King Saud University
Department Of Mathematics.
M-203 [Final Examination]
(Differential and Integral Calculus)
(II-Semester 1439/1440)

Max. Marks: 40 Time: 3 hrs

Marking Scheme: **Q1[4+4+4]; Q2[4+4+4]; Q3[4+4+4+4].**

Q. No: 1 (a) Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty} \frac{n^2-1}{2^n(n^2+5n)}$.

(b) Find the interval of convergence and radius of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{\ln^2(n)}{n}(x-1)^n.$$

(c) Find the MacLaurin series for $f(x) = \cos(x^3)$ and use its first three non-zero terms to approximate the integral $\int_0^1 \frac{1-\cos(x^3)}{x^6}\, dx$.

Q. No: 2 (a) A lamina has the shape of the region bounded by the graphs of the semi-circle $x = \sqrt{1-y^2}$ and the y-axis. If the density at a point $P(x,y)$ is directly proportional to the distance of P to the origin, find the moment of inertia about the y-axis.

(b) Find the centroid of the solid bounded by the graphs of the equations $z = 9 - x^2 - y^2$ and $z = 1 + x^2 + y^2$.

(c) Evaluate the triple integral by changing to spherical coordinates:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} (x^2+y^2+z^2)^{\frac{3}{2}} dz dy dx.$$

Q. No: 3 (a) Using Green's theorem, calculate the line integral $\oint_C y^2 dx + (x+y)^2\, dy$ where C is the triangle with vertices $A(1,0), B(1,1)$, and $D(0,1)$.

(b) Show that the following integral is independent of path and evaluate it:

$$\int_{(1,0,0)}^{(3,\frac{\pi}{2},1)} (2x\sin y + e^{3z})dx + (x^2\cos y)dy + (3xe^{3z}+5)dz$$

(c) Use the divergence theorem and cylindrical coordinates to find the Flux integral of the vector field $\vec{F}(x,y,z) = xy\,\vec{i} + yz\vec{j} + zx\vec{k}$ over the boundary S of the closed region Q bounded by the graphs of the equations $x^2 + y^2 = 1, z = 0$, and $z = 2$. (Provided S is oriented by the unit normal directed outward).

(d) Use Stoke's theorem to evaluate the line integral $\oint_C \vec{F} \cdot \overrightarrow{dr}$ for the vector field $\vec{F}(x,y,z) = (y-z)\,\vec{i} + (z-x)\vec{j} + (x-y)\vec{k}$, C is the boundary of the part of the plane $2x + 3y + z = 6$ in the first octant oriented in a counterclockwise direction when viewed from above.

M- 203

II - Semester (1439/1440) Final Exam.

Max: Marks: 40 (Solutions) Time: 3 Hours

Q# 1(a) Determine the convergence or divergence of the Series: $\sum_{n=1}^{\infty} \frac{n^2-1}{2^n(n^2+5n)}$ [Marks: 4]

Soln. Let $a_n = \frac{n^2-1}{2^n(n^2+5n)}$ and $b_n = \frac{1}{2^n}$

Applying LCT: $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \neq 0$ and (1)

$\sum b_n = \sum \frac{1}{2^n}$ is a conv. Geom. series. $r=\frac{1}{2}<1$ (2)

Hence by LCT, $\sum_{n=1}^{\infty} \frac{n^2-1}{2^n(n^2+5n)}$ is also conv. (1)

Q# 1(b) Find the Interval of Convergence and radius of Convergence of the power series $\sum_{n=1}^{\infty} \frac{\ln^2(n)}{n}(x-1)^n$. [Marks: 4]

Soln. we apply absolute Ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty} \left| \frac{\ln^2(n+1)}{n+1}(x-1)^{n+1} \cdot \frac{n}{\ln^2(n)} \cdot \frac{1}{(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left(\frac{\ln(n+1)}{\ln(n)}\right)^2 |x-1| = |x-1|$$

For abs. conv. $|x-1|<1 \iff -1<x-1<1$

$\iff 0<x<2$; $(0,2)$ (1)

At $x=0$, we have $\sum_{n=1}^{\infty} \frac{\ln^2(n)}{n}(-1)^n$, which is conv by AST (1)

At $x=2$, we have $\sum_{n=1}^{\infty} \frac{\ln^2(n)}{n}$ which by Integral test

$\lim_{t\to\infty} \int_1^t \ln^2(x)\cdot\frac{1}{x}\,dx = \lim_{t\to\infty} \left[\frac{\ln^3(x)}{3}\right]_1^t = \infty$; Divg. (1)

$\therefore$ Interval of conv: $[0,2)$ and radius of Conv: $\frac{2-0}{2} = 1$ (1)

Q#1(c). Find the Maclaurin Series for $f(x) = \cos(x^3)$ and use its first three non-zero terms to approximate the integral $\int_0^1 \frac{1-\cos(x^3)}{x^6}\,dx$. [Marks: 4]

Soln. we have $f(x) = \cos x \Rightarrow f(0) = 1$

$$f'(x) = -\sin x \Rightarrow f'(0) = 0$$
$$f''(x) = -\cos x \Rightarrow f''(0) = -1$$
$$f'''(x) = +\sin x \Rightarrow f'''(0) = 0$$
$$f^{(iv)}(x) = \cos x \Rightarrow f^{(iv)}(0) = 1$$

Hence $f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{2n!} + \cdots$ ②

$$\therefore \cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{2n!}$$

$$\cos(x^3) = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} + \cdots = 1 - \frac{x^6}{2} + \frac{x^{12}}{24} - \cdots \quad ①$$

$$\therefore \int_0^1 \frac{1-\cos(x^3)}{x^6}\,dx = \int_0^1 \frac{1}{x^6}\left[\frac{x^6}{2} - \frac{x^{12}}{24} + \cdots\right]dx = \int_0^1 \left(\frac{1}{2} - \frac{x^6}{24}\right)dx$$

$$= \left[\frac{x}{2} - \frac{x^7}{7 \times 24}\right]_0^1 = \frac{1}{2} - \frac{1}{168} \approx 0.5 - 0.00595$$

$$\approx \underline{\underline{0.494}} \quad ①$$

Q#2(a) A lamina has the shape of the region bounded by the graphs of the semi-circle $x = \sqrt{1-y^2}$ and the y-axis. If the density at a point $P(x, y)$ is directly prop. to the distance of the P from the origin, find the moment of inertia about the y-axis. [Marks: 4]

Soln. Density: $\delta(x, y) = k\sqrt{x^2+y^2}\,dA$ ①

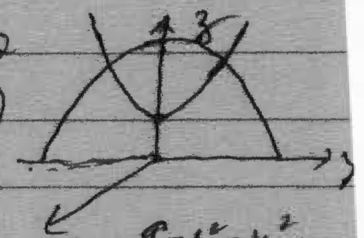
$$\therefore I_y = \iint_R x^2 k\sqrt{x^2+y^2}\,dA = k\int_{-\pi/2}^{\pi/2}\int_0^1 r^2\cos^2\theta\, r.\, r\,dr\,d\theta \quad ②$$

$$= k\int^{\pi/2} \cos^2\theta \left[\frac{r^5}{5}\right]_0^1 d\theta = k.\frac{1}{5}\int^{\pi/2} \cos^2\theta = \frac{1}{\ldots}k\int^{\pi/2}(1+\cos 2\theta \ldots$$

Q#2(b) Find the Centroid of the solid bounded by the graphs of the equations $z = 9 - x^2 - y^2$ and $z = 1 + x^2 + y^2$

[Marks: 4]

Soln. Mass: $m = \int_0^{2\pi}\int_0^{2}\int_{1+r^2}^{9-r^2} r\,dz\,dr\,d\theta$ (2)



$9 - x^2 - y^2 = 1 + x^2 + y^2$

$\therefore 2x^2 + 2y^2 = 8$

$\therefore x^2 + y^2 = 4$

$= \int_0^{2\pi}\int_0^{2} (9 - r^2 - 1 - r^2)\, r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^{2} (8 - 2r^2)\, r\,dr\,d\theta$

$= 2\int_0^{2\pi}\int_0^{2} (4 - r^2)\, r\,dr\,d\theta = 2\int_0^{2\pi}\left[4\frac{r^2}{2} - \frac{r^4}{4}\right]_0^2 d\theta$

$= 16\pi$ (1)

Centroid: $(\bar{x}, \bar{y}, \bar{z})$: By symmetry $\bar{x} = \bar{y} = 0$

$\therefore M_{xy} = \int_0^{2\pi}\int_0^{2}\int_{1+r^2}^{9-r^2} z\, r\,dz\,dr\,d\theta = 80\pi$

$\therefore \bar{z} = \frac{M_{xy}}{m} = \frac{80\pi}{16\pi} = 5$ $\therefore (\bar{x}, \bar{y}, \bar{z}) = (0, 0, 5)$ (1)

Q#2(c) Evaluate the triple integral by changing to Spherical coordinates: $\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} (x^2+y^2+z^2)^{3/2}\,dz\,dy\,dx$

[Marks: 4]

Soln. $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{2}} (\rho^2)^{3/2} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$ (3)

$= \int_0^{\pi/2}\int_0^{\pi/4} \left[\frac{\rho^6}{6}\right]_0^{\sqrt{2}} \sin\varphi\, d\varphi\, d\theta$

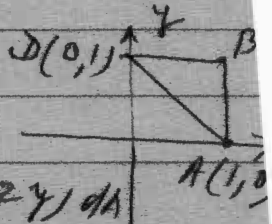
$= \frac{4}{3}\int_0^{\pi/2} \left[-\cos\varphi\right]_0^{\pi/4} d\theta = \frac{4}{3}\int_0^{\pi/2}\left(-\frac{1}{\sqrt{2}} + 1\right) d\theta$

$= \frac{4}{3} \times \frac{\pi}{2}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) = \frac{1}{3}(2-\sqrt{2})\pi$ (1)

Q# 3(a) Using Green's theorem, calculate the line integral

$$\oint_C y^2 dx + (x+y)^2 dy,$$ where C is the triangle with vertices $A(1,0)$, $B(1,1)$, and $D(0,1)$. [Marks: 4

Soln. By Green's theorem we have



$$\oint_C y^2 dx + (x+y)^2 dy = \iint_R \left(2(x+y) - 2y\right) dA$$
$$= \iint_R 2x \, dA \quad (1)$$

Equation: $(1,0)$ to $(0,1)$

$$\frac{y-0}{1-0} = \frac{x-1}{0-1} \quad \therefore y = -x+1$$

Hence $$\iint_R 2x \, dA = 2\int_0^1 \int_{1-x}^1 x \, dy \, dx \quad (2)$$

$$= 2\int_0^1 x(\cancel{1}-\cancel{1}+x)\, dx = 2\int_0^1 x^2 dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3} \approx 0.666 \quad (1)$$

Q# 3(b) Show that the ~~line~~ following integral is indp. of path and evaluate it:

$$\int_{(1,0,0)}^{(3,\frac{\pi}{2},1)} (2x\sin y + e^{3z})dx + (x^2\cos y)dy + (3xe^{3z}+5)dz$$

[Marks: 4]

Soln. Here $M = 2x\sin y + e^{3z}$, $N = x^2\cos y$, $P = 3xe^{3z}+5$

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} = 0; \quad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z} = 3e^{3z}; \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = 2x\cos y \quad (1)$$

Now, $$\vec{F}(x,y,z) = (2x\sin y + e^{3z})\vec{i} + (x^2\cos y)\vec{j} + (3xe^{3z}+5)\vec{k}$$
$$= f_x(x,y,z)\vec{i} + f_y(x^2\cos y)\vec{j} + f_z(x,y,z)\vec{k}$$

Equating, we have $f_x(x,y,z) = 2x\sin y + e^{3z}$ —— (1)

$$f_y(x,y,z) = x^2\cos y \text{ —— } (2)$$

$$f_z(x,y,z) = 3xe^{3z} + 5 \text{ —— } (3)$$

Integrating (1) w.r.t. to x, (2) w.r.t. to y and (3) w.r.t. to z, we

$$f(x,y,z) = x^2\sin y + xe^{3z} + g(y,z) \text{ —— } (4)$$

$$f(x,y,z) = x^2\sin y + g(x,z) \text{ —— } (5)$$

$$f(x,y,z) = xe^{3z} + 5z + g(x,y) \text{ —— } (6)$$

Finally: $f(x,y,z) = x^2\sin y + xe^{3z} + 5z + c$ (2)

Hence, we have $\int_{(1,0,0)}^{(3,\frac{\pi}{2},1)} (2x\sin y + e^{3z})dx + (x^2\cos y)dy + (3xe^{3z}+5)dz$

$$= \Big[f(x,y,z)\Big]_{(1,0,0)}^{(3,\frac{\pi}{2},1)} = \Big[x^2\sin y + xe^{3z} + 5z\Big]_{(1,0,0)}^{(3,\frac{\pi}{2},1)}$$

$$= (9 + 3e^3 + 5) - (0 + 1)$$

$$= \underline{\underline{13 + 3e^3}} \approx 73.25 \quad (1)$$

Q# 3(c) Use the divergence theorem and cylindrical Coordinates to find the Flux integral of the vector field $\vec{F}(x,y,z) = xy\,\vec{i} + yz\,\vec{j} + zx\,\vec{k}$ over the boundary S of the closed region Q bounded by the graphs of the equations $x^2+y^2=1$, $z=0$, $z=2$ (provided S is oriented by the unit normal directed outward). [Marks: 4]

Soln. By Divergence theorem $\iint_S \vec{F}\cdot d\vec{n} = \iiint_Q \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dV$

$$= \int_0^{2\pi}\int_0^1\int_0^2 (r\cos\theta + r\sin\theta + z)\, r\, dz\, dr\, d\theta \quad (2)$$

$$= \int_0^{2\pi}\int_0^1 \left[r\cos\theta\, z + r\sin\theta\, z + \frac{z^2}{2}\right]_0^2 r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (2r\cos\theta + 2r\sin\theta + 2)\, r\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^1 (r\cos\theta + r\sin\theta + 1)\, r\, dr\, d\theta$$

$$= 2\int_0^{2\pi} \left[\frac{r^3}{3}\cos\theta + \frac{r^3}{3}\sin\theta + \frac{r^2}{2}\right]_0^1 d\theta$$

$$= 2\int_0^{2\pi} \left(\frac{1}{3}\cos\theta + \frac{1}{3}\sin\theta + \frac{1}{2}\right) d\theta$$

$$= 2\left[\frac{1}{3}\sin\theta - \frac{1}{3}\cos\theta + \frac{1}{2}\theta\right]_0^{2\pi}$$

$$= 2\pi \quad (1)$$

Q# 3(d) Use Stokes' theorem to evaluate the line integral $\oint_C \vec{F}\cdot d\vec{r}$ for the vector field $\vec{F}(x,y,z) = (y-z)\vec{i} + (z-x)\vec{j} + (x-y)\vec{k}$, C is the boundary of the part of the plane $2x+3y+z=6$ in the first octant oriented in a counterclockwise direction when viewed from above. [Marks: 4]

Soln. curl of $\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x-y \end{vmatrix}$

$$= -2\vec{i} - 2\vec{j} - 2\vec{k} = M_1\vec{i} + N_1\vec{j} + P_1\vec{k} \quad (1)$$

Now, $z = 6 - 2x - 3y = g(x,y)$ $\therefore g_x = -2,\ g_y = -3$ (1)

$$\therefore \oint_C \vec{F}\cdot d\vec{r} \overset{\text{By Stokes' theorem}}{=} \iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\, dS = -12\iint_{R_{xy}} dy\, dx$$

$$= -12\int_0^3\int_0^{2-\frac{2}{3}x} dy\, dx = -12\int_0^3 \left(2 - \frac{2}{3}x\right) dx = \underline{\underline{-36}} \quad (2)$$