{b}$ with $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$,

(a) Show that A is positive definite.

(b) Use Jacobi method with $\mathbf{x}^{(\mathbf{0})} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ to compute the second approximation $\mathbf{x}^{(\mathbf{2})}$ of the system's solution.

(c) Estimate the number of iterations needed to solve the system by Jacobi method with accuracy 10^{-4}.

OVER

[VI]

(a) Suppose that $\widetilde{\mathbf{x}}$ is an approximation to the solution of $A\mathbf{x} = \mathbf{b}$, A is nonsingular and $\mathbf{r}$ is the residual vector for $\widetilde{\mathbf{x}}$. Prove that if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, then

$$\frac{\|\mathbf{x} - \widetilde{\mathbf{x}}\|}{\|\mathbf{x}\|} \leq K(A)\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}.$$

(b) For $A = \begin{bmatrix} 1.0001 & 1 \\ 0.5 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3.0002 \\ 2 \end{bmatrix}$,

(i) Compute the exact solution $\mathbf{x}$ for $A\mathbf{x} = \mathbf{b}$ by Gaussian elimination with partial pivoting.

(ii) Is A ill-conditioned? Justify your answer.

(iii) **(BONUS)** Show that if $\|\mathbf{r}\| < \epsilon$ then $\|\mathbf{x} - \widetilde{\mathbf{x}}\| < 5.4\epsilon$, for any $\epsilon > 0$.

Good Luck