Final Exam – Semester I, 1447

Department of Mathematics, College of Science, KSU

Course: Math 481   Total Marks: 40   Duration: 3 Hours

Question 1 [6 points]

(a) Let $f : [a, b] \to \mathbb{R}$ be an increasing function. Prove that f is Riemann integrable on $[a, b]$. [2]

(b) Using the definition of the Riemann integral as the limit of Riemann sums, evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3 + k^3}.$$

[4]

Question 2 [8 points]

(a) For each $n \in \mathbb{N}$, define the function

$$f_n(x) = \frac{\sin(nx)}{nx}, \qquad x \in (0, \infty).$$

(i) Find the pointwise limit of the sequence (f_n) on $(0, \infty)$. [2]

(ii) Determine whether (f_n) converges uniformly on $(0, \infty)$, and justify your answer. [2]

(iii) Determine whether (f_n) converges uniformly on $[a, \infty)$ for any $0 < a$. Justify your answer. [2]

(b) Justify the interchange of summation and integration, and evaluate

$$\int_0^{\pi} \left(\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} \right) dx.$$

[2]

Question 3 [8 points]

(a) Consider the series of functions

$$\sum_{n=1}^{\infty} \frac{nx}{1+n^3x^3}, \qquad x \geq 0.$$

Show that this series does not converge uniformly on $[0,\infty)$, but does converge uniformly on every closed interval $[a,b] \subset (0,\infty)$. [5]

(b) Using part (a), prove that the sum function

$$f(x) = \sum_{n=1}^{\infty} \frac{nx}{1+n^3x^3}$$

is continuous on $(0,\infty)$. [3]

Question 4 [8 points]

(i) (a) Define what it means for a set $E \subset \mathbb{R}$ to be *Lebesgue measurable* in terms of the Lebesgue outer measure m^*.

(b) Prove that if a set $E \subset \mathbb{R}$ satisfies

$$m^*(E) = 0,$$

then both E and its complement E^c are Lebesgue measurable.

[5]

(ii) Give a concrete example of a function that is Lebesgue integrable but not Riemann integrable on $[0,1]$, and briefly explain why it satisfies these properties. [3]

Question 5 [10 points]

(a) Let $(f_n)_{n\geq 1}$ be the sequence of functions

$$f_n(x) = xe^{-nx}, \qquad x \in [0,\infty).$$

(i) Show that each f_n is Lebesgue measurable and compute

$$\int_0^{\infty} f_n(x)\, dx.$$

[3]

(ii) Show that the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges *pointwise* for all $x > 0$. Then, justify the interchange of summation and integration by applying the *Monotone Convergence Theorem*, verifying its hypotheses. Finally, conclude that

$$\int_0^{\infty} \left(\sum_{n=1}^{\infty} f_n(x) \right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

[5]

(b) Let

$$g_n(x) = \frac{(\sin x)^n}{1+x^2}, \qquad x \geq 0.$$

(i) Show that (g_n) converges pointwise to 0 *almost everywhere* on $[0, \infty)$.

(ii) Using the Dominated Convergence Theorem, prove that

$$\lim_{n \to \infty} \int_0^{\infty} g_n(x)\, dx = 0.$$

[2]