Solution Key:

King Saud University
College of Sciences
Department of Mathematics
Math-244 (Linear Algebra); Final Exam; Semester 441

Max. Marks: 40 **Time: 3 hours**

Note: ***Attempt all the five questions. Scientific calculators are not allowed***!

Question 1 [Marks: 5× 1+5× 1]**:**

I. Choose the correct answer:

(i) If W is the subspace $\{(a,b,c,d) \in \mathbb{R}^4 : b = a - c\}$ of Euclidean space $\mathbb{R}^4$, then $\dim(W)$ is:

a) 1 b) 2 **c)** ✓ 3 d) 4.

(ii) If $rank(A) = 3$ where A is a matrix of size 5×9, then $nullity(A^T)$ is:

a) 1 b) ✓ 2 c) 3 d) 6.

(iii) If θ is the angle between A and B with respect to the standard inner product on M_{22} where $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$, then $cos\,\theta$ is:

a) $\frac{1}{\sqrt{2}}$ b) $\frac{1}{2}$ c) $\frac{15}{2\sqrt{30}}$ d) ✓ 0.

(iv) The values of k for which the vectors $\boldsymbol{u} = (u_1 = 2, u_2 = -4)$ and $\boldsymbol{v} = (v_1 = 1, v_2 = 3)$ in $\mathbb{R}^2$ are orthogonal with respect to the inner product $\langle \boldsymbol{u},\boldsymbol{v} \rangle = 2u_1v_1 + ku_2v_2$:

a) $\frac{1}{\sqrt{2}}$ b) $\frac{1}{2}$ c) $\frac{15}{2\sqrt{30}}$ d) ✓ $\frac{1}{3}$.

(v) If $B = \{(2,1), (-3,4)\}$ and $C = \{(1,1), (0,3)\}$ are bases of $\mathbb{R}^2$, then the transition matrix ${}_{\boldsymbol{B}}\boldsymbol{P}_{\boldsymbol{C}}$ is:

a) $\begin{bmatrix} 7/11 & 1/11 \\ 9/11 & 6/11 \end{bmatrix}$ b) ✓ $\begin{bmatrix} 7/11 & 9/11 \\ 1/11 & 6/11 \end{bmatrix}$ c) $\begin{bmatrix} 7/11 & 9/11 \\ 6/11 & 1/11 \end{bmatrix}$ d) $\begin{bmatrix} 9/11 & 7/11 \\ 1/11 & 6/11 \end{bmatrix}$.

II. Determine whether the following statements are true or false; justify your answer.

(i) If $A, B \in M_n(\mathbb{R})$, then $det(A^T B) = det(B^T A)$.

True: $det(A^T B) = det(A)det(B) = det(B^T A)$.

(ii) A basis for solution space of the following linear system is $\{(4,1,0,0), (-3,0,1,0)\}$:

$$x_1 - 4x_2 + 3x_3 - x_4 = 0$$
$$2x_1 - 8x_2 + 6x_3 - 3x_4 = 0.$$

True: the solution space $= \{(4s-3t, s, t, 0): s,t \in \mathbb{R}\}$; $(4s-3t, s, t, 0) = s(4,1,0,0) + t(-3,0,1,0)$ and $\{(4,1,0,0), (-3,0,1,0)\}$ is linearly independent.

(iii) If $W = \{A \in M_2(\mathbb{R}) : A \text{ is singular}\}$, then W is vector subspace of $M_2(\mathbb{R})$.

False: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are singular matrices but their sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not singular.

(iv) If u, v and w are vectors in an inner product space such that $\langle u,v \rangle = 3$, $\langle v,w \rangle = -5$, $\langle u,w \rangle = -1$ and $\|u\| = 2$, then $\langle u - 2w, 3u + v \rangle = 25$.

False: $25 \neq 31$ $(\because \langle u - 2w, 3u + v \rangle = 3(2^2) + 3 + (-2)(3)(-1) + (-2)(-5) = 31)$

(v) If the characteristic polynomial of 2×2 matrix A is $q_A(\lambda) = \lambda^2 - 1$, then A is diagonalizable.

True: ∓ 1 are two different eigen-values of the 2×2 matrix $\boldsymbol{A}$.

Question 2 [Marks: 2+2+2]**:** Consider the matrices $A=\begin{bmatrix}1&1&0&0\\0&1&1&1\\1&0&1&2\\1&1&1&2\end{bmatrix}$ and $B=\begin{bmatrix}1&1&0&0&2\\0&1&1&1&-1\\1&0&1&2&1\\1&1&1&2&-2\end{bmatrix}$. Then:

a) Find A^{-1} by the elementary matrix method.

Solution: Since $\left[\begin{array}{cccc|cccc}1&1&0&0&1&0&0&0\\0&1&1&1&0&1&0&0\\1&0&1&2&0&0&1&0\\1&1&1&2&0&0&0&1\end{array}\right]\sim\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&1&-1\\0&1&0&0&0&0&-1&1\\0&0&1&0&1&2&2&-3\\0&0&0&1&-1&-1&-1&2\end{array}\right]$, $A^{-1}=\begin{bmatrix}1&0&1&-1\\0&0&-1&1\\1&2&2&-3\\-1&-1&-1&2\end{bmatrix}$.

b) Show that $nullity(A)\neq nullity(B)$.

Solution: Since $RREF(B)=\begin{bmatrix}1&0&0&0&5\\0&1&0&0&-3\\0&0&1&0&8\\0&0&0&1&-6\end{bmatrix}$, $nullity(B)=1$; but $nullity(A)=0$ because A is invertible.

c) Find a basis for the null space spaces $N(B)$.

Solution: Since $N(B)=\{(-5t,3t,-8t,6t,t):t\in\mathbb{R}\}$, $\{(-5,3,-8,6,1)\}$ is a basis for $N(B)$.

Question 3 [Marks: 3+3]**:**

a) Find the values of x so that the set $\{(1,-2,x),(1,-x,2),(1,-4,2x)\}$ is linearly independent in the Euclidean space $\mathbb{R}^3$.

Solution: $\because$ $\alpha(1,-2,x),+\beta(1,-x,2)+\gamma(1,-4,2x)=(0,0,0)\Longrightarrow\begin{cases}\alpha+\beta+\gamma=0\\-2\alpha-x\beta-4\gamma=0\\x\alpha+2\beta+2x\gamma=0.\end{cases}$

$\therefore$ The given set would be linearly independent *iff* $\begin{vmatrix}1&1&1\\-2&-x&-4\\x&2&2x\end{vmatrix}\neq 0$ *iff* $x\in\mathbb{R}\setminus\{\mp 2\}$.

b) Let $\boldsymbol{F}=span(\{(1,-1,0,1),(0,1,0,-1),(-1,2,0,-1)\})$ in $\mathbb{R}^4$. Find a basis for $\boldsymbol{F}$ and show that $(0,1,0,0)\in\boldsymbol{F}$.

Solution: Since $\{(1,-1,0,1),(0,1,0,-1),(-1,2,0,-1)\}$ is linearly independent in $\mathbb{R}^4$, the same set is a basis of $\boldsymbol{F}$. Next, we observe that $(0,1,0,0)=(-1,2,0,-1)+(1,-1,0,1)\in\boldsymbol{F}$.

Question 4: [Marks: 2+4]

a) Let u and v be any two vectors in an inner product space. Show that:

$$2(\|u\|^2+\|v\|^2)=\|u+v\|^2+\|u-v\|^2.$$

Solution: $\|u+v\|^2+\|u-v\|^2=<u+v,u+v>+<u-v,u-v>=2<u,u>+2<v,v>=2(\|u\|^2+\|v\|^2)$.

b) Let the set $B=\{\boldsymbol{u}_1=(1,0,0),\boldsymbol{u}_2=(3,1,-1),\boldsymbol{u}_3=(0,3,1)\}$ be linearly independent in the Euclidean inner product space $\mathbb{R}^3$. Construct an orthonormal basis for $\mathbb{R}^3$ by applying the Gram-Schmidt algorithm on B.

Solution: Put $e_1 = v_1 = u_1 = (1,0,0)$. Then $v_2 = u_2 - \langle u_2, e_1 \rangle e_1 = u_2 - 3e_1 = (0,1,-1)$ and so $e_2 = \frac{1}{||v_2||} v_2 = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Finally, $v_3 = u_3 - \langle u_3, e_2 \rangle e_2 - \langle u_3, e_1 \rangle e_1 = u_3 - \sqrt{2}e_2 = (0,2,2)$ and so $e_3 = \frac{1}{||v_3||} v_3 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Thus, $\{e_1 = (1,0,0), e_2 = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), e_3 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\}$ is the required orthonoral basis of the inner product space $\mathbb{R}^3$.

Question 5: [Marks: (4+2) + (2+2+~~1~~ 2)]

a) Let $\boldsymbol{B} = \{(1,1,0), (0,1,1), (1,0,1)\}$ be a basis for $\mathbb{R}^3$, $\boldsymbol{C} = \{x+1, x-1, x^2+1\}$ be a basis for P_2 (the vector space of all real polynomials (in variable x) of degree ≤ 2 . Let $\boldsymbol{T}: \mathbb{R}^3 \to P_2$ be the linear transformation: $\boldsymbol{T}(a,b,c) = (a+b) + (b+c)x + (a+c)x^2$, $\forall (a,b,c) \in \mathbb{R}^3$.

(i) Find the values of q, r, s in the transformation matrix $[\boldsymbol{T}]_{\boldsymbol{B}}^{\boldsymbol{C}} = \begin{bmatrix} 1 & q & 0 \\ r & 1 & 1 \\ 1 & 1 & s \end{bmatrix}$ with respect to the bases $\boldsymbol{B}$ and $\boldsymbol{C}$.

Solution: Since $[\boldsymbol{T}]_{B}^{C} = \begin{bmatrix} 1 & q & 0 \\ r & 1 & 1 \\ 1 & 1 & s \end{bmatrix}$, we get $[T(1,1,0)]_C = \begin{bmatrix} 1 \\ r \\ 1 \end{bmatrix}$, $[T(0,1,1)]_C = \begin{bmatrix} q \\ 1 \\ 1 \end{bmatrix}$ and $[T(1,0,1)]_C = \begin{bmatrix} 0 \\ 1 \\ s \end{bmatrix}$.

Now, $T(1,1,0) = 2 + x + x^2 = 1(x+1) + 0(x-1) + 1(x^2+1)$ gives $\begin{bmatrix} 1 \\ r \\ 1 \end{bmatrix} = [T(1,1,0)]_C = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; hence, $\boldsymbol{r} = \boldsymbol{0}$.

Similarly, $q = \boldsymbol{1}$ and $\boldsymbol{s} = \boldsymbol{2}$.

(ii) Find the coordinate vector $[T(1,1,1)]_{\boldsymbol{C}}$.

Solution: Since $[\boldsymbol{T}]_{B}^{C} = \begin{bmatrix} 1 & q & 0 \\ r & 1 & 1 \\ 1 & 1 & s \end{bmatrix}$ and $[(1,1,1)]_B = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, we get $[T(1,1,1)]_C = [\boldsymbol{T}]_{B}^{C}[(1,1,1)]_B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

b) Consider the matrix $A = \begin{bmatrix} 1 & 7 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$ is diagonalizable.

(i) Show that the matrix A is diagonalizable.

Solution: The given matrix $A = \begin{bmatrix} 1 & 7 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$ being upper triangular has eigen-values -1, 1 and 2; so, it is diagonalizable.

(ii) Find an invertible matrix P and a diagonal matrix D satisfying $P^{-1}AP = D$.

Solution: $\boldsymbol{P} = \begin{bmatrix} 1 & 7 & \frac{+7}{3} \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 \end{bmatrix}$ with $\boldsymbol{P}^{-1} = \begin{bmatrix} 1 & -7 & \frac{-7}{~~6~~} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$ and $\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(iii) Find A^7.

Solution: $A^7 = P\, D^7\, P^{-1} = \begin{bmatrix} 1 & 7 & \frac{-7}{2} \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^7 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -7 & \frac{-7}{6} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$.

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