KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

MATH-244 (Linear Algebra); Final Exam; Semester 432

Max. Marks: 40 Max. Time: 3 hours

Note: *Attempt all the five questions!*

Question 1 [3+2+2 marks]:

a) If $A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, then find $adj(adj(A))$.

b) Find the values of k that makes the matrix $\begin{bmatrix} 2 & 3k-2 \\ k^2 & -1 \end{bmatrix}$ symmetric.

c) Let $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 4 \\ 0 & 0 & 6 \end{bmatrix}$. Explain! Why the matrix B can be expressed as a product of elementary matrices?

Question 2 [3+3 marks]:

a) Solve the linear system of equations with augmented matrix:

$$[A:B] = \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ -1 & 2 & 0 & -2 & 2 \\ 3 & 1 & 0 & 3 & -1 \end{array}\right].$$

b) Solve the following linear system of equations by Cramer's Rule:

$$\begin{aligned} x - y \quad\quad &= 1 \\ -2x + 3y - 4z &= 0 \\ -2x + 3y - 3z &= 1 \end{aligned}$$

Question 3 [2+2+2+3 marks]:

a) Show that $E = \{ax - 2ax^4 + (a-b)x^6 + (3a+2b)x^7 : a, b \in \mathbb{R}\}$ is a subspace of the real vector space P_7 of polynomials with *degree* ≤ 7.

b) Find a basis and dimension of the vector space E.

c) Show that $< (x_1, x_2, x_3), (y_1, y_2, y_3) > = x_1y_1 + 2x_2y_2 + 3x_3y_3$ defines an inner product on the vector space $\mathbb{R}^3$.

d) Find an orthogonal basis of $\mathbb{R}^3$, with respect to the inner product defined above in Part c), by using the Gram-Schmidt algorithm on $\{u_1 = (1,1,1),\ u_2 = (1,1,0),\ u_3 = (0,1,0)\}$.

Question 4 [3+3+3 marks]:

Let $B = \{u_1 = (1,-1), u_2 = (1,1)\}$ and $C = \{v_1 = (1,1,0), v_2 = (1,0,1), v_3 = (1,1,1)\}$ be bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Let $\mathbf{T}: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that $\mathrm{T}(1,-1) = (3,5,2)$, $\mathrm{T}(1,1) = (2,-1,-3)$. Then find:

a) $\mathbf{T(1,0)}$ and $\mathbf{T(0,1)}$.

b) The matrix $[\mathbf{T}]_B^C$ of the linear transformation $\mathbf{T}$ with respect to the bases B and C.

c) The coordinate vector $[\mathrm{T}(0,1)]_C$ by using $[\mathbf{T}]_B^C$.

Question 5 [2+2+2+3 marks]:

Let $\boldsymbol{\lambda_1 = 0}$, $\boldsymbol{\lambda_2 = 1}$ and $\boldsymbol{\lambda_3 = -1}$ be the eigenvalues of 8×8 matrix A with algebraic multiplicities **3**, **2** and **3**, respectively. Let $\mathbf{dim}(E_{\lambda_1}) = 3$, $\mathbf{dim}(E_{\lambda_2}) = 2$ and $\mathbf{dim}(E_{\lambda_3}) = 3$, where E_{λ_j} denotes the eigenspace with respect to the eigenvalue λ_j.

a) Find the characteristic polynomial $q_A(\lambda)$ of the matrix A.

b) Explain, why the matrix A is diagonalizable?

c) Find the diagonal matrix D such that $A = PDP^{-1}$, where P is an invertible matrix.

d) Find A^{11}.

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