King Saud University
College of Sciences
Department of Mathematics
MATH-244 (Linear Algebra); Final Exam; Semester 442

Max. Marks: 40 **Time: 3 hours**

Name: **ID:** **Section:** **Signature:**

Note: ***Attempt all the five questions. Scientific calculators are not allowed!***

Question 1 [Marks: 5+5]:

a) Choose the correct answer:

(i) Let B and C be ordered bases of a vector space V with transition matrix ${}_CP_B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. If the coordinate vector $[v]_C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ then the coordinate vector $[\boldsymbol{v}]_B$ is:

(a) (1,3,6) (b) (6,3,1) (c) (1,1,1) (d) (1,1,2).

(ii) The dimension of the column space $col(A^t)$ of $A = \begin{bmatrix} 2 & 3 & 1 & -1 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is:

(a) 1 (b) 2 (c) 4 (d) 5.

(iii) If $U = \begin{bmatrix} -1 & 3 \\ y & 1 \end{bmatrix}$ and $V = \begin{bmatrix} 5 & 2y \\ -4 & 1 \end{bmatrix}$ are two orthogonal matrices with respect to the inner product $<A, B> = trace\,(AB^t)$ on the vector space M_2 of 2x2 real matrices, then:

(a) $y = 2$ (b) $y = -2$ (c) $y = 0$ (d) $y = 1$.

(iv) If the inner product on the vector space $\boldsymbol{P_2}$ of polynomials with degree ≤ 2 is defined by $\langle p,q \rangle = aa_1 + 2bb_1 + cc_1,\ \forall\, p = a + bx + cx^2,\ q = a_1 + b_1x + c_1x^2 \in \boldsymbol{P_2}$ and θ is the angle between the polynomials $1 + x - x^2$ and $2 + x - 2x^2$, then:

(a) $\cos\theta = \frac{5}{3\sqrt{3}}$ (b) $\cos\theta = \frac{2}{\sqrt{3}}$ (c) $\cos\theta = \frac{3}{\sqrt{10}}$ (d) $\cos\theta = 1$.

(v) If $S = \{v_1 = (2,1), v_2 = (1,0)\}$ is a basis for Euclidean space $\mathbb{R}^2$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation defined by $T(v_1) = (1,5)$ and $T(v_2) = (0,3)$, then $T(4,6)$ is equal to:

(a) (6,6) (b) $(-8,-22)$ (c) $(-10,-8)$ (d) (4,23).

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b) Determine whether the following statements are true or false; justify your answer.

(i) If $\{(-3r + 4s, r - s, r, s): r, s \in \mathbb{R}\}$ is the solution space of homogeneous system $AX = 0$, then $nullity(A) = 2$.

(ii) For any $m \times n$ matrix A, $dim(N(A^t)) + dim(col(A)) = m$.

(iii) The transformation $T: \mathbb{R} \to \mathbb{R}$ given by $T(r) = |r|$ is linear.

(iv) Eigenvalues of any matrix are same as the eigenvalues of its reduced row echelon form.

(v) If the characteristic polynomial of a matrix A is $q_A(\lambda) = \lambda^2 - 2$, then A is diagonalizable.

Question 2 [Marks: 2.5+1+2-5]: Consider the matrix $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}$. Then:

(i) Find a basis for $col(A)$.
(ii) Find $dim(row(A))$.
(iii) Find a basis for the null space $N(A)$.

Question 3 [Marks: 3+3]: Let $E = \{v_1 = (1,1,-4,-3), v_2 = (2,0,2,-2), v_3 = (2,-1,3,2)\}$. Then:

(i) Find a basis B for the vector space $span(E)$ such that $B \subseteq E$. If $E - B \neq \phi$, then express each element of $E - B$ as linear combination of the basic vectors.
(ii) Use the basis B (as in Part (i)) to find a basis $\boldsymbol{C}$ for the Euclidean space $\mathbb{R}^4$.

Question 4: [Marks: 2+4]

a) Let $\{u, v, w\}$ be an orthogonal set of vectors in an inner product space. Then show that:
$$\|u\|^2 + \|v\|^2 + \|w\|^2 = \|u + v + w\|^2.$$
b) Let $A = \{\boldsymbol{u_1} = (1,1,1), \boldsymbol{u_2} = (0,1,-1), \boldsymbol{u_3} = (3,-2,2)\}$. Use the Gram-Schmidt algorithm to obtain an orthonormal set B of vectors such that $span(B) = span(A)$.

Question 5: [Marks: (2+1.5+2.5) + (2.5+1+2.5)]

a) Let the linear transformation T: $M_2 \rightarrow \mathbb{R}^2$ be defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b)$, $\forall\, a,b,c,d \in \mathbb{R}$, Then find:

(i) A basis for $ker(T)$.
(ii) $rank(T)$.
(iii) The standard matrix $[T]_B^C$, where B and C are the standard bases of M_2 and $\mathbb{R}^2$, respectively.

b) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ x & 2 & 0 \\ y & z & -3 \end{bmatrix}$. Then:

(i) Find the values of $\boldsymbol{x}$, $\boldsymbol{y}$ and $\mathbf{z}$ such that $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = -3$ are the eigenvalues of A with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, respectively.

(ii) Use the values of $\boldsymbol{x}$, $\boldsymbol{y}$ and $\mathbf{z}$ (as in Part (i)), show that the matrix A is diagonalizable.

(iii) Find A^5.

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