King Saud University
College of Sciences
Department of Mathematics
Semester 452 / Final Exam / MATH-244 (Linear Algebra)

Max. Marks: 40 **Time: 3 hours**

Name: **ID:** **Section:** **Signature:**

Note: *Attempt all the five questions. Scientific calculators are not allowed.
* Please fill in the above columns and give your answer to Question 1 on this page of the paper.

Question 1 [Marks: 10×1]: Choose the correct answer:

(i) If B is a 5×7 matrix and $nullity(B) = 3$, then $nullity(B^T)$ equals:

a) 2 b) 5 c) 3 d) 1

(ii) If C is a 3×3 matrix with $det(C) = 2$, then $rank(C)$ equals:

a) 1 b) 3 c) 2 d) 0

(iii) Which of the following matrices cannot be a transition matrix:

a) $\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & -3 \\ 0 & 1 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 0 & 0 \\ -2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 5 \\ 2 & 6 & 0 \end{bmatrix}$

(iv) If u and v are orthogonal unit vectors in an inner product space, then $\|v - u\|$ equals:

a) 1 b) $\sqrt{2}$ c) 0 d) $\sqrt{3}$

(v) If $u = (1,-1,0)$, $v = (3,-1,2)$ and $w = (-2,0,1)$ in Euclidean space $\mathbb{R}^3$, then $\frac{<u,v>}{<w,w>} v$ equals:

a) $\frac{4}{5}(3,-1,2)$ b) $\frac{4}{5}$ c) $\frac{5}{4}(-2,0,1)$ d) $\frac{4}{5}(1,-1,0)$

(vi) If v and w are orthogonal nonzero vectors in any Euclidean space $\mathbb{R}^n$, then the angle θ between v and w equals:

a) 45° b) 90° c) 180° d) 0°

(vii) The set $\{(x, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0)\}$ of vectors in the Euclidean space $\mathbb{R}^3$ is (ortho)normal if x equals:

a) 0 b) $\frac{1}{\sqrt{6}}$ c) $\frac{-1}{\sqrt{6}}$ d) $\frac{1}{2}$

(viii) Consider the vector space P_n of real polynomials of degree $\leq n$. If the linear transformation $T: P_2 \to P_1$ is given by $T(a_0 + a_1 x + a_2 x^2) = a_0 + a_1 + a_2 x$, then $ker(T)$ equals:

a) $\{0\}$ b) $span\{-1 + x\}$ c) $span\{-1, x\}$ d) P_2

(ix) Consider the basis $\{(1,2),(3,0)\}$ for the space $\mathbb{R}^2$. If the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(1,2) = (1,5)$ and $T(3,0) = (-4,6)$, then $T(7,8)$ equals:

a) (13,26) b) (−15,35) c) (−3,11) d) (0,26)

(x) If $A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$, then the eigenvalues of A^4 are:

a) 2, 16 b) -1, 8 c) 1, 16 d) 4, 16

Question 2 [Marks: 2+2+3]:

(a) Given: the polynomial $p(x) = x^3 - 3x^2 - 4x + 13$ and the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Compute $p(A)$.

(b) Show that $\begin{vmatrix} 1 & 2 & 2 \\ x+1 & 2x+1 & 2x+2 \\ x+1 & x+1 & 2x+1 \end{vmatrix}$ is constant, for all $x \in \mathbb{R}$.

(c) Let A be $m \times n$ matrix and let B and C be linearly independent vectors in $\mathbb{R}^m$. Suppose X_1 is a solution of $AX_1 = B$ and X_2 is a solution of $AX_2 = C$. Then, show that the vectors X_1 and X_2 are linearly independent in $\mathbb{R}^n$. Explain why $rank(A) \geq 2$.

Question 3 [Marks: 2+3+3]:

Consider the matrix $A = \begin{bmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 2 \end{bmatrix}$. Then:

(a) Compute the *RREF* of A.
(b) Find a basis of the column space *col* (A) and a basis of the null space $N(A)$.
(c) Find $rank(A)$ and $nullity(A)$.

Question 4: [Marks: 3+2+3]

(a) Consider the vector space P_2 of real polynomials of degree ≤ 2 equipped with the inner product:
$$< p(x), q(x) > = p(-1)q(-1) + p(o)q(0) + p(1)q(1) \text{ for all } p(x), q(x) \in P_2$$
Explain why $\{1, x, x^2\}$ is not orthogonal. Apply the Gram-Schmidt process to transform the basis $\{1, x, x^2\}$ of P_2 to an orthogonal basis.

(b) Let $S = \{u, v, w\}$ be any orthonormal subset of the above inner product space P_2. Show that S is a basis for P_2.

(c) Consider the basis $\{v_1 = (2,2,1), v_2 = (2,1,0), v_3 = (1,0,0)\}$ for the vector space $\mathbb{R}^3$. Let the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ be given by: $T(v_1) = (3,-1)$, $T(v_2) = (6,2)$ and $T(v_3) = (4,3)$. Then find:

(i) $ker(T)$ (ii) $rank(T)$.

Question 5: [Marks: 3+4]

(a) Let A, B and C be square matrices of size n, where C is invertible satisfying $B = C^{-1}AC$. If λ is an eigenvalue of A and X is its corresponding eigenvector, then find a nonzero vector $Y \in \mathbb{R}^n$ such that $BY = \lambda Y$.

(b) Show that the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ is diagonalizable and find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

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