King Saud University
College of Sciences
Department of Mathematics
MATH-244 (Linear Algebra); Final Exam; Semester 443

Max. Marks: 40 **Time: 3 hours**

Name: **ID:** **Section:** **Signature:**

Note: ***Attempt all the five questions. Scientific calculators are not allowed!***

Question 1 [Marks:10 ×1]:

Choose the correct answer:

(i) If the matrix $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -4 \\ 1 & 3 & h \end{bmatrix}$ is non-invertible, then h is equal to:
a) 5 b) -3 c) -5 d) 3.

(ii) Which of the following matrices cannot be a transition matrix?
a) $\begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

(iii) Let F denote the set of nontrivial solutions of homogenous linear system $AX = O$, where the matrix of coefficients A is invertible and $X \epsilon \mathbb{R}^3$. Then, F is equal to:
a) $\mathbb{R}^3$ b) $\{(0,0,0)\}$ c) $\{\}$ d) $\mathbb{R}^3 - \{(0,0,0)\}$.

(iv) If $W = span\,\{(1,-1,0,1), (-1,1,1,0), (2,-2,1,3)\}$, then $dim W$ is equal to:
a) 1 b) 2 c) 3 d) 4.

(v) If $B = \{(2,-4), (3,-3)\}$ is an ordered basis of the vector space $\mathbb{R}^2$, then the coordinate vector $[(1,1)]_B$ is equal to:
a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ b) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ c) $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(vi) If $U = \begin{bmatrix} 2 & 5 \\ -1 & x \end{bmatrix}$ and $V = \begin{bmatrix} -3 & 2x \\ 3 & -1 \end{bmatrix}$ are orthogonal matrices in $M_2(\mathbb{R})$ with respect to the inner product $< A, B > = trace\,(AB^T)$, then x is equal to:
a) 2 b) -2 c) 0 d) 1.

(vii) If $\langle\,,\rangle$ is an inner product on $\mathbb{R}^3$ such that $\|u\|^2 = 2$, $\|v\|^2 = 3$, $\langle u,v\rangle = 1$, then $\langle 3u - v\,, 2u - 4v\rangle$ is equal to:
a) $\sqrt{13}$ b) -14 c) 10 d) 38.

(viii) If the inner product on the vector space P_2 of polynomials with degree ≤ 2 is defined by $\langle p,q\rangle = aa_1 + 2bb_1 + cc_1$ for all $p = a + bx + cx^2, q = a_1 + b_1x + c_1x^2 \in P_2$ and θ denote the angle between polynomials $2 + x + x^2$ and $-1 + x + 2x^2$, then $\cos\theta$ is equal to
a) $\frac{2}{\sqrt{7}}$ b) $\frac{2}{7}$ c) 0 d) $\frac{6}{7}$.

(ix) If the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x, y) = (2x - y, -8x + 4y)$. Then $ker(T)$ includes:
a) (5,10) b) (10,2) c) $(-5,10)$ d) (10,5).

(x) If the characteristic polynomial of a matrix A is $q_A(\lambda) = \lambda^2 - 9$, then the matrix A is:
a) not diagonalizable b) diagonalizable c) 3×3 d) not invertible.

Question 2 [Marks: 2+2+2]: Let the matrix $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$. Then:

a) Find a basis of the null space of A.
b) Find a basis for the column space of A.
c) Verify that $rank(A) + nullity(A) = 3$.

Question 3 [Marks: 3+3]:

a) Let the set $B = \{u_1, u_2, u_3\}$ be a basis for a vector space V. If $u, v \in V$ are linearly independent vectors. Then show that $\{[u]_B, [v]_B\}$ is a linearly independent subset of $\mathbb{R}^3$.
b) Find a basis for the Euclidean space $\mathbb{R}^4$ that includes the vectors $(0,0,0,1), (1,1,1,0), (0,1,1,0)$.

Question 4: [Marks: 2+4]

a) Let $\{u, v, w\}$ be an orthonormal set of vectors in an inner product space. Use Pythagorean Theorem to evaluate $\|u + v + w\|^2$.
b) Let the set $\boldsymbol{B} = \{\boldsymbol{u}_1 = (1,1,1), \boldsymbol{u}_2 = (0,1,1)), \boldsymbol{u}_3 = (0,0,1)\}$ be linearly independent in the Euclidean inner product space $\mathbb{R}^3$. Construct an orthonormal set $\boldsymbol{C}$ for $\mathbb{R}^3$ by applying the Gram-Schmidt algorithm on $\boldsymbol{B}$ such that $span(\boldsymbol{C}) = span(\boldsymbol{B})$.

Question 5: [Marks: (4+2) + (2+2+2)]

a) Let $A = \{v_1, v_2, v_3\}$ be a basis for vector space V, $B = \{w_1, w_2, w_3, w_4, w_5\}$ be a basis for vector space W. Let $T : V \rightarrow W$ be the linear transformation such that: $w_2 - 3w_4 + T(v_1) = 2w_1 + 7w_5$; $T(v_2) + w_4 = 2w_3 + w_5$; $w_3 + T(v_3) = 2w_2 + 4w_4 - w_5$. Then:
   (i) Find the transformation matrix $[T]_A^B$ with respect to the ordered bases A and B.
   (ii) Find the coordinate vector $[T(v_1 + v_2 + v_3)]_B$.

b) Let $A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$. Then:
   (i) Show that the matrix A is diagonalizable.
   (ii) Find an invertible matrix P and a diagonal matrix D satisfying $P^{-1}AP = D$.
   (iii) Find A^5.

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