Chapter 3: Exponential Distribution &Poisson Process

- Definitions
- Properties
- Modeling and Parameters
- Applications

3.3 Poisson Process Modeling

Data

AT(n) = arrival time of customer number n.

S(n) = service time of customer number n.

n	AT(n)	S(n)
1	AT(1)	S(1)
2	AT(2)	S(2)
3	AT(3)	S(3)
.	.	.
.	.	.
.	.	.
N	AT(N)	S(N)

3.3 Poisson Process Modeling

3.3.1 Modeling Arrival Process

$AT(n)$ = arrival time of customer number n.

T_n = time between arrivals n and $n-1$

$= AT(n) - AT(n-1)$ where $AT(0) = 0$

n	$AT(n)$	T_n
1	$AT(1)$	$T_1 = AT(1) - 0$
2	$AT(2)$	$T_2 = AT(2) - AT(1)$
3	$AT(3)$	$T_3 = AT(3) - AT(2)$
.	.	.
.	.	.
.	.	.
N	$AT(N)$	$T_N = AT(N) - AT(N-1)$

3.3 Poisson Process Modeling

3.3.1 Modeling Arrival Process

no. of events Poisson Process with rate λ $\Rightarrow$ $\Leftarrow$ **Time between events is Exponential with mean 1/λ**

If number arrivals is Poisson process then

$$T_n \sim \text{Exponential}$$

3.3 Poisson Process Modeling

3.3.1 Modeling Arrival Process

Average time between arrivals $\overline{T}$

$$\overline{T} = \frac{\sum_{n=1}^{N} T_n}{N}$$

Then the arrival process is Poisson with rate $\hat{\lambda}$

$$\hat{\lambda} = \frac{1}{\overline{T}}$$

3.3 Poisson Process Modeling

3.3.1 Modeling Arrival Process

Example:

$T_n = AT(n) - AT(n-1)$

Cust.#	AT(n)	S(n)
1	1.4	3.6
2	2.1	0.7
3	5.4	1.7
4	7.3	1.1
5	25.0	1.1
6	26.9	5.8
7	28.5	8.2
8	29.4	1.7
9	34.6	0.6
10	36.8	5.1

Cust.#	AT(n)	S(n)
11	42.2	3.5
12	50.6	0.9
13	52.4	15.9
14	56.4	14.3
15	56.5	10.4
16	64.6	0.8
17	68.2	6.0
18	71.5	4.1
19	74.6	1.5
20	74.7	4.9

3.3 Poisson Process Modeling

3.3.1 Modeling Arrival Process

Example:

$$T_n = AT(n) - AT(n-1)$$

Assume Poisson Arrivals ⇔ Exponential interarrival time

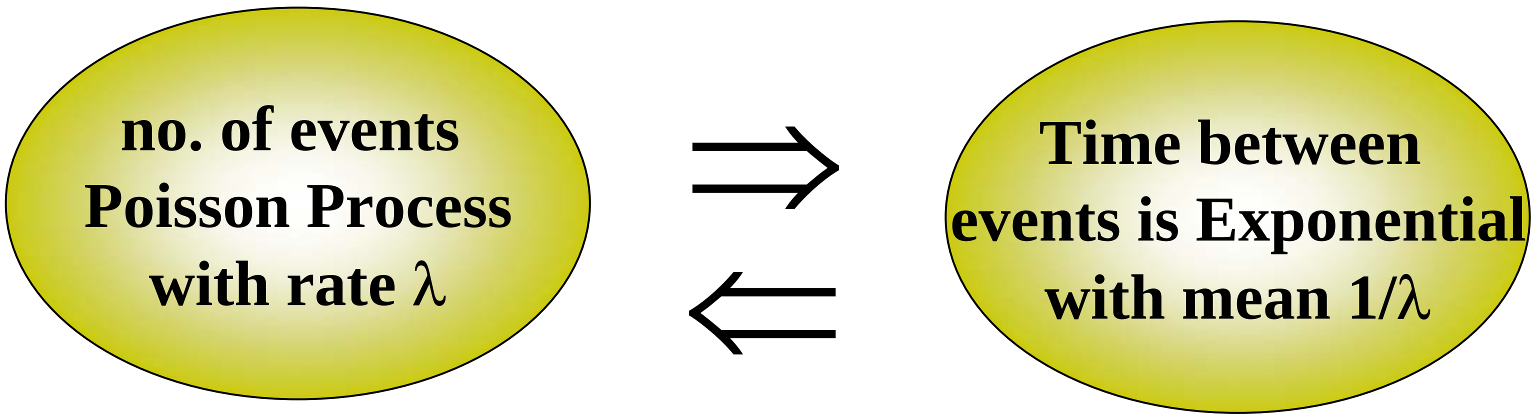
$$\text{Average interarrival time} = \overline{T} = \frac{\sum_{n=1}^{N} T_n}{N}$$

$$= \frac{74.7}{20} = 3.735 \text{ min.}$$

$$\text{Expected arrival rate} = \hat{\lambda} = \frac{1}{3.735} = 0.268 \text{ Cust/min} = 16.06 \text{ Cust/hr}$$

3.3 Poisson Process Modeling

3.3.2 Modeling Service Process



If service time is exponential then Departure time is Poisson process

$$S(n) \sim \text{Exponential}$$

3.3 Poisson Process Modeling

3.3.2 Modeling Service Process

Example:

Assume Exponential Service time $\Leftrightarrow$ Poisson Departures

$$\text{Average Service time} = \overline{S} = \frac{\sum_{n=1}^{N} S(n)}{N}$$

$$= \frac{91.9}{20} = 4.595 \text{ min.}$$

$$\text{Expected arrival rate} = \hat{\mu} = \frac{1}{4.959} = 0.202 \text{ Cust/min} = 12.1 \text{ Cust/hr}$$

3.3.3 Goodness of Fit

I. Variance

no. of events Poisson Process with rate λ $\Rightarrow$ **Poisson Process Mean = Variance**

no. of events Poisson Process with rate λ $\Leftarrow$ **Poisson Process Mean = Variance**

From data, using interarrival times

If $$E[T] = \frac{\sum_{n=1}^{N} T_n}{N} \approx \sqrt{\frac{\sum_{n=1}^{N} (T_n - \bar{T})^2}{N-1}} = SD[T]$$

$\Rightarrow$ Poisson Process is good to model the arrival process

3.3 Poisson Process Modeling

3.3.3 Goodness of Fit

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If $$E[T] = \frac{\sum_{n=1}^{N} T_n}{N} \approx \sqrt{\frac{\sum_{n=1}^{N}(T_n - \bar{T})^2}{N-1}} = SD[T]$$

$\Rightarrow$ Poisson Process is good to model the arrival process

If $$E[S] = \frac{\sum_{n=1}^{N} S(n)}{N} \approx \sqrt{\frac{\sum_{n=1}^{N}(S(n) - \bar{S})^2}{N-1}} = SD[S]$$

$\Rightarrow$ Poisson Process is good to model the service process

3.3.3 Goodness of Fit

Example:

Average interarrival time = $\overline{T} = 3.735$ min.

SD interarrival time = $\sqrt{16.22} = 4.03$

Poisson Arrival Process with Rate $\lambda = \overline{T}^{-1} = 0.268$ cust/min

Average service time = $\overline{S} = 4.595$ min.

SD of service time = $\sqrt{20.26} = 4.5$

Poisson Service Process with Rate $\mu = \overline{S}^{-1} = 0.202$ cust/min

3.3 Poisson Process Modeling

3.3.3 Goodness of Fit

II. Average Arrivals Function at *t*

If a process is Poisson with rate λ then the plot of $(n\ ,\ AT(n))$ will follow the average λt

<u>Steps:</u>

1. Plot $(n, AT(n))$ where x-axes is time, y-axes is cust. #
2. Find E[T] and let $\hat{\lambda}$
3. Draw the function $y = \hat{\lambda} t$
4. If the plots close to the line $\Rightarrow$ Poisson Process

3.3 Poisson Process Modeling

3.3.3 Goodness of Fit

Example

$\overline{T} = 3.735$ min

$\lambda = 1/3.735$

$= 0.268$ cust/min

Plot $(n, AT(n))$

and $y = \lambda t$

$y = 0.268\, t$

Cust.#	AT(n)
1	1.4
2	2.1
3	5.4
4	7.3
5	25.0
6	26.9
7	28.5
8	29.4
9	34.6
10	36.8

Cust.#	AT(n)
11	42.2
12	50.6
13	52.4
14	56.4
15	56.5
16	64.6
17	68.2
18	71.5
19	74.6
20	74.7

3.3 Poisson Process Modeling

3.3.3 Goodness of Fit

Example