King Saud University
Department of Mathematics
1st Semester 1435-1436 H

MATH 244 (Linear Algebra)
Final Exam
Duration: 3 Hours

Student's Name	Student's ID	Group No.

Question No.	I	II	III	IV	V	VI	Total
Mark							

[**I**] **Determine** whether the following is **True** or **False**. [9 *Points*]

(1) If $A = \begin{bmatrix} 3 & -1 \\ 0 & -2 \end{bmatrix}$, then $A^2 - A - 6I_2 = 0_{2\times 2}$. ( )

(2) If $(X - I_2)^{-1} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$, then $X = \begin{bmatrix} 2 & -3 \\ -2 & 6 \end{bmatrix}$. ( )

(3) The matrix $\begin{bmatrix} 5 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ is invertible. ( )

(4) If the matrix $\begin{bmatrix} 1 & x+4 & 1 \\ 3 & 0 & 4 \\ 1 & x+y & 2 \end{bmatrix}$ is symmetric, then $x = -1$ and $y = 5$. ( )

(5) If $V = \mathbb{R}^2$ with the following addition and scalar multiplication on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:
$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$, $k\mathbf{u} = (2ku_1, 2ku_2)$, then V is a vector space. ( )

(6) $W = \{A \in M_{nn}, tr(A) = 0\}$ is a subspace of M_{nn}. ( )

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(7) $\left\{x^2+1, -x^2+1, 6x^2+5x+2\right\}$ spans P_2. ( )

(8) $S=\{(0,-1,1),(2,1,3),(1,2,0)\}$ is linearly independent. ( )

(9) $\{x, \cos x\}$ is linearly dependent. ( )

(10) If $W=\{(a,b,c,d): a+b=d, a-b=c\}$, then $dim(W)=3$. ( )

(11) If B is a 7×5 matrix, then $rank(B)\leqslant 6$. ( )

(12) If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in a vector space V, then $\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2$.. ( )

[II] Choose the correct answer. [8 *Points*]

(1) If $\det(2AB^T)=8,\ A=\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}$ and B is a 2×2 matrix, then $\det(B)$ equals.

(a) $\frac{1}{2}$ **(b)** $\frac{1}{4}$ **(c)** 4 **(d)** None of the previous

(2) The condition on b_1, b_2 and b_3 which guarantees that the following system is consistent

$$\begin{aligned} x_1-2x_2+5x_3 &= b_1 \\ 4x_1-5x_2+8x_3 &= b_2 \\ -3x_1+3x_2-3x_3 &= b_3 \end{aligned}$$

is

(a) $b_3+b_2-b_1=0$ **(b)** $b_3-b_2+b_1=0$ **(c)** $b_2-4b_1=0$ **(d)** None of the previous

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(3) If $A^{-1} = \begin{bmatrix} -3 & -2 & 2 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$, then $adj(A)$ equals

(a) $\frac{1}{2}\begin{bmatrix} -3 & -2 & 2 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ **(b)** $2\begin{bmatrix} -3 & -2 & 2 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 2 & 0 \\ -5/2 & -4 & 1/2 \\ -1/2 & -1 & 1/2 \end{bmatrix}$ **(d)** None of the previous

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(4) If $S = \{x+1, x+2, x^2\}$ is a basis for P_2 and the coordinate vector of $\mathbf{p}(x) \in P_2$ is given by $(\mathbf{p})_S = (1, 2, 3)$, then

(a) $\mathbf{p}(x) = 1 + 2x + 3x^2$ **(b)** $\mathbf{p}(x) = 3 + 2x + 3x^2$ **(c)** $\mathbf{p}(x) = 5 + 3x + 3x^2$ **(d)** None of the previous

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(5) If $T(x_1, x_2, x_3) = (x_1 + 2x_2, -x_3, 4x_1 - x_3)$, then $[T]$ is given by

(a) $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 4 & 0 & -1 \end{bmatrix}$ **(d)** None of the previous

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(6) The image of the vector $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ if its rotated $60°$ about the z-axis is

(a) $\begin{bmatrix} 1 \\ \sqrt{3} \\ -2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 \\ -\sqrt{3} \\ -2 \end{bmatrix}$ **(c)** $\begin{bmatrix} 2 \\ \sqrt{3} \\ -1 \end{bmatrix}$ **(d)** None of the previous

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(7) If T_1 is the reflection operator about the line $y = x$ in $\mathbb{R}^2$ and T_2 is the orthogonal projection on the y-axis in $\mathbb{R}^2$, then $[T_1 \circ T_2]$ is

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ **(d)** None of the previous

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[III] [7 *Points*]

(i) Let $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -1 & 2 & -1 \\ 1 & -3 & 4 & -5 \end{bmatrix}$.

(a) **Find** a basis for the solution space of $A\mathbf{x} = \mathbf{0}$.

(b) **Evaluate** $nullity(A)$ and $nullity(A^T)$.

(ii) Let $S = \{(1, 1, 2, 3), (2, 3, 1, 0), (1, 3, -4, -9)\}$.

(a) **Find** a subset of S that forms a basis for the subspace $W = Span(S)$.

(b) **Express** each vector not in the basis as a linear combination of the basis vectors.

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[IV] [8 *Points*]

(i) Let $A = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) **Find** the eigenvalues and bases for the corresponding eigenspaces of A.

(b) **What** are the eigenvalues of A^3.

(c) **Is** A^3 invertible? **Justify** your answer.

(ii) For $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$, **show** that $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 2$.

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[V] [5 *Points*]

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a matrix transformation with $[T] = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) **Prove** that T is one-to-one.

(b) **Compute** $T^{-1}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$

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[VI] [6 *Points*]

(i) Let $\begin{bmatrix} a & 0 & b & | & 2 \\ a & a & 4 & | & 4 \\ 0 & a & 2 & | & b \end{bmatrix}$ be the augmented matrix for a linear system. **Find** the values for a and b for which the system has infinitely many solutions.

(ii) (**BONUS**)

(a) **Find** the values of a for which the following system has a unique solution

$$\begin{aligned} 6ax + 4y &= 5 \\ 9x + 2ay &= -2 \end{aligned}$$

(b) **Find** a unit vector $\mathbf{u}$ that is orthogonal to both $(1, 2, 0)$ and $(-1, 0, 2)$.

Good Luck