Department of Statistics and Operations Research
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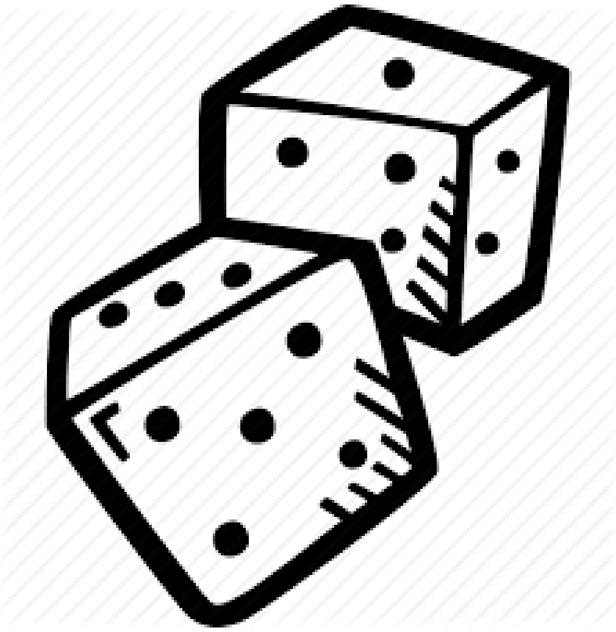
Probability (2)

Exercises



Editing by: Kholoud Basalim

Exercises 1

Q1: A fair coin is tossed four times, and the sequence of heads and tails is observed.

(a) List each of the 16 sequences in the sample space S

(b) Let events A, B, C, and D be given by A = {at least 3 heads}, B = {at most 2 heads}, C = {heads on the third toss}, and D = {1 head and 3 tails}

If the probability set function assigns $\frac{1}{16}$ to each outcome in the sample space, find

(i) $P(A)$,

(ii) $P(A \cap B)$,

(iii) $P(B)$,

(iv) $P(A \cap C)$, **H.W**

(v) $P(D)$,

(vi) $P(A \cup C)$ **H.W**

(vii) $P(B \cap D)$. **H.W**

Q2: If $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.3$, Find a. $P(A \cup B)$, b. $P(A \cap B')$, and c. $P(A' \cup B')$.

Q3: Given that $P(A \cup B) = 0.76$ and $P(A \cup B') = 0.87$, find $P(A)$. **H.W**

Q4: A survey is made to determine the number of households having electric appliances in a certain city. It is found that 75% have radios (R), 65% have irons (I), 55% have electric toasters (T), 50% have (IR), 40% have (RT), 30% have (IT), and 20% have all three.

1) Find the probability that a household has at least one of these appliances ($P(R \cup I \cup T)$).

2) Find $P(R^c U T^c)$.

Q5: Let A and B be independent events with $P(A) = 0.7$ and $P(B) = 0.2$.
Compute (a) $P(A \cap B)$, (b) $P(A \cup B)$, and (c) $P(A' \cup B')$.

Q6: Let $P(A) = 0.3$ and $P(B) = 0.6$.

a. Find $P(A \cup B)$ when A and B are independent.

b. Find $P(A \mid B)$ when A and B are mutually exclusive.

Q7: Bowl B1 contains two white chips, bowl B2 contains two red chips, bowl B3 contains two white and two red chips, and bowl B4 contains three white chips and one red chip. The probabilities of selecting bowl B1, B2, B3, or B4 are 1/2, 1/4, 1/8, and 1/8, respectively. A bowl is selected using these probabilities and a chip is then drawn at random. Find

(a) P(W), the probability of drawing a white chip.

(b) P(B1 |W), the conditional probability that bowl B1 had been selected, given that a white chip was drawn.

Q8: Find the mean and variance for the following discrete distributions:

(c) $f(x) = \frac{4-x}{6}$, $x = 1,2,3$

(a) $f(x) = \frac{1}{5}$, $x = 5,10,15,20,25$ **H.W**

Q9: Given $E(X + 4) = 10$ and $E[(X + 4)^2] = 116$, determine
(a) $Var(X + 4)$, (b) $\mu = E(X)$, and (c) $\sigma^2 = Var(X)$.

Q10: If the mean and the variance of a binomial distribution are 10 and 5 respectively, then :
1) Determine the probability mass function.
2) Calculate the probability P(X = 0), P(X = 1) and P(X = 2).
3) Calculate the probability P(X ≥ 0).

Q11: **H.W** Suppose that the probability that a person dies when he or she contracts a certain disease is 0.4. A sample of 10 persons who contracted this disease is randomly chosen. Find the following
1) The probability that exactly 4 persons will die among this sample.
2) The probability that less than 3 persons will die among this sample.
3) The probability that more than 8 persons will die among this sample.
4) The expected number of persons who will die in this sample.
5) The variance of the number of persons who will die in this sample.

Q12: Let X have a Poisson distribution with a mean of 4. Find
(a) P(2 ≤ X ≤ 5).
(b) P(X ≥ 3). **H.W**

Q13: Suppose that the probability of suffering a side effect from a certain flu vaccine is 0.005. If 1000 persons are inoculated, find the approximate probability that:
(a) At most 1 person suffers.
(b) 4, 5, or 6 persons suffer. **H.W**

SOLUTION

$$n = 1000$$
$$p = 0.005$$
$$\lambda = \mu = np = 1000(0.005) = 5$$

(a) Evaluate the formula of Poisson probability at $k = 0, 1$:

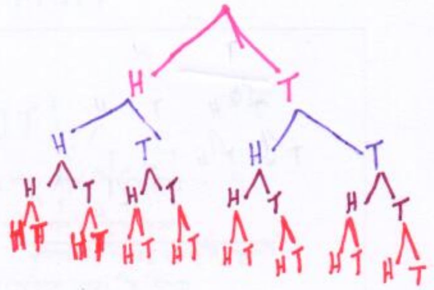
$$P(X = 0) = \frac{5^0 e^{-5}}{0!} = e^{-5} \approx 0.0067$$
$$P(X = 1) = \frac{5^1 e^{-5}}{1!} = 5e^{-5} \approx 0.0337$$

Use the addition rule for mutually exclusive events:

$$\begin{aligned} P(X \le 1) &= P(X = 0) + P(X = 1) \\ &= 0.0067 + 0.0337 \\ &= 0.0404 \end{aligned}$$

STAT415

Exercice 1



Q1: S = { HHHH , HHHT, HHTH , HHTT, HTHH , HTHT , HTTH, HTTT,
THHH, THHT, THTH , THTT, TTHH , TTHT, TTTH, TTTT }

n(S)=16

على الأقل A= {at least 3H} = { HHHH, HHHT, HHTH , HTHH, THHH }   n(A)=5 (≥ 3)

على الأكثر B = {at most 2H} = { HHTT , HTHT , HTTH, HTTT, THHT , THTH, THTT , TTHH, TTHT, TTTH, TTTT } (≤ 2)

n(B)=11

C = { H on the third toss } = { HHHH, HHHT , HTHH , HTHT, THHH, THHT, TTHH , TTHT }

n(C)=8

D = { 1H and 3T } = { HTTT, THTT, TTHT, TTTH }   n(D)=4

- $p(A) = \frac{5}{16}$
- $p(A \cap B) = \frac{0}{16} = 0$

  $A \cap B = \{\ \} = \emptyset$
- $p(B) = \frac{11}{16}$
- $p(A \cap C) = \frac{4}{16} = \frac{1}{4} = 0.25$
- $p(D) = \frac{4}{16} = \frac{1}{4} = 0.25$
- $p(A \cup C) = p(A) + p(C) - p(A \cap C) = \frac{5}{16} + \frac{8}{16} - \frac{4}{16} = \frac{9}{16}$
- $p(B \cap D) = \frac{4}{16} = \frac{1}{4} = 0.25$

Q2: $p(A) = 0.4$ $p(B) = 0.5$ $p(A \cap B) = 0.3$

- $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

  $= 0.4 + 0.5 - 0.3$

  $= 0.6$
- $p(A \cap B') = p(A) - p(A \cap B)$

  $= 0.4 - 0.3$

  $= 0.1$
- $p(A' \cup B') = p(A \cap B)'$

  $= 1 - p(A \cap B)$

  $= 1 - 0.3$

  $= 0.7$

Q3: $p(A \cup B) = 0.76$ $p(A \cup B') = 0.87$

- $p(A)$

$p(A \cup B') = p(A) + p(B') - p(A \cap B')$

$= \cancel{p(A)} + [1 - p(B)] - [\cancel{p(A)} - p(A \cap B)]$

$= 1 - p(B) + p(A \cap B)$

$= 1 - \cancel{p(B)} + [p(A) + \cancel{p(B)} - p(A \cup B)]$

$= 1 + p(A) - p(A \cup B)$

$0.87 = 1 + p(A) - 0.76$

$\boxed{p(A) = 0.63}$ #

$\boxed{\begin{array}{l} p(A \cup B) = p(A) + p(B) - p(A \cap B) \\ \Rightarrow p(A \cap B) = p(A) + p(B) - p(A \cup B) \end{array}}$

Q4: $P(R)=0.75$ $P(I)=0.65$ $P(T)=0.55$ $P(IR)=0.5$ $P(RT)=0.4$
$P(IT)=0.3$ $P(RIT)=0.20$

- $P(R\cup I\cup T) = P(R)+P(I)+P(T) - P(RI) - P(RT) - P(IT) + P(RIT)$
$= 0.75 + 0.65 + 0.55 - 0.5 - 0.4 - 0.3 + 0.2$
$= 0.95$

- $P(R^c \cup T^c) = P(R\cap T)^c = 1 - P(R\cap T)$
$= 1 - 0.4$
$= 0.6$

~~Q5: $P(A_1) = \frac{1041}{1456}$ $P(A_1|S_1) = \frac{P(A_1\cap S_1)}{P(S_1)} = \frac{n(A_1\cap S_1)}{n(S_1)} = \frac{392}{633}$~~

~~$P(A_1|S_2) = \frac{P(A_1\cap S_2)}{P(S_2)} = \frac{649}{823}$~~

Q5: $P(A)=0.7$ $P(B)=0.2$ A, B indep.

- $P(A\cap B) = P(A)P(B) = 0.7\times 0.2 = 0.14$
- $P(A\cup B) = P(A) + P(B) - P(A\cap B)$
$= 0.7 + 0.2 - (0.7\times 0.2)$
$= 0.7 + 0.2 - 0.14 = 0.76$
- $P(A'\cup B') = P(A\cap B)'$
$= 1 - P(A\cap B) = 1 - 0.14 = 0.86$

Q6: $P(A)=0.3$ $P(B)=0.6$

- $P(A\cup B)$, A B indep.
$P(A\cup B) = P(A) + P(B) - P(A\cap B)$
$= 0.3 + 0.6 - [0.3\times 0.6]$
$=$

- $P(A|B)$, A, B mutully exclusive.
$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{0}{0.6} = 0$

Q4: a- $f(x) = \frac{x}{c}$ $x = 1, 2, 3, 4$

$\sum_x f(x) = 1$

$\sum_{x=1}^{4} \frac{x}{c} = 1$

$\frac{1}{c} + \frac{2}{c} + \frac{3}{c} + \frac{4}{c} = 1$

$\frac{10}{c} = 1$

$\boxed{c = 10}$

b- $f(x) = cx$ $x = 1, 2, \ldots 10$

$\sum_x f(x) = 1$

$\sum_{x=1}^{10} cx = 1$

$c[1 + \cdots + 10] = 1$

$c\,55 = 1$

$\boxed{c = \frac{1}{55}}$

Q7: a- $f(x) = \frac{4-x}{6}$ $x = 1, 2, 3$

x	1	2	3	Total
f(x)	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	1

mean: $\mu = \sum x f(x)$

$= \frac{3}{6} + \frac{4}{6} + \frac{3}{6} = \frac{10}{6}$, $E[x^2] = \sum x^2 f(x) = \frac{3}{6} + \frac{8}{6} + \frac{9}{6} = \frac{20}{6}$

variance:

$\sigma^2 = E[x^2] - (E[x])^2$

$= \frac{20}{6} - \left(\frac{10}{6}\right)^2$

$=$

b- $f(x) = \frac{1}{5}$

mean: variance:

Q8: $E[x+4] = 10$, $E[(x+4)^2] = 116$

• $Var(x+4) =$ ~~Var(x)~~ $E[(x+4)^2] - E[(x+4)]^2 = 116 - 10^2 = 16$

• $\mu = E[x] \Rightarrow E[x+4] = 10$

$E[x] + 4 = 10$

$\boxed{E[x] = 6}$

• $\sigma^2 = Var(x) \Rightarrow E[x^2] - (E[x])^2$

$= 52 - 6^2$

$= 16$

$E[(x+4)^2] = 116$

$E[x^2 + 8x + 16] = 116$

$E[x^2] + 8E[x] + 16 = 116$

$E[x^2] = 116 - 16 - 8(6)$

$= 52$

Q9:

Binomial mean $= \mu = np = 10$ • Variance $= npq = 5$

• $f(x) = \binom{n}{x} p^x q^{n-x}$ $x = 0, \ldots, n$

$= \binom{20}{x} (0.5)^x (0.5)^{20-x}$ $x = 0, \ldots 20$

$10q = 5$

$q = \frac{5}{10}$

$\therefore \boxed{q = \frac{1}{2}}$

$q = 1 - p \Rightarrow \boxed{p = \frac{1}{2}}$

• $\mu = 10$

$np = 10$

$n\frac{1}{2} = 10$

$\boxed{n = 20}$

• $p(x=0) = 9.5 \times 10^{-7}$ • $p(x=1) = 1.9 \times 10^{-5}$ • $p(x=2) = 1.8 \times 10^{-4}$

• $p(X \geq 0) = 1 - p(X < 0) = 1$

Q_{10}: $X \sim Bin(n=10, p=0.4)$

$f(x) = \binom{n}{x} p^x q^{n-x} \quad x=0,\ldots,n \quad \Rightarrow f(x) = \binom{10}{x}(0.4)^x(0.6)^{10-x} \; ; \; x=0,1,\ldots 10$

~~p(x)~~ $P(X=4) = 0.2142$

$P(X<3) = 0.1673$

$P(X>3) = 0.0017$

mean: $\mu = np = 0.4 \times 10 = 4$

Variance: $\sigma^2 = npq = 10 \times 0.4 \times 0.6 = 2.4$

Q_{11}: $\lambda = 4$, $f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad x=0,1,\ldots$

$P(2 \leq X \leq 5) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= e^{-4}\left[\frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}\right]$

$= 0.6936$

$P(X \geq 3) = 1 - P(X<3) = 1 - [P(0) + P(1) + P(2)]$

$= 0.76$

Q_{12}: $P = 0.05 \quad n = 1000 \quad \Rightarrow \lambda = np = 1000(0.005) = 5 \qquad f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \; ; \; x=0,1,\ldots$

a. $P(X \leq 1) = P(X=0) + P(X=1)$

$= 0.0067 + 0.0337$

$= 0.0404$

b. $P(X=4) + P(X=5) + P(X=6) =$

$= 0.1755 + 0.1755 + 0.1462$

$= 0.4972$

Exercises 2

Q1: Let X be a continuous random variable on the interval (0, 1) with density function

$$f(x) = \begin{cases} 3x^2, & \text{for } 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the cumulative function F of X .

Q2: The proportion of time per day that all checkout counters in a supermarket are busy follows a distribution

$$f(x) = \begin{cases} kx^2(1-x)^9, & \text{for } 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

What is the value of the constant k so that f(x) is a valid probability density function ?

Q3: For each of the following functions:

(a) $\mathbf{f(x) = \frac{x^3}{4}}$, $\mathbf{0 < x < c}$, **(b)** $\mathbf{f(x) = \frac{3}{16}x^2}$, $\mathbf{-c < x < c}$ **H. W**

(i) find the constant c so that f (x) is a pdf of a random variable X,
(ii) find the cdf , F(x) = P(X ≤ x),
(iii) find μ and σ^2

Q4: Let $f(x) = \frac{1}{2}$, $-1 < x < 1$, be the pdf of X. Find the mean and variance of X.

Q5: Let X have an exponential distribution with mean $\theta > 0$. Show that :

$$P(X > x + y | X > x) = P(X > y)$$

Q6: Let X1,X2,X3,X4,X5 are independent and identically distribution exponential random variables with the parameter λ. Compute P{min(X1,X2,X3,X4,X5) ≤ a}

Maximum and minimum of independent random variables

- Let the random variables $X_1,\ldots,X_n$ be **totally independent**
- Denote: $X^{\min} := \min\{X_1,\ldots,X_n\}$. Then

$$P\{X^{\min} > x\} = P\{X_1 > x, \ldots, X_n > x\} = P\{X_1 > x\}\cdots P\{X_n > x\}$$

Exercise 2

Q1: $F(x) = P(X \leq x) = \int_0^x 3t^2 \, dt = 3\frac{t^3}{3}\Big]_0^x = x^3$

$$\therefore F(x) = \begin{cases} 0 & x < \\ x^3 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

Q2: Recall that for $f(x)$ to be a density function $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \int_0^1 k x^2 (1-x)^9 \, dx = 1$$

Let $u = 1-x \Rightarrow du = -dx$

if $x = 0 \rightarrow u = 1$
$x = 1 \rightarrow u = 0$

$$\int_1^0 k\,(1-u)^2 u^9 \, -du = 1$$

$$\int_0^1 k\,[1-2u+u^2]\, u^9 \, du = 1$$

$$k \int_0^1 (u^9 - 2u^{10} + u^{11})\, du = 1$$

$$k \left[\frac{u^{10}}{10} - 2\frac{u^{11}}{11} + \frac{u^{12}}{12} \right]_0^1 = 1$$

$$k\left[\frac{1}{10} - \frac{2}{11} + \frac{1}{12}\right] = 1$$

$$\frac{k}{660} = 1$$

$$\therefore \boxed{k = 660} \quad \#$$

Q_3: (a) $f(x) = \frac{x^3}{4} \qquad 0 < x < c$

i- $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$\Rightarrow \int_0^c \frac{x^3}{4}\, dx = 1$

$\Rightarrow \left[\frac{x^4}{16}\right]_0^c = 1$

$\Rightarrow \frac{c^4}{16} = 1$

$\Rightarrow c^4 = 16 \Rightarrow \boxed{c = 2}$ #

ii- $F(x) = p(X \leq x) = \int_0^x \frac{t^3}{4}\, dt = \left[\frac{t^4}{16}\right]_0^x = \frac{x^4}{16}$

iii- mean (μ) $(E(x))$:

$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_0^2 x\, \frac{x^3}{4}\, dx = \left[\frac{x^5}{20}\right]_0^2 = \frac{2^5}{20} = \frac{8}{5} = 1.6$

$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^2 x^2 \frac{x^3}{4}\, dx = \left[\frac{x^6}{24}\right]_0^2 = \frac{2^6}{24} = \frac{8}{3} = 2.667$

Variance (σ^2):

$\sigma^2 = E[x^2] - (E[x])^2$

$= \frac{8}{3} - \left(\frac{8}{5}\right)^2$

$= \frac{8}{75} \simeq 0.1067$

H.w

(b) $f(x) = \frac{3}{16} x^2 \qquad -c < x < c$

i- $\int_{-c}^{c} \frac{3}{16} x^2\, dx = 1$

$\left[\frac{\cancel{3}}{16} \frac{x^3}{\cancel{3}}\right]_{-c}^{c} = 1$

$\frac{c^3}{16} + \frac{c^3}{16} = 1$

$\frac{2c^3}{16} = 1 \Rightarrow c^3 = 8 \Rightarrow \boxed{c = 2}$ #

$F(x) = \frac{x^3}{16} + \frac{1}{2}$

$\mu = 0 \qquad E(x^2) = \frac{12}{5}$

$\sigma^2 = 2.4$

Q_4: $f(x) = \frac{1}{2} \quad -1 < x < 1$

$X \sim U(-1, 1)$

$X \sim U(a,b) \quad , a < b$

- $f(x) = \frac{1}{b-a} \quad a < x < b$
- $F(x) = \frac{x-a}{b-a}$
- $\mu = \frac{a+b}{2}$ • $\sigma^2 = \frac{(b-a)^2}{12}$

mean :

$$\mu = \frac{a+b}{2} = \frac{-1+1}{2} = 0$$

variance :

$$\sigma^2 = \frac{(b-a)^2}{12} = \frac{(1+1)^2}{12} = \frac{4}{12} = \frac{1}{3} = 0.333$$

Q_5: $P(X > x+y \mid X > x) = P(X > y)$

$X \sim \exp\left(\frac{1}{\theta}\right) \quad , \theta > 0$

* Conditional probability :

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

pdf : $f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \quad , \quad x > 0$

- $P(X > x) = \int_x^\infty \frac{1}{\theta} e^{-\frac{x}{\theta}} \, dx = \left[-e^{-\frac{x}{\theta}} \right]_x^\infty = e^{-\frac{x}{\theta}}$

Similary $P(X > x+y) = e^{-\frac{(x+y)}{\theta}}$ and $P(X > y) = e^{-\frac{y}{\theta}}$

- $P(X > x+y \mid X > x) = \frac{P(X > x+y \cap X > x)}{P(X > x)}$

$$= \frac{P(X > x+y)}{P(X > x)} = \frac{e^{-\frac{(x+y)}{\theta}}}{e^{-\frac{x}{\theta}}} = \frac{e^{-\frac{x}{\theta} - \frac{y}{\theta}}}{e^{-\frac{x}{\theta}}} = \frac{\cancel{e^{-\frac{x}{\theta}}} \, e^{-\frac{y}{\theta}}}{\cancel{e^{-\frac{x}{\theta}}}} = e^{-\frac{y}{\theta}}$$

$$= P(X > y) \quad \#$$

Q_6:

$$P(\min(X_1, X_2, X_3, X_4, X_5) \leq a) = 1 - P(\min(X_1, \dots, X_5) \geq a)$$

$$= 1 - P(X_1 \geq a, X_2 \geq a, \dots X_5 \geq a)$$

$$= 1 - \left[P(X_1 \geq a) \, P(X_2 \geq a) \dots P(X_5 \geq a) \right]$$

$$= 1 - \left[e^{-\lambda a} \; e^{-\lambda a} \; \dots \; e^{-\lambda a} \right]$$

$$= 1 - e^{-5\lambda a}$$

$X \sim \exp(\lambda)$

$f(x) = \lambda e^{-\lambda} \; ; \; \begin{matrix} x \geq 0 \\ \lambda > 0 \end{matrix}$

$F(x) = P(X \leq x) = 1 - e^{-\lambda x}$

$P(X \geq x) = e^{-\lambda x}$

Exercises 3 – chapter 2

Discrete

Q1: For each of the following functions, determine the constant **c** so that f (x, y) satisfies the conditions of being a joint pmf for two discrete random variables X and Y:

(a) $f(x, y) = c(x + 2y) \quad x = 1,2 \quad y = 1,2,3$

(b) $f(x, y) = c(x + y) \quad x = 1,2,3 \quad y = 1, \dots, x$

Q2: Let the joint pmf of X and Y be defined by:

$$f(x, y) = \frac{x + y}{32} \quad x = 1,2 \quad y = 1,2,3,4$$

(a) Find $f_X(x)$, the marginal pmf of X.
(b) Find $f_Y(y)$, the marginal pmf of Y.
(c) Find P(X > Y).
(d) Find P(Y = 2X).
(e) Find P(X + Y = 3).
(f) Find P(X ≤ 3 − Y).
(g) Are X and Y independent or dependent? Why or why not?
(h) Find the means and the variances of X and Y, cov(x,y).
(i) Find P(1 ≤ Y ≤ 3 |X = 1), P(Y ≤2 |X = 2), and P(X = 2 |Y = 3).
(j) Find E(Y |X = 1) and Var(Y |X = 1).

Q3: H.W

Suppose that X_1 and X_2 are discrete random variables with joint pmf of the form

$$f(x_1, x_2) = c(x_1 + x_2) \qquad x_1 = 0, 1, 2;\ x_2 = 0, 1, 2$$

and zero otherwise. Find the constant c.

Q4:

If X and Y are discrete random variables with joint pmf

$$f(x, y) = c\,\frac{2^{x+y}}{x!\,y!} \qquad x = 0, 1, 2, \dots;\ y = 0, 1, 2, \dots,$$

and zero otherwise.

(a) Find the constant c.
(b) Find the marginal pdf's of X and Y.
(c) Are X and Y independent? Why or why not?

Q1: Let $f(x,y) = \frac{3}{16}xy^2$, $0 \le x \le 2$, $0 \le y \le 2$, be the joint pdf of X and Y.

(a) Find $f_X(x)$ and $f_Y(y)$, the marginal probability density functions.
(b) Are the two random variables independent? Why or why not?
(c) Compute the means and variances of X and Y.
(d) Find $P(X \le Y)$.

Q2: Let X and Y have the joint pdf $f(x, y) = cx(1-y)$, $0 < y < 1$, and $0 < x < 1 - y$.
(a) Determine c.
(b) Compute $P(Y < X \mid X \le 1/4)$.

" Exercise 3 "

Q1: a- $f(x,y) = c(x+2y) \qquad x=1,2 \qquad y=1,2,3$

$$\sum_x \sum_y f(x,y) = 1$$

$$f(1,1)+f(1,2)+f(1,3)+f(2,1)+f(2,2)+f(2,3) = 1$$

$$3c + 5c + 7c + 4c + 6c + 8c = 1$$

$$33c = 1$$

$$\Rightarrow \boxed{c = \frac{1}{33}}$$

b- $f(x,y) = c(x+y) \qquad x=1,2,3 \qquad y=1,\dots,x$

$$\sum_x \sum_y f(x,y) = 1$$

$$f(1,1)+f(2,1)+f(2,2)+f(3,1)+f(3,2)+f(3,3) = 1$$

$$2c + 3c + 4c + 4c + 5c + 6c = 1$$

$$24c = 1$$

$$\Rightarrow \boxed{c = \frac{1}{24}}$$

Q2: $f(x,y) = \frac{x+y}{32} \qquad x=1,2 \qquad y=1,2,3,4$

a- $f_X(x)$:

$$f(x) = \sum_y f(x,y) = \sum_{y=1}^{4} \frac{x+y}{32}$$

$$= \frac{x+1}{32} + \frac{2+x}{32} + \frac{x+3}{32} + \frac{x+4}{32}$$

$$= \frac{4x+10}{32}$$

$$\therefore f(x) = \frac{4x+10}{32} \quad ; \quad x=1,2$$

b- $f_Y(y)$:

$$f(y) = \sum_x f(x,y) = \sum_{x=1}^{2} \frac{x+y}{32}$$

$$= \frac{1+y}{32} + \frac{2+y}{32}$$

$$= \frac{3+2y}{32}$$

$$\therefore f_Y(y) = \frac{3+2y}{32} \quad ; \quad y=1,2,3,4$$

c- $p(X>Y)$: $x = 1, 2$ $y = 1, 2, 3, 4$

$X>Y \Rightarrow$ true if $x = 2$ and $y = 1$

$p(X>Y) = f(2,1)$

$= \frac{2+1}{32} = \frac{3}{32} = 0.09375$

~~d- p(X+Y=3)~~ :

d- $P(Y=2X)$:

$Y = 2X$

$y=2 = 2(1)$, $\therefore x = 1$

$y = 4 = 2(2)$ $\therefore x = 4$

$P(Y=2X) = f(1,2) + f(2,4)$

$= \frac{1+2}{32} + \frac{2+4}{32}$

$= \frac{3}{32} + \frac{6}{32} = \frac{9}{32} = 0.28125$

e- $P(X+Y=3)$:

$X+Y=3$

if $x=1$ $y=2$

$x=2$ $y=1$

$P(X+Y=3) = f(1,2) + f(2,1)$

$= \frac{1+2}{32} + \frac{2+1}{32}$

$= \frac{3}{32} + \frac{3}{32} = \frac{6}{36} = 0.1875$

f- $P(X \leq 3-Y)$:

$X \leq 3-Y$

$X+Y \leq 3$

$1+1 \leq 3$

$1+2 \leq 3$

$2+1 \leq 3$

$P(X \leq 3-Y) = f(1,1) + f(1,2) + f(2,1)$

$= \frac{1+1}{32} + \frac{1+2}{32} + \frac{2+1}{32}$

$= \frac{2}{32} + \frac{3}{32} + \frac{3}{32}$

$= \frac{8}{32} = \frac{1}{4} = 0.25$

g- X and Y independent if $f(x,y) = f(x)\,f(y)$:

$f(x,y) = f(x)\,f(y)$

$\frac{x+y}{32} \neq \frac{4x+10}{32} \cdot \frac{3+2y}{32}$

$\therefore$ X and Y not independent (dependent) h- mean :

$$\mu_x = E[x] = \sum_x x f(x) = \sum_{x=1}^{2} x \cdot \frac{4x+10}{32} = \left[1 \cdot \frac{4+10}{32} + 2 \cdot \frac{8+10}{32}\right] = \frac{25}{16} = 1.5625$$

$$\mu_Y = E[Y] = \sum_{y=} y f(y) = \sum_{y=1}^{4} y \, \frac{3+2y}{32} = \left[1 \cdot \frac{3+2}{32} + 2 \cdot \frac{3+4}{32} + 3\,\frac{3+6}{32} + 4\,\frac{3+8}{32}\right] = \frac{45}{16} = 2.8125$$

$$E[x^2] = \sum_x x^2 f(x) = \sum_{x=1}^{2} x^2 \, \frac{4x+10}{32} = \frac{43}{16}$$

$$E[Y^2] = \sum_y y^2 f(y) = \sum_{y=1}^{4} y^2 \, \frac{3+2y}{32} = \frac{145}{16}$$

variance :

$$\sigma_X^2 = E[X^2] - \mu_x^2 =$$

$$= \frac{43}{16} - \left(\frac{25}{16}\right)^2$$

$$= \frac{63}{256} = 0.246$$

$$\sigma_Y^2 = E[Y^2] - \mu_Y^2$$

$$= \frac{145}{16} - \left(\frac{45}{16}\right)^2$$

$$= \frac{295}{256} = 1.152$$

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$= \frac{35}{8} - \left(\frac{25}{16} \cdot \frac{45}{16}\right)$$

$$= -\frac{5}{256} = -0.0195$$

$$E[XY] = \sum_x \sum_y xy \; f(x,y)$$

$$= \sum_{x=1}^{2} \sum_{y=1}^{4} xy \; \frac{x+y}{32}$$

$$= \Big[\underset{(1,1)}{\frac{2}{32}} + \underset{(1,2)}{2\,\frac{3}{32}} + \underset{(1,3)}{3\,\frac{4}{32}} + \underset{(1,4)}{4\,\frac{5}{32}} + \underset{(2,1)}{2\,\frac{3}{32}} + \underset{(2,2)}{4\,\frac{4}{32}} + \underset{(2,3)}{6\,\frac{5}{32}} + \underset{(2,4)}{8\,\frac{6}{32}} \Big]$$

$$= \frac{35}{8}$$

$$\cancel{i- P(1 \le Y \le 3 \mid X=1) = \sum_{Y=1}^{3} f(Y \mid x)}$$

$$\cancel{= \sum_{Y=1}^{3} \frac{f(x,y)}{f(x)}} =$$

i- $P(1 \leq Y \leq 3 \mid X=1) = \sum_{y=1}^{3} \frac{f(y \mid x=1)}{\cancel{f(x=1)}}$

$$= \sum_{y=1}^{3} \frac{f(y, x=1)}{f(x=1)}$$

$$= \frac{f(1,1)}{f(x=1)} + \frac{f(2,1)}{f(x=1)} + \frac{f(3, x=1)}{f(x=1)}$$

$$= \frac{2/\cancel{32}}{14/\cancel{32}} + \frac{3/\cancel{32}}{14/\cancel{32}} + \frac{4/\cancel{32}}{14/\cancel{32}}$$

$$= \frac{2}{14} + \frac{3}{14} + \frac{4}{14}$$

$$= \frac{9}{14}$$

$$f(x) = \frac{4x+10}{32} \quad x = 1, 2$$

$$f(x=1) = \frac{14}{32} \quad f(x=2) = \frac{18}{32}$$

$$f(x,y) = \frac{x+y}{32} \quad x = 1, 2 \quad y = 1, 2, 3, 4$$

$P(Y \leq 2 \mid X=2) = \sum_{y=1}^{2} f(y \mid x=2)$

$$= \sum_{y=1}^{2} \frac{f(y, x=2)}{f(x=2)}$$

$$= \frac{f(y=1, x=2)}{f(x=2)} + \frac{f(y=2, x=2)}{f(x=2)}$$

$$= \frac{3/\cancel{32}}{18/\cancel{32}} + \frac{4/\cancel{32}}{18/\cancel{32}}$$

$$= \frac{3}{18} + \frac{4}{18}$$

$$= \frac{7}{18}$$

$P(X=2 \mid Y=3) = \frac{f(x=2, y=3)}{f(y=3)} = \frac{5/\cancel{32}}{9/\cancel{32}}$

$$= \frac{5}{9}$$

$$f(y) = \frac{3+2y}{32} \quad y = 1, 2, 3, 4$$

$$f(3) = \frac{9}{32}$$

j- $E[Y|X=1] = \sum_y y\, f(y|X=1)$

$= \sum_{y=1}^{4} y \frac{f(y, x=1)}{f(x=1)}$

$= \left[1 \cdot \frac{2/32}{14/32}\right] + \left[2 \cdot \frac{3/32}{14/32}\right] + \left[3 \cdot \frac{4/32}{14/32}\right] + \left[4 \cdot \frac{5/32}{14/32}\right]$

$= \frac{2}{14} + \frac{6}{14} + \frac{12}{14} + \frac{20}{14}$

$= \frac{40}{14} = \frac{20}{7} = 2.857$

$V(Y|X=1) = E[Y^2|X=1] - \left(E[Y|X=1]\right)^2$

$= \frac{130}{14} - \left(\frac{20}{7}\right)^2$

$= \frac{55}{49} = 1.22$

$E[Y^2|X=1] = \sum_{y=1}^{4} y^2 f(y|X=1)$

$= \sum_{y=1}^{4} y^2 \frac{f(y, x=1)}{f(x=1)}$

$= \frac{2}{14} + \frac{12}{14} + \frac{36}{14} + \frac{80}{14}$

$= \frac{130}{14}$

Q3: $f(x_1, x_2) = c(x_1 + x_2)$ $\quad x_1 = 0, 1, 2 \quad x_2 = 0, 1, 2$

$\sum_{x_1} \sum_{x_2} f(x_1, x_2) = 1$

$\sum_{x_1=0}^{2} \sum_{x_2=0}^{2} c(x_1 + x_2) = 1$

$c(0+0) + c(0+1) + c(0+2) + c(1+0) + c(1+1) + c(1+2) + c(2+0) + c(2+1) + c(2+2) = 1$

$0 + c + 2c + c + 2c + 3c + 2c + 3c + 4c = 1$

$18c = 1$

$\Rightarrow \boxed{c = \frac{1}{18}}$

Q4: $f(x,y) = c\,\frac{2^{x+y}}{x!\,y!}$ $x = 0,1,2,\dots$ $y = 0,1,2,\dots$

a- $\sum_{x=0}^{\infty}\sum_{y=0}^{\infty} f(x,y) = 1$

$$\sum_{x=0}^{\infty}\sum_{y=0}^{\infty} c\,\frac{2^{x+y}}{x!\,y!} = 1$$

$$c \sum_{x=0}^{\infty}\frac{2^x}{x!}\left(\sum_{y=0}^{\infty}\frac{2^y}{y!}\right) = 1$$

$$\boxed{\sum_{x=0}^{\infty}\frac{k^x}{x!} = e^k}$$

$$c \sum_{x=0}^{\infty}\frac{2^x}{x!}\,e^2 = 1$$

$$c e^2 \sum_{x=0}^{\infty}\frac{2^x}{x!} = 1$$

$$c e^2 e^2 = 1 \Rightarrow c e^4 = 1 \Rightarrow c = \frac{1}{e^4}$$

$$\Rightarrow \boxed{c = e^{-4}}$$

b- $f_x(x) = \sum_{y=0}^{\infty} f(x,y) = \sum_{y=0}^{\infty} e^{-4}\frac{2^{x+y}}{x!\,y!} = \frac{2^x e^{-4}}{x!}\sum_{y=0}^{\infty}\frac{2^y}{y!}$

$$= \frac{2^x e^{-4}}{x!}\,e^2$$

$$= \frac{2^x e^{-2}}{x!} \quad ; \quad x = 0,1,2,\dots$$

$\therefore X \sim \text{poisson}(\lambda = 2)$

$f_y(y) = \sum_x f(x,y) = \sum_{x=0}^{\infty} e^{-4}\frac{2^{x+y}}{x!\,y!} = e^{-4}\frac{2^y}{y!}\sum_{x=0}^{\infty}\frac{2^x}{x!}$

$$= e^{-4}\frac{2^y}{y!}\,e^2$$

$$= \frac{2^y e^{-2}}{y!} \quad ; \quad y = 0,1,2,\dots$$

$\therefore Y \sim \text{poisson}(\lambda = 2)$

c- X and Y independend if $f(x,y) = f(x)\,f(y)$

$$\frac{e^{-4}\,2^{x+y}}{x!\,y!} = \frac{2^x e^{-2}}{x!}\cdot\frac{2^y e^{-2}}{y!}$$

$$\frac{e^{-4}\,2^{x+y}}{x!\,y!} = \frac{2^{x+y}\,e^{-4}}{x!\,y!}$$

$\therefore$ X and Y are independent

Exercises 4

Q1:Let $f(x,y) = \frac{3}{16}xy^2$, $0 \le x \le 2$, $0 \le y \le 2$ be the joint pdf of X and Y.

(a) Find $f_X(x)$ and $f_Y(y)$,the marginal probability density functions.

(b) Are the two random variables independent? Why or why not?

(c) Compute the means and variances of X and Y. **H.W**

(d) Find $P(X \le Y)$.

Q2: Let X and Y have the joint pdf $f(x,y) = x + y$, $0 \le x \le 1$, $0 \le y \le 1$.

(a) Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.

(b) show that $f(x,y) \equiv f_X(x) f_Y(y)$ .Thus, X and Y are dependent.

Q3:Let $f(x,y) = 2e^{-x-y}$, $0 \le x \le y \le \infty$ be the joint pdf of X and Y. Find $f_X(x)$ and $f_Y(y)$, the marginal pdfs of X and Y, respectively. Are X and Y independent? H.W

Q4: Let X and Y be continuous random variables with a joint pdf of the form

$$f(x, y) = k(x + y) \qquad 0 \leqslant x \leqslant y \leqslant 1$$

and zero otherwise.

(a) Find k so that $f(x, y)$ is a joint pdf.

(b) Find the marginals, $f_1(x)$ and $f_2(y)$.

(c) Find the conditional pdf $f(y \mid x)$.

(d) Find the conditional pdf $f(x \mid y)$.

Exercise 4

Q1: $f(x,y) = \frac{3}{16}xy^2 \qquad 0 \le x \le 2 \qquad 0 \le y \le 2$

ⓐ $f_X(x) = \int_y f(x,y)\,dy = \int_0^2 \frac{3}{16}xy^2\,dy = \frac{3}{16}x\left[\frac{y^3}{3}\right]_0^2 = \frac{\cancel{3}}{\cancel{16}_2}x\cdot\frac{\cancel{2^3}}{\cancel{3}} = \frac{x}{2}\;;\; 0 \le x \le 2$

$f_Y(y) = \int_x f(x,y)\,dx = \int_0^2 \frac{3}{16}xy^2\,dx = \frac{3}{16}y^2\left[\frac{x^2}{2}\right]_0^2 = \frac{3}{\cancel{16}_4}y^2\frac{\cancel{2^2}}{2} = \frac{3}{8}y^2\;;\; 0 \le y \le 2$

ⓑ X and Y independent $\Leftrightarrow f(x,y) = f(x)f(y)$

$\frac{3}{16}xy^2 = \frac{x}{2}\cdot\frac{3}{8}y^2$

$\frac{3}{16}xy^2 = \frac{3}{16}xy^2$

$\therefore$ X, Y independent

ⓒ $\mu_X = \int_x x\,f(x)\,dx = \int_0^2 x\cdot\frac{x}{2}\,dx = \frac{x^3}{2\cdot 3}\Big]_0^2 = \frac{2^3}{6} = \frac{4}{3}$

$E[X^2] = \int_x x^2 f(x)\,dx = \int_0^2 x^2\cdot\frac{x}{2}\,dx = \frac{x^4}{2\cdot 4}\Big]_0^2 = \frac{2^4}{2\cdot 4} = 2$

$\sigma_X^2 = E[X^2] - \mu_X^2 = 2 - \left(\frac{4}{3}\right)^2 = \frac{2}{9}$

$\mu_Y = \int_y y\cdot f(y)\,dy = \int_0^2 \frac{3}{8}y^3\,dy = \frac{3}{8}\frac{y^4}{4}\Big]_0^2 = \frac{3\cdot 2^4}{8\cdot 4} = \frac{3}{2}$

$E[X^2] = \int_y y^2 f(y)\,dy = \int_0^2 y^2\cdot\frac{3}{8}y^2\,dy = \frac{3}{8}\frac{y^5}{5}\Big]_0^2 = \frac{3}{8}\frac{2^5}{5} = \frac{12}{5}$

$\sigma_Y^2 = E[Y^2] - \mu_Y^2 = \frac{12}{5} - \left(\frac{3}{2}\right)^2 = \frac{3}{20}$

ⓓ $P(X \le Y) = \int_0^2\int_0^y f(x,y)\,dx\,dy$

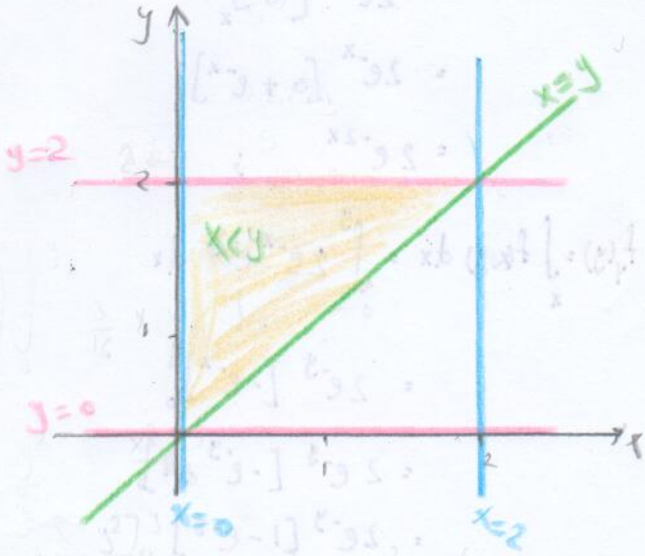
$= \int_0^2\int_0^y \frac{3}{16}xy^2\,dx\,dy$

$= \frac{3}{16}\int_0^2 y^2\frac{x^2}{2}\Big|_0^y\,dy$

$= \frac{3}{16}\int_0^2 \frac{y^4}{2}\,dy$

$= \frac{3}{16}\frac{y^5}{2\cdot 5}\Big]_0^2$

$= \frac{3}{16}\cdot\frac{2^5}{2\cdot 5} = \frac{3}{5}$



Q_2: $f(x,y) = x+y \qquad 0 \le x \le 1 \qquad 0 \le y \le 1$

ⓐ $f_x(x) = \int_y f(x,y)\,dy = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2} \; ; \; 0 \le x \le 1$

$f_y(y) = \int_x f(x,y)\,dx = \int_0^1 (x+y)\,dx = \left[\frac{x^2}{2} + yx\right]_0^1 = \frac{1}{2} + y \; ; \; 0 \le y \le 1$

ⓑ
if $f(x,y) \ne f(x)\,f(y) \Rightarrow x, y$ dependent

$(x+y) \ne (x+\frac{1}{2})(\frac{1}{2}+y)$

$\therefore$ X, Y dependent

[redacted]

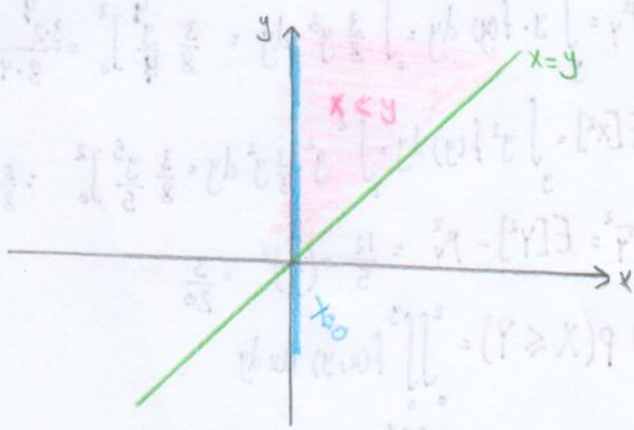
Q_3: $f(x,y) = 2e^{-x-y} \qquad 0 \le x \le y \le \infty$

$f_x(x) = \int_y f(x,y)\,dy = \int_{y=x}^{\infty} 2e^{-x}e^{-y}\,dy$

$= 2e^{-x}\left[-e^{-y}\right]_x^{\infty}$

$= 2e^{-x}\left[0 + e^{-x}\right]$

$= 2e^{-2x} \;;$



$f_y(y) = \int_x f(x,y)\,dx = \int_0^y 2e^{-x}e^{-y}\,dx$

$= 2e^{-y}\left[-e^{-x}\right]_0^y$

$= 2e^{-y}\left[-e^{-y} + 1\right]$

$= 2e^{-y}\left[1 - e^{-y}\right]$

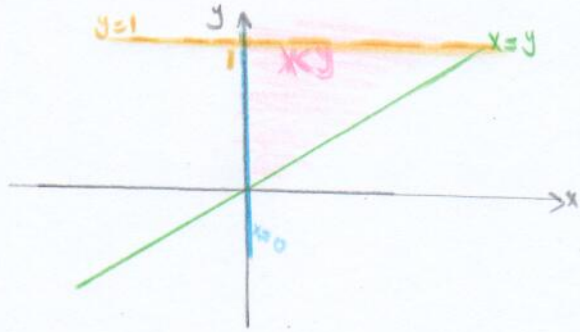
X, Y independent $\iff f(x,y) = f_x(x)\,f_y(y)$

$2e^{-x-y} \ne 2e^{-2x}\;2e^{-y}\left[1-e^{-y}\right]$

$\therefore$ X, Y not independent.

Kholoud Basalim

$Q_4: \; f(x,y) = k(x+y) \qquad 0 \leqslant x \leqslant y \leqslant 1$



ⓐ $\iint\limits_{y\;x} f(x,y)\,dx\,dy = 1$

$\int_0^1 \int_0^y k(x+y)\,dx\,dy = 1$

$\int_0^1 k\left[\frac{x^2}{2} + yx\right]_0^y dy = 1$

$k\int_0^1 \frac{y^2}{2} + y^2\,dy = 1$

$k\left[\frac{y^3}{2\cdot 3} + \frac{y^3}{3}\right]_0^1 = 1$

$k\left[\frac{1}{6} + \frac{1}{3}\right] = 1$

$\therefore \boxed{k = 2}$

ⓑ $f_x(x) = \int\limits_y f(x,y)\,dy = \int_x^1 2(x+y)\,dy = 2\left[xy + \frac{y^2}{2}\right]_x^1 = 2\left[(x+\tfrac{1}{2}) - (x^2 + \tfrac{x^2}{2})\right] = 2x + 1 - 3x^2 \quad ; 0<x<1$

$f_y(y) = \int\limits_x f(x,y)\,dx = \int_0^y 2(x+y)\,dx = 2\left[\frac{x^2}{2} + yx\right]_0^y = 2\left[(\frac{y^2}{2} + y^2) - (0+0)\right] = 3y^2 \quad ; \; 0<y<1$

ⓒ $f(y|x) = \frac{f(x,y)}{f(x)} = \frac{2(x+y)}{2x+1-3x^2}$

ⓓ $f(x|y) = \frac{f(x,y)}{f(y)} = \frac{2(x+y)}{3y^2}$

Exercises 5

Q1: Let $f(x,y) = \frac{3}{2}$, $x^2 \le y \le 1$, $0 \le x \le 1$, be the joint pdf of X and Y.

(a) Find $P(0 \le X \le \frac{1}{2})$.

(b) Find $P(\frac{1}{2} \le Y \le 1)$.

(c) Find $P(X \ge \frac{1}{2}, Y \ge \frac{1}{2})$.

(d) Are X and Y independent ? Why or why not?

Q2: Let $f(x, y) = \frac{4}{3}$, $0 < x < 1$, $x^3 < y < 1$, zero elsewhere.

(a) Find $P(X > Y)$.

Q3: Let X and Y have the joint pdf $f(x, y) = cx(1 - y)$, $0 < y < 1$, $0 < x < 1 - y$.

(a) Determine c.

(b) Compute $P(Y < X \mid X \le \frac{1}{4})$.

Q4: Let $f(x,y) = \frac{1}{40}$, $0 \le x \le 10$, $10 - x \le y \le 14 - x$, be the joint pdf of X and Y.

(a) Find $f_X(x)$,the marginal pdf of X.

(b) Determine $h(y \mid x)$, the conditional pdf of Y, given that X = x.

(c) Calculate $E(Y \mid x)$, the conditional mean of Y, given that X = x.

Q5: Let $f(x, y) = \frac{1}{8}$, $0 \le y \le 4$, $y \le x \le y + 2$, be the joint pdf of X and Y.

(a) Find $f_X(x)$,the marginal pdf of X.

(b) Find $f_Y(y)$,the marginal pdf of Y.

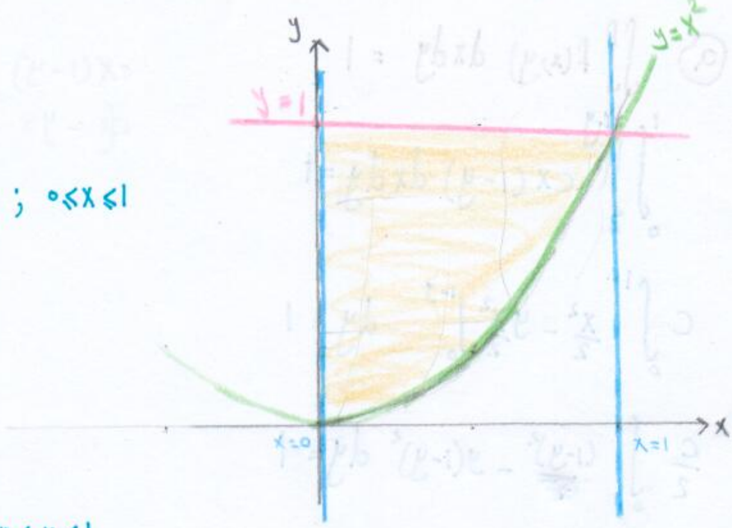
(c) Determine $h(y \mid x)$, the conditional pdf of Y, given that X = x.

(d) Determine $g(x \mid y)$, the conditional pdf of X, given that Y = y.

(e) Compute $E(Y \mid x)$, the conditional mean of Y, given that X = x.

(f) Compute $E(X \mid y)$, the conditional mean of X, given that Y = y

" Exercises 5 "



Q_1: $f(x,y)=\frac{3}{2}$, $x^2\leq y\leq 1$, $0\leq x\leq 1$

ⓐ $P(0\leq X\leq \frac{1}{2})$:

$f_X(x)=\int f(x,y)\,dy=\int_{x^2}^{1}\frac{3}{2}\,dy=\frac{3}{2}y\Big]_{x^2}^{1}=\frac{3}{2}(1-x^2)$; $0\leq x\leq 1$

$P(0\leq X\leq \frac{1}{2})=\int_0^{\frac{1}{2}}\frac{3}{2}(1-x^2)\,dx=\frac{3}{2}\left[x-\frac{x^3}{3}\right]_0^{\frac{1}{2}}=\frac{11}{16}$

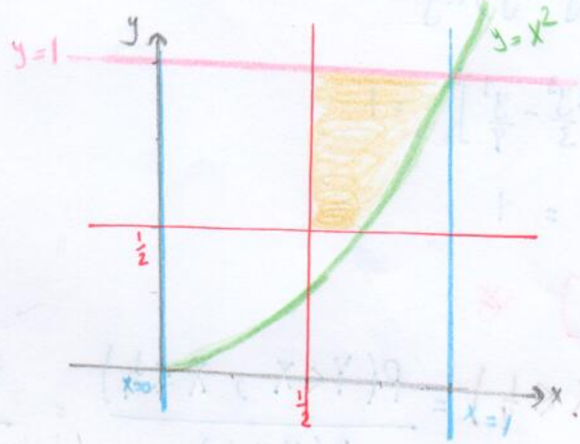
ⓑ $P(\frac{1}{2}\leq Y\leq 1)$:

$f_Y(y)=\int f(x,y)\,dx=\int_0^{\sqrt{y}}\frac{3}{2}\,dx=\frac{3}{2}x\Big]_0^{\sqrt{y}}=\frac{3}{2}\sqrt{y}$; $0\leq y\leq 1$

$P(\frac{1}{2}\leq Y\leq 1)=\int_{\frac{1}{2}}^{1}\frac{3}{2}\sqrt{y}\,dy=\cancel{\frac{3}{2}}\frac{y^{\frac{3}{2}}}{\cancel{\frac{3}{2}}}\Big]_{\frac{1}{2}}^{1}=1-(\frac{1}{2})^{\frac{3}{2}}=0.646$

$y^{\frac{1}{2}}$

ⓒ $P(X\geq \frac{1}{2}, Y\geq \frac{1}{2})$:



$=\iint f(x,y)\,dx\,dy$

$=\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{\sqrt{y}}\frac{3}{2}\,dx\,dy$

$=\int_{\frac{1}{2}}^{1}\frac{3}{2}(\sqrt{y}-\frac{1}{2})\,dy$

$=0.27144$

ⓓ X, Y independent $\Leftrightarrow$ $f_{XY}(x,y)=f_X(x)\,f_Y(y)$

$\frac{3}{2}\neq\frac{3}{2}(1-x^2)\cdot\frac{3}{2}\sqrt{y}$

$\therefore$ X, Y not independent

Q_2: $f(x,y)=\frac{4}{3}$, $0<x<1$, $x^3<y<1$

ⓐ $P(X>Y)=\iint f(x,y)\,dy\,dx$

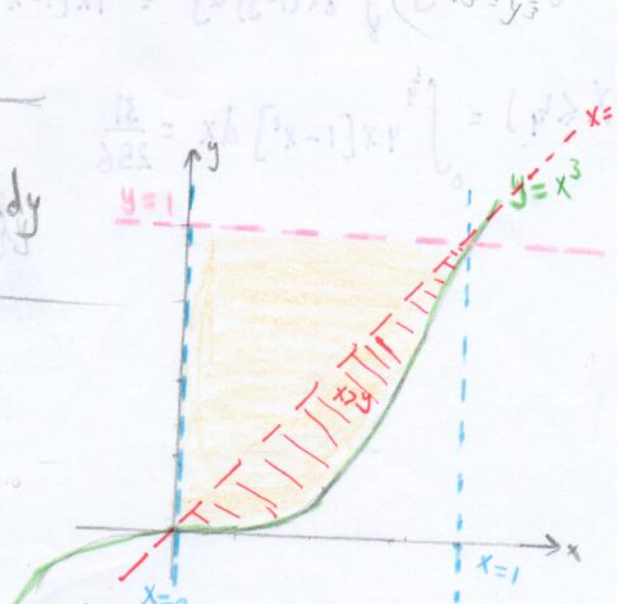
$=\int_0^1\int_{x^3}^{x}\frac{4}{3}\,dy\,dx$

$=\int_0^1 \frac{4}{3}y\Big]_{x^3}^{x}\,dx$

$=\frac{1}{3}$

or

$\boxed{\int_0^1\int_y^{\sqrt[3]{y}} f(x,y)\,dx\,dy}$



Q_3: $f(x,y) = cx(1-y)$, $0<y<1$, $0<x<1-y$

ⓐ $\iint f(x,y)\,dx\,dy = 1$

$cx(1-y)$

$c[x - yx]$

$$\int_0^1\int_0^{1-y} cx(1-y)\,dx\,dy = 1$$

$$c\int_0^1 \frac{x^2}{2} - y\frac{x^2}{2}\Big|_0^{1-y}\,dy = 1$$

$$\frac{c}{2}\int_0^1 (1-y)^2 - y(1-y)^2\,dy = 1$$

$$\frac{c}{2}\int_0^1 (1-2y+y^2) - y(1-2y+y^2)\,dy = 1$$

$$\frac{c}{2}\int_0^1 (1-3y+3y^2-y^3)\,dy = 1$$

$$\frac{c}{2}\left[y - \frac{3y^2}{2} + \frac{3y^3}{3} - \frac{y^4}{4}\right]_0^1 = 1$$

$$\frac{c}{2}\left[\frac{1}{4}\right] = 1$$

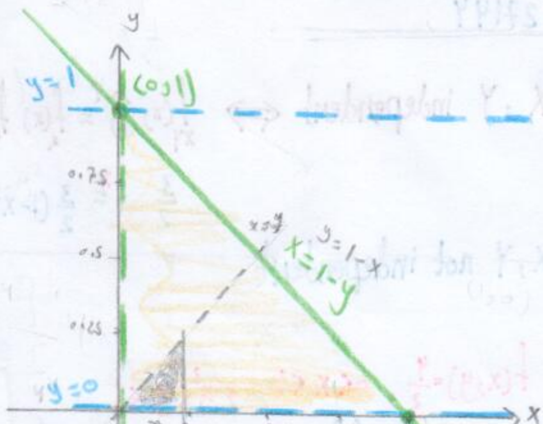
$$\Rightarrow \boxed{c = 8}$$

ⓑ $P(Y<X \mid X \leq \frac{1}{4}) = \dfrac{P(Y<X\,,\ X\leq\frac{1}{4})}{P(X\leq\frac{1}{4})} = \dfrac{P(Y<X\leq\frac{1}{4})}{P(X\leq\frac{1}{4})} = \dfrac{29/768}{31/256} = \dfrac{29}{93} = 0.3118$

- $P(Y<X\leq\frac{1}{4}) = \int_0^{\frac{1}{4}}\int_y^{\frac{1}{4}} 8x(1-y)\,dx\,dy = \frac{29}{768}$

- $f_X(x) = \int f(x,y)\,dy = \int_0^{1-x} 8x(1-y)\,dy = 4x[1-x^2]$, $0<x<1$

- $P(X\leq\frac{1}{4}) = \int_0^{\frac{1}{4}} 4x[1-x^2]\,dx = \frac{31}{256}$



$X = 1-y$

$x=0 \rightarrow y=1$

$x=1 \rightarrow y=0$

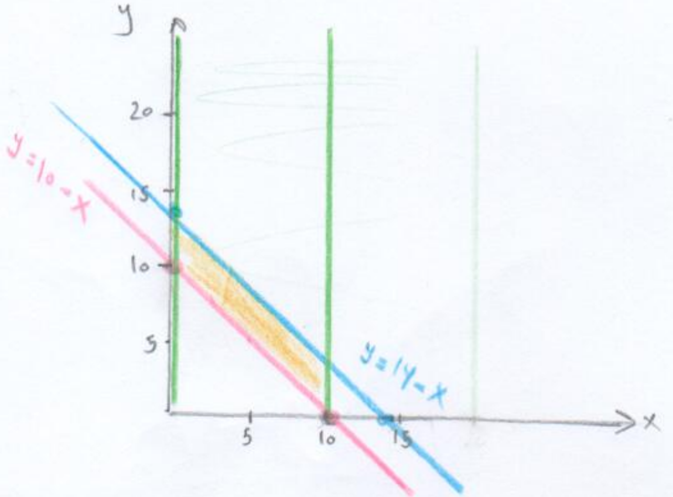
$x=-1 \rightarrow y=2$

Q_4 : $f(x,y) = \frac{1}{40}$ $\quad 0 \le x \le 10$, $\quad 10-x \le y \le 14-x$

ⓐ $f_x(x) = \int_y f_{xy}(x,y)\, dy = \int_{10-x}^{14-x} \frac{1}{40}\, dy = \frac{1}{40}\, y \Big]_{10-x}^{14-x} = \frac{1}{40}\left[(14-x)-(10-x)\right] = \frac{4}{40} = \frac{1}{10}$; $0 \le x \le 10$

ⓑ $h(y|x) = \frac{f(x,y)}{f_x(x)} = \frac{1/40}{1/10} = \frac{1}{4}$

ⓒ $E[Y|x] = \int_y y\, f(y|x)\, dy = \int_{10-x}^{14-x} y\, \frac{1}{4}\, dy = \left[\frac{y^2}{8}\right]_{10-x}^{14-x} = \frac{1}{8}\left[(14-x)^2-(10-x)^2\right] = 12-x$



$y = 10 - x$

x	10	0	5
y	0	10	5

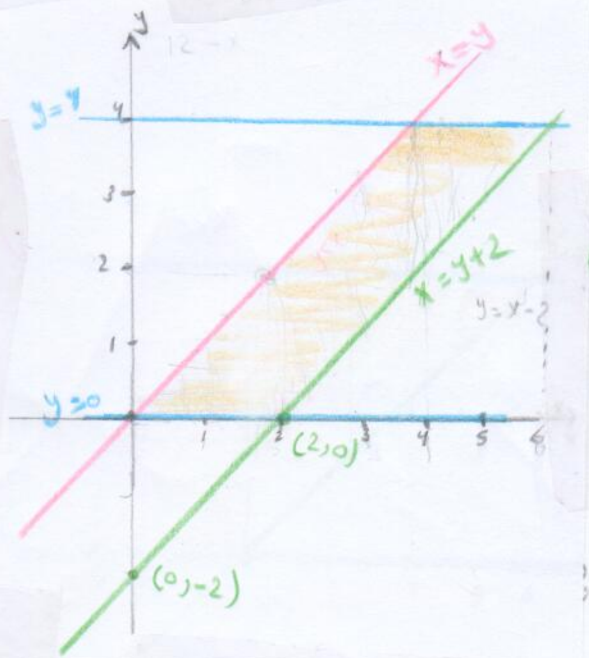
$y = 14 - x$

x	14	0	10
y	0	14	4

Q_5 : $f(x,y) = \frac{1}{8}$, $0 \le y \le 4$, $y \le x \le y+2$

ⓐ $f_x(x) = \int_y f(x,y)\, dy$

$$f_x(x) = \begin{cases} \int_0^x \frac{1}{8}\, dy = \frac{x}{8} & 0 \le x < 2 \\ \int_{x-2}^{x} \frac{1}{8}\, dy = \frac{1}{4} & 2 \le x < 4 \\ \int_{x-2}^{4} \frac{1}{8}\, dy = \frac{1}{8}[6-x] & 4 \le x \le 6 \end{cases}$$



ⓑ $f_y(y) = \int_x f(x,y)\, dx = \int_y^{y+2} \frac{1}{8}\, dx = \frac{1}{8} x \Big]_y^{y+2} = \frac{1}{8}[y+2-y] = \frac{1}{4}$; $0 \le y \le 4$

ⓒ $h(y|x) = \frac{f(x,y)}{f_x(x)}$

ⓓ $g(x|y) = \frac{f(x,y)}{f_y(y)} = \frac{1/8}{1/4} = \frac{1}{2}$

Exercises 6

Q1: Let X be a random variable with pdf $f(x) = 4x^3$ if $0 < x < 1$ and zero otherwise . Use the cumulative (CDF) technique to determine the pdf of the following random variable :

(a) $Y = X^4$
(b) $W = e^X$
(c) $Z = \ln X$

Q2: Let X be a random variable that is uniformly distributed , $X \sim UNIF(0,1)$. Use the CDF technique to determine the pdf of the following:

(a) $Y = X^{\frac{1}{4}}$
(b) $W = e^{-X}$ **(H.W)**
(c) $Z = 1 - e^{-X}$

Q3: The pdf of X is $f(x) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$. Let $Y = -2\,\theta\, lnX$. Use the cumulative (CDF) technique to determine the pdf of Y ? **(H.W)**

Q4: Let X have a logistic distribution with pdf

$$f(x) = \frac{e^{-x}}{(1 + e^{-x})^2}, \qquad -\infty < x < \infty$$

Show that

$$Y = \frac{1}{1 + e^{-X}}$$

has a U(0, 1) distribution. (Use the CDF technique)

Exercises 7

Q1: Let X have the pdf $f(x) = 4x^3$, $0 < x < 1$. Find the pdf of $Y = X^2$. (Use direct transformation method)

Q2: Let X have the pdf $f(x) = xe^{-\frac{x^2}{2}}$, $0 < x < \infty$. Find the pdf of $Y = X^2$. (Use direct transformation method)

Q3: Let X have a gamma distribution with $\alpha = 3$ and $\beta = \frac{1}{2}$. Determine the pdf of $Y = \sqrt{X}$.
(Use direct transformation method)

Q4: Rework (Question 1 in Exercise 6) using transformation methods (direct transformation method) **H.W**
Q5: Rework (Question 2 in Exercise 6) using transformation methods (direct transformation method) **H.W**

Q6: Let X_1 , X_2 denote two independent random variables, each with a $\chi^2_{(2)}$ distribution.
Find the joint pdf of $Y_1 = X_1$ and $Y_2 = X_1 + X_2$. Note that the support of Y_1 , Y_2 is $0 < y_1 < y_2 < \infty$.
Also, find the marginal pdf of each of Y_1 and Y_2.
Are Y_1 and Y_2 independent?

Q7: Let X_1 *and* X_2 be independent random variables, each with pdf

$$\boldsymbol{f(x) = e^{-x}} \quad , \quad \boldsymbol{0 < x < \infty}$$

Find the joint pdf of $Y_1 = X_1 - X_2$ *and* $Y_2 = X_1 + X_2$.

Q8: Let X and Y have joint pdf $f(x, y) = 4e^{-2(x+y)}$, $0 < x < \infty$, $0 < y < \infty$, and zero otherwise. Find the joint pdf of $U = X/Y$ and $V = X$.

Q9: Suppose that X_1 and X_2 are independent gamma variables,

$$f(x_1, x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x_1^{\alpha-1} x_2^{\beta-1} e^{-x_1-x_2} \qquad 0 < x_i < \infty$$

Find the joint pdf of $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$. **H.W**

" Exercises 7 "

Q1: $f(x) = 4x^3 \;,\; 0<x<1 \;,\; Y = X^2$

1. domain of Y:

$0<X<1 \Rightarrow 0<X^2<1$

$0<y<1$

2. Inverse function (invese transformation):

$Y = X^2 \Rightarrow X = \sqrt{y}$

$\therefore g^{-1}(y) = \sqrt{y}$

3. derivative $g^{-1}(y)$ with respect to y:

$$\frac{d}{dy} g^{-1}(y) = \frac{d}{dy}\sqrt{y}$$

$$= \frac{1}{2\sqrt{y}}$$

4. Use the formula to find pdf $(f_Y(y))$:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$f_Y(y) = \overset{2}{\cancel{4}}(\sqrt{y})^{\cancel{3}} \frac{1}{\cancel{2\sqrt{y}}}$$

$$f_Y(y) = 2y \;;\; 0<y<1$$

Q2: $f(x) = x e^{-\frac{x^2}{2}} \;,\; 0<x<\infty \;,\; Y = X^2$

1. domain of Y:

$0<X<\infty \Rightarrow 0<y<\infty$

2. inverse transformation:

$Y = X^2 \Rightarrow g^{-1}(y) = \sqrt{y}$

3. derivative $g^{-1}(y)$ with respect to y:

$$\frac{d}{dy} g^{-1}(y) = \frac{d}{dy}\sqrt{y} = \frac{1}{2\sqrt{y}}$$

4. use the formula to find pdf of Y $(f_Y(y))$:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= \sqrt{y}\, e^{-\frac{y}{2}} \frac{1}{2\sqrt{y}} \Rightarrow f_Y(y) = \frac{1}{2} e^{-\frac{y}{2}} \quad 0<y<\infty$$

$\therefore y \sim \exp(\lambda = 2)$

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Q_3 : $X \sim \text{gamma}(\alpha = 3, \beta = \frac{1}{2})$, $Y = \sqrt{X}$

$$f_X(x) = \frac{\beta^{\alpha} x^{\alpha - 1} e^{-\beta x}}{\Gamma(\alpha)} \quad ; \; 0 \leq x < \infty$$

$$= \frac{x^2 e^{-\frac{x}{2}}}{2^3 \, \Gamma 3} \quad ; \; 0 \leq x < \infty$$

$\boxed{1}$ domains of Y :

$0 \leq x < \infty \Rightarrow 0 \leq \sqrt{x} < \infty \Rightarrow 0 \leq y < \infty$

$\boxed{2}$ inverse transformation :

$Y = \sqrt{X} \Rightarrow X = Y^2$

$\therefore g^{-1}(y) = y^2$

$\boxed{3}$ derivative $g^{-1}(y)$ with respect to y :

$$\frac{d}{dy} g^{-1}(y) = \frac{d}{dy} y^2 = 2y$$

$\boxed{4}$ Use the formula to find pdf of Y :

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= \frac{y^4 e^{-\frac{y^2}{2}}}{2^3 \, \Gamma 3} \cdot 2y$$

$$= \frac{y^5 e^{-\frac{y^2}{2}}}{8} \quad ; \; 0 \leq y < \infty$$

Q_6: X_1, X_2 independent $\sim \chi^2_{(2)}$, $Y_1 = X_1$, $Y_2 = X_1 + X_2$

chi-square distribution:

$$f(x) = \frac{x^{\frac{r}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{r}{2}} \Gamma(\frac{r}{2})} \sim \chi^2_r \quad ; x \geq 0$$

① Find $f(x_1, x_2)$:

$$f_{X_1,X_2}(x_1, x_2) = f(x_1)\, f(x_2)$$

$$= \frac{e^{-\frac{x_1}{2}}}{2} \cdot \frac{e^{-\frac{x_2}{2}}}{2}$$

$$= \frac{e^{-(x_1+x_2)/2}}{4} \quad ; \quad x_1 \geq 0 \quad x_2 \geq 0$$

② Domain of Y_1, Y_2.

$Y_1 = X_1$ $\quad$ $Y_2 = X_1 + X_2 \Rightarrow X_2 = Y_2 - X_1 \Rightarrow X_2 = Y_2 - Y_1$

- $X_1 \geq 0 \Rightarrow y_1 \geq 0$
- $X_2 \geq 0 \Rightarrow y_2 - y_1 \geq 0 \Rightarrow y_2 > y_1 > 0$

③ inverse transformations:

$Y_1 = X_1$ $\quad$ $Y_2 = X_1 + X_2$

- $x_1 = y_1 \Rightarrow \therefore g_1^{-1}(y_1, y_2) = y_1$
- $x_2 = y_2 - x_1 \Rightarrow x_2 = y_2 - y_1 \Rightarrow \therefore g_2^{-1}(y_1, y_2) = y_2 - y_1$

④ Jacobian J:

$$J = \begin{vmatrix} \frac{\partial g_1^{-1}(y_1,y_2)}{\partial y_1} & \frac{\partial g_1^{-1}(y_1,y_2)}{\partial y_2} \\ \frac{\partial g_2^{-1}(y_1,y_2)}{\partial y_1} & \frac{\partial g_2^{-1}(y_1,y_2)}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

⑤ Use the formula to find $f_{Y_1,Y_2}(y_1, y_2)$:

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\left(g_1^{-1}(y_1,y_2)\ ,\ g_2^{-1}(y_1,y_2)\right) |J|$$

$$= \frac{e^{-\left(\frac{y_1 + y_2 - y_1}{2}\right)}}{4} \ (1)$$

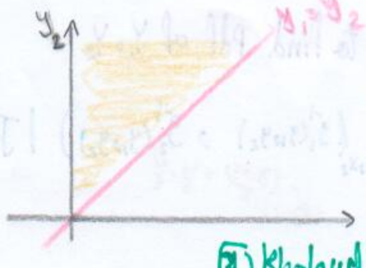
$$= \frac{e^{-\frac{y_2}{2}}}{4} \quad ; 0 < y_1 < y_2 < \infty$$

marginal pdf of Y_1:

$$f_{Y_1}(y_1) = \int f(y_1, y_2)\, dy_2$$

$$= \int_{y_1}^{\infty} \frac{e^{-\frac{y_2}{2}}}{4}\, dy_2 = -\frac{1}{2} e^{-\frac{y_2}{2}} \Big|_{y_1}^{\infty} = (0 + \frac{1}{2} e^{-\frac{y_1}{2}}) = \frac{e^{-\frac{y_1}{2}}}{2}$$

$0 < y_1 < \infty$



(3) Kholoud Basalim

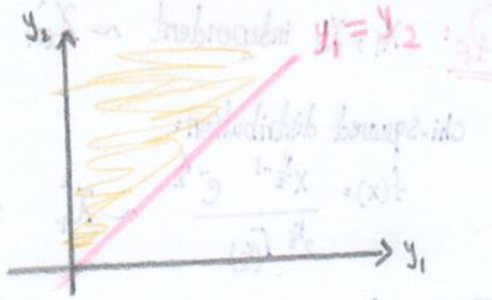
marginal pdf of Y_2 :

$$f_{Y_2}(y_2) = \int f(y_1, y_2)\, dy_1$$

$$= \int_0^{y_2} \frac{e^{-\frac{y_2}{2}}}{4}\, dy_1$$

$$= \frac{e^{-\frac{y_2}{2}}}{4}\; y_1 \Big|_0^{y_2}$$

$$= \frac{y_2 e^{-\frac{y_2}{2}}}{4} \quad ; \quad 0 < y_2 < \infty$$



* Y_1, Y_2 independent $\Leftrightarrow$ $f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1)\, f_{Y_2}(y_2)$

$$\frac{e^{-\frac{y_2}{2}}}{4} \neq \frac{e^{-\frac{y_1}{2}}}{2}\; \frac{y_2 e^{-\frac{y_2}{2}}}{4}$$

Thus Y_1, Y_2 are not independent

$\underline{Q_7}$: $f(x) = e^{-x}$; $0 < x < \infty$

X_1, X_2 independent , $Y_1 = X_1 - X_2$, $Y_2 = X_1 + X_2$

1) find $f(x_1, x_2)$:

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2)$$

$$= e^{-(x_1+x_2)} \quad ; \quad \begin{matrix} 0 < x_1 < \infty \\ 0 < x_2 < \infty \end{matrix}$$

2) Domain of Y_1, Y_2 :

$Y_1 = X_1 - X_2$	$Y_2 = X_1 + X_2$
$0 < X_1 < \infty$	$0 < X_2 < \infty$
$0 < \frac{y_1+y_2}{2} < \infty$	$0 < \frac{y_2 - y_1}{2} < \infty$
$-y_2 < y_1 < \infty$	$0 < y_2 - y_1 < \infty$
	$y_1 < y_2 < \infty$

$$-y_2 < y_1 < y_2 \quad , \quad 0 < y_2 < \infty$$

3) ~~Domain of Y_1, Y_2~~ inverse transformation :

$Y_1 = X_1 - X_2$ $\qquad$ $Y_2 = X_1 + X_2$

$y_1 + y_2 = \cancel{x_1} - x_2 + \cancel{x_1} + x_2$ $\qquad$ $y_1 - y_2 = \cancel{x_1} - x_2 - \cancel{x_1} - x_2$

$y_1 + y_2 = 2x_1$ $\qquad$ $y_1 - y_2 = -2x_2$

$\boxed{\frac{y_1+y_2}{2} = x_1}$ $\qquad$ $\boxed{\frac{y_2 - y_1}{2} = x_2}$

$\therefore g_1^{-1}(y_1, y_2) = \frac{y_1+y_2}{2}$ $\qquad$ $\therefore g_2^{-1}(y_1, y_2) = \frac{y_2-y_1}{2}$

4) Jacobian J :

$$J = \begin{vmatrix} \frac{\partial g_1^{-1}(y_1,y_2)}{\partial y_1} & \frac{\partial}{\partial y_2} g_1^{-1}(y_1,y_2) \\ \frac{\partial g_2^{-1}(y_1,y_2)}{\partial y_1} & \frac{\partial}{\partial y_2} g_2^{-1}(y_1,y_2) \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

5) Use the formula to find pdf of Y_1, Y_2 :

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\left(g_1^{-1}(y_1,y_2),\; g_2^{-1}(y_1,y_2)\right) |J| = \frac{1}{2} e^{-y_2} \quad ; \quad -y_2 < y_1 < y_2 \quad , \quad 0 < y_2 < \infty$$

$$e^{-\left(\frac{y_1+y_2}{2} + \frac{y_2 - y_1}{2}\right)}$$

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Q_8: $f(x,y) = 4e^{-2(x+y)}$, $0<x<\infty$, $0<y<\infty$, $U = \frac{X}{Y}$, $V = X$

1. Domain of U, V:

- $\boxed{V = X} \Rightarrow 0 < X < \infty \Rightarrow 0 < V < \infty$
- $\boxed{U = \frac{X}{Y} \quad Y = \frac{X}{U}} \Rightarrow 0 < Y < \infty \Rightarrow 0 < \frac{X}{U} < \infty \Longrightarrow \frac{1}{0} > \frac{U}{X} > \frac{1}{\infty} \Rightarrow \frac{U}{X} > 0 \Rightarrow U > 0$

2. inverse transformation:

$V = X \qquad \therefore g_1^{-1}(V,U) = V$

$U = \frac{X}{Y} \qquad Y = \frac{X}{U} = \frac{V}{U} \quad \therefore g_2^{-1}(V,U) = \frac{V}{U}$

3. Jacobian:

$$J = \begin{vmatrix} \frac{\partial g_1^{-1}(V,U)}{\partial V} & \frac{\partial g_1^{-1}(V,U)}{\partial U} \\ \frac{\partial g_2^{-1}(V,U)}{\partial V} & \frac{\partial g_2^{-1}(V,U)}{\partial U} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{U} & -\frac{V}{U^2} \end{vmatrix} = -\frac{V}{U^2}$$

4. Use the formula to find pdf of V, U:

$$f_{V,U}(V,U) = f_{X,Y}(g_1^{-1}(V,U), g_2^{-1}(V,U)) \, |J|$$

$$= 4e^{-2(V+\frac{V}{U})} \frac{V}{U^2}$$

$$= 4\frac{V}{U^2} e^{-2V(1+\frac{1}{U})} \quad ; \quad V > 0, \; U > 0$$

Q_9: $f(x_1,x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x_1^{\alpha-1} x_2^{\beta-1} e^{-x_1-x_2}$ $\quad 0 < X_i < \infty$, $Y_1 = X_1 + X_2$, $Y_2 = \frac{X_1}{X_1+X_2}$

1. Domain of Y_1, Y_2:

- $\boxed{Y_1 = X_1 + X_2}$ and $\boxed{Y_2 = \frac{X_1}{X_1+X_2}} \Rightarrow Y_1 Y_2 = \cancel{(X_1+X_2)} \frac{X_1}{\cancel{(X_1+X_2)}} = X_1$

$\therefore X_1 = Y_1 Y_2 \Rightarrow 0 < X_1 < \infty \Rightarrow 0 < y_1 y_2 < \infty \Rightarrow \boxed{0 < y_1 < \infty}$

$Y_1 = X_1 + X_2 \Rightarrow Y_1 = Y_1 Y_2 + X_2 \Rightarrow X_2 = Y_1 - Y_1 Y_2 \Rightarrow X_2 = Y_1(1-Y_2)$

$\therefore X_2 = Y_1(1-Y_2) \Rightarrow 0 < X_2 < \infty \Rightarrow 0 < y_1(1-y_2) < \infty \Rightarrow \boxed{0 < y_2 < 1}$

2. inverse transformation:

$Y_1 = X_1 + X_2 \qquad Y_2 = \frac{X_1}{X_1+X_2}$

from step(1):
$\Rightarrow \quad X_1 = y_1 y_2 \qquad X_2 = y_1(1-y_2)$

$\therefore g_1^{-1}(y_1,y_2) = y_1 y_2 \qquad g_2^{-1}(y_1,y_2) = y_1(1-y_2)$

3. Jacobian:

$$J = \begin{vmatrix} \frac{\partial g_1^{-1}(y_1,y_2)}{\partial y_1} & \frac{\partial g_1^{-1}(y_1,y_2)}{\partial y_2} \\ \frac{\partial g_2^{-1}(y_1,y_2)}{\partial y_1} & \frac{\partial g_2^{-1}(y_1,y_2)}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix} = -y_1 y_2 - y_1(1-y_2) = -y_1$$

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4. Use the formula to find pdf of Y_1, Y_2:

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}\left(g_1^{-1}(y_1,y_2) , g_2^{-1}(y_1,y_2)\right) \ |J|$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (y_1 y_2)^{\alpha-1} (y_1(1-y_2))^{\beta-1} e^{-y_1y_2 - y_1(1-y_2)} \ |-y_1|$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} y_1^{\alpha+\beta-1} y_2^{\alpha-1} (1-y_2)^{\beta-1} e^{-y} \ ; \quad 0 < y_1 < \infty, \quad 0 < y_2 < 1$$

Exercises 8

Q1: Let X_1, X_2, X_3, X_4, X_5 be a random sample of size 5 from a geometric distribution with $p = \frac{1}{3}$.
(a) Find the Moment-generating function (MGF) of $Y = X_1 + X_2 + X_3 + X_4 + X_5$.
(b) How is Y distributed?

Q2: Let X_1, X_2, X_3 denote a random sample of size 3 from a gamma distribution with $\alpha = 7$ and $\beta = 5$.
(a) Find the MGF of $Y = X_1 + X_2 + X_3$.
(b) How is Y distributed?

Q3: Let $W = X_1 + X_2 + \cdots \ldots \ldots + X_h$, a sum of h mutually independent and identically distributed exponential random variables with parameter θ. Show that W has a gamma distribution with parameters $\alpha = h$ and $\beta = \theta$, respectively.

Q4: Let $X_1, X_2, \ldots \ldots \ldots, X_{10}$ be a random sample of size n=10 from exponential distribution with parameter $\theta = 2$, $X_i \sim Exp(\theta = 2)$.
- **(a)** Find the MGF of $Y = \sum_{i=1}^{10} X_i$.
- **(b)** What is the pdf of Y?

Q5: Let X_1, X_2, X_3 be mutually independent random variables with Poisson distributions having means 2, 1, and 4, respectively.
(a) Find the MGF of the sum $Y = X_1 + X_2 + X_3$.
(b) How is Y distributed?

Q6: Generalize **Q5** by showing that the sum of n independent Poisson random variables with respective means $\mu 1, \mu 2, \ldots, \mu n$ is Poisson with mean $\mu_1 + \mu_2 + \cdots . + \mu_n$ **H.W**

Markov's inequality
If X is a random variable that takes only nonnegative values, then, for any value $a > 0$,
$$P\{X \geq a\} \leq \frac{E[X]}{a}$$

Q7: Let $X_1, X_2, \ldots \ldots, X_{20}$ be independent Poisson random variables with mean 1.

Use the Markov inequality to obtain a bound on $P\left(\sum_{i=0}^{20} X_i \geq 15\right)$.

Q8: Let X be a Poisson random variable with mean 20. Use the Markov inequality to obtain a bound on $P(X \geq 26)$. **H.W**

Q9 : Suppose that it is known that the number of items produced in factory during a week is a random variable with mean 50. Give an upper bound on the probability that this week's production will be more than or equal 75 ?

	Probability mass function, $p(x)$	Moment generating function, $M(t)$	Mean	Variance
Binomial with parameters n, p; $0 \le p \le 1$	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$(pe^t + 1 - p)^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda}\frac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \le p \le 1$	$p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Negative binomial with parameters r, p; $0 \le p \le 1$	$\binom{n-1}{r-1} p^r (1-p)^{n-r}$, $n = r, r+1, \ldots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$

	Probability mass function, $f(x)$	Moment generating function, $M(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters $(s, \lambda), \lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{s-1}}{\Gamma(s)} & x \ge 0 \\ 0 & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda - t}\right)^s$	$\frac{s}{\lambda}$	$\frac{s}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$ $\quad -\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2

Q1: $X_1, X_2, \dots, X_5 \sim \text{Geometric}(P = \frac{1}{3})$

a- moment-generating function of $Y = X_1 + X_2 + X_3 + X_4 + X_5$.

if $X \sim \text{Geometric}(P) \Rightarrow f_X(x) = p^x (1-p)^{1-x} \quad x = 1, 2, \dots$

$$M_X(t) = \frac{Pe^t}{1-(1-P)e^t}$$

$$M_Y(t) = E[e^{tY}] = E\left[e^{t(X_1+X_2+X_3+X_4+X_5)}\right]$$
$$= E\left[e^{tX_1+tX_2+tX_3+tX_4+tX_5}\right]$$
$$= E[e^{tX_1}] \cdot E[e^{tX_2}] \cdot E[e^{tX_3}] \cdot E[e^{tX_4}] \cdot E[e^{tX_5}]$$
$$= M_{X_1}(t) \cdot M_{X_2}(t) \cdot M_{X_3}(t) \cdot M_{X_4}(t) \cdot M_{X_5}(t)$$
$$= \frac{Pe^t}{1-(1-P)e^t} \cdot \frac{Pe^t}{1-(1-P)e^t} \cdot \frac{Pe^t}{1-(1-P)e^t} \cdot \frac{Pe^t}{1-(1-P)e^t} \cdot \frac{Pe^t}{1-(1-P)e^t}$$

$$M_Y(t) = \left[\frac{Pe^t}{1-(1-P)e^t}\right]^5 = \left[\frac{\frac{1}{3}e^t}{1-(1-\frac{1}{3})e^t}\right]^5$$

$Y = X_1 + X_2 + X_3 + X_4 + X_5 \sim \text{Negative Binomial}(r = 5, P = \frac{1}{3})$

Q2: $X_1, X_2, X_3 \sim \text{Gamma}(\alpha = 7, B = 5)$

if $X \sim \text{Gamma}(\alpha = 7, \beta = 5) \Rightarrow f_X(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-x\beta}}{\Gamma(\alpha)} \quad x$

$$M_X(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha$$

- MGF of $Y = X_1 + X_2 + X_3$:

$$M_Y(t) = E[e^{tY}] = E\left[e^{t(X_1+X_2+X_3)}\right]$$
$$= E\left[e^{tX_1} e^{tX_2} e^{tX_3}\right]$$
$$= E[e^{tX_1}] \cdot E[e^{tX_2}] \cdot E[e^{tX_3}]$$
$$= M_{X_1}(t) \cdot M_{X_2}(t) \cdot M_{X_3}(t)$$
$$= \left(\frac{5}{5-t}\right)^7 \cdot \left(\frac{5}{5-t}\right)^7 \cdot \left(\frac{5}{5-t}\right)^7 = \left(\frac{5}{5-t}\right)^{7+7+7} = \left(\frac{5}{5-t}\right)^{21}$$

$\therefore Y = X_1 + X_2 + X_3 \sim \text{Gamma}(\alpha = 21, B = 5)$

 Kholoud Basalim

$\theta e^{-\theta x}$

mean $= \frac{1}{\lambda} = \theta$

$\Rightarrow \lambda = \frac{1}{\theta}$

Q3: $X_1, X_2, \dots, X_n \sim exp(\theta) \frac{1}{\theta}$

$X \sim Exponential(\theta) \Rightarrow f_X(x) = \theta e^{-\theta x} \quad ; x \geq 0$

$$M_X(t) = \frac{\theta}{\theta - t}$$

$W = X_1 + X_2 + \dots + X_n$

$$M_W(t) = E[e^{tW}] = E\left[e^{t(X_1+X_2+\dots+X_n)}\right]$$

$$= E\left[e^{tX_1+tX_2+\dots+tX_n}\right]$$

$$= E[e^{tX_1}] \cdot E[e^{tX_2}] \cdot \dots \cdot E[e^{tX_n}]$$

$$= M_{X_1}(t) \cdot M_{X_2}(t) \cdot \dots \cdot M_{X_n}(t)$$

$$= \frac{\theta}{\theta-t} \cdot \frac{\theta}{\theta-t} \cdot \dots \cdot \frac{\theta}{\theta-t}$$

$$= \left(\frac{\theta}{\theta-t}\right)^n$$

$\therefore W = X_1 + X_2 + \dots + X_n \sim Gamma(\alpha = n, \beta = \theta)$

Q4: $X_1, X_2, \dots, X_{10} \sim exp(\theta = 2)$

$X_i \sim Exp(2) \Rightarrow f_X(x) = 2e^{-2x} \quad ; x \geq 0$

$$M_X(t) = \frac{2}{2-t}$$

$Y = \sum_{i=1}^{10} X_i$

$$M_Y(t) = E[e^{tY}] = E\left[e^{t(X_1+X_2+\dots+X_{10})}\right]$$

$$= E[e^{tX_1}] \cdot E[e^{tX_2}] \cdot \dots \cdot E[e^{tX_{10}}]$$

$$= M_{X_1}(t) \cdot M_{X_2}(t) \cdot \dots \cdot M_{X_{10}}(t)$$

$$= \frac{2}{2-t} \cdot \frac{2}{2-t} \cdot \dots \cdot \frac{2}{2-t}$$

$$= \left(\frac{2}{2-t}\right)^{10}$$

$\therefore Y = \sum_{i=1}^{10} X_i \sim Gamma(\alpha = 10, \beta = 2)$

$\left(\frac{\beta}{\beta - t}\right)^{\alpha} \Rightarrow$ MGF of Gamma

Q_5: $X_1 \sim \text{poission}(\lambda_1 = 2)$

$X_2 \sim \text{poisson}(\lambda_2 = 1)$

$X_3 \sim \text{poisson}(\lambda_3 = 4)$

if $X \sim \text{poisson}(\lambda) \Rightarrow f_X(x) = \dfrac{\lambda^x e^{-\lambda x}}{x!}$; $x = 0, 1, 2, \dots$

$$M_X(t) = e^{\lambda(e^t - 1)}$$

- MGF of $Y = X_1 + X_2 + X_3$

$$M_Y(t) = E[e^{tY}] = E\left[e^{t(x_1 + x_2 + x_3)}\right]$$

$$= E[e^{tx_1}]\, E[e^{tx_2}]\, E[e^{tx_3}]$$

$$= M_{X_1}(t) \cdot M_{X_2}(t) \cdot M_{X_3}(t)$$

$$= e^{2(e^t - 1)} \cdot e^{1(e^t - 1)} \cdot e^{4(e^t - 1)}$$

$$= e^{2(e^t - 1) + 1(e^t - 1) + 4(e^t - 1)}$$

$$= e^{(e^t - 1)[2 + 1 + 4]}$$

$$= e^{7(e^t - 1)}$$

$\therefore Y = X_1 + X_2 + X_3 \sim \text{Poisson}(\lambda = 7)$

$\therefore f_Y(y) = \dfrac{7^y e^{-7y}}{y!}$, $y = 0, 1, 2, \dots$

Q_6:

if $X_1 \sim \text{poisson}(\mu_1)$

$X_2 \sim \text{poisson}(\mu_2)$

$\vdots$

$X_n \sim \text{poisson}(\mu_n)$

$Y = X_1 + X_2 + \dots + X_n \Rightarrow Y = \sum_{i=1}^{n} x_i \sim \text{poisson}(\mu_1 + \mu_2 + \dots + \mu_n)$

markov's inequality:

Q7: $X_1, X_2, \dots X_{20}$ indep. poisson with mean = 1

$$P\left(\sum_{i=1}^{20} X_i \geq 15\right) \leq \frac{E\left[\sum_{i=1}^{20} X_i\right]}{a}$$

$$\leq \frac{20}{15}$$

$$\leq \frac{4}{3}$$

$X \sim$ poisson $(\lambda = 1)$

$Y = \sum_{i=1}^{n} X_i \sim$ poisson $(n\lambda)$

$\Rightarrow E[Y] = n\lambda = 20(1) = 20$

Q8: $X \sim$ poisson $(\lambda = 20)$

$$P(X \geq 26) \leq \frac{E[X]}{a}$$

$$\leq \frac{20}{26} = 0.7692$$

Q9:

X = number of items that will be produced in a week.

By Markov's inequality:

$$P(X \geq 75) \leq \frac{E[X]}{a} = \frac{20}{75} = \frac{2}{3}$$

Exercises 9

Chapter 4
Probability inequalities

- Chebyshev's inequality

$$\mathbb{P}\left(|X - \mathbb{E}(X)| \geq \alpha\right) \leq \frac{V(X)}{\alpha^2}$$

$$\mathbb{P}\left(|X - \mathbb{E}(X)| \leq \alpha\right) \geq 1 - \frac{V(X)}{\alpha^2}$$

One-Sided Chebyshev

$$\mathbb{P}\left(X \geq \mu + \alpha\right) \leq \frac{\sigma^2}{\alpha^2 + \sigma^2}$$

$$\mathbb{P}\left(X \leq \mu - \alpha\right) \leq \frac{\sigma^2}{\alpha^2 + \sigma^2}$$

Q1: If X is a random variable with mean 33 and variance 16, use Chebyshev's inequality to find :
(a) A **lower bound** for $P(23 < X < 43)$.
(b) An **upper bound** for $P(|X - 33| \geq 14)$.

Q2: If $E(X) = 17$ and $E(X^2) = 298$, use Chebyshev's inequality to determine
(a) A **lower bound** for $P(10 < X < 24)$.
(b) An **upper bound** for $P(|X - 17| \geq 16)$.

Q3: Let X be a Poisson random variable with mean 20. Use the Chebyshev inequality to obtain **an upper bound** on $P(X \geq 26)$.

Q4: If the number of items produced in factory during a week is random variable with mean 100 and variance 400 ,use Chebyshev inequality compute **an upper bound** on the probability that this week's production will be **at least 120** . **H.W**

Chapter 5 Order Statistics

- **Density of the maximum**

$$f_{(n)}(x) = nf(x)F(x)^{n-1}$$

- **Density of the minimum**

$$f_{(1)}(x) = nf(x)(1-F(x))^{n-1}$$

Density of the kth Order Statistic

$$f_{(k)}(x) = nf(x)\binom{n-1}{k-1}F(x)^{k-1}(1-F(x))^{n-k}$$

Cumulative Distribution of the min and max

$$F_{(1)}(x) = 1-(1-F(x))^n$$

$$F_{(n)}(x) = F(x)^n$$

Q5: Let $Y_{(1)} < Y_{(2)} < Y_{(3)} < Y_{(4)} < Y_{(5)}$ be the order statistics of five independent observations from an exponential distribution that has a mean of $\theta = 3$.

(a) Find the pdf of the sample median $Y_{(3)}$.

(b) Compute the probability that $Y_{(4)}$ is less than 5.

(c) Determine $P(1 < Y_{(1)})$.

Q6: Let $Y_{(1)} < Y_{(2)} < \cdots \ldots\ldots\ldots\ldots\ldots < Y_{(19)}$ be the order statistics of $n = 19$ independent observations from the exponential distribution with mean θ. What is the **pdf of $\mathbf{Y_{(1)}}$**? **H.W**

Q7: Consider a random sample of size n from a distribution with pdf $f(x) = \frac{1}{x^2}\ if\ 1 \le x < \infty$; zero otherwise .

(a) Find the pdf of the smallest order statistic, $Y_{(1)}$

(b) Find the pdf of the largest order statistic, $Y_{(n)}$

Q8: Let $Y_{(1)} < Y_{(2)} < Y_{(3)} < Y_{(4)} < Y_{(5)} < Y_{(6)}$ be the order statistics associated with $n = 6$ independent observations each from the distribution with probability density function:

$$f(x) = \frac{1}{2}x$$

for $0 < x < 2$. What is the **probability density function of the first, fourth, and sixth order statistics**? **H.W**

Answer:

$g_1(y) = 3y\left(1-\frac{y^2}{4}\right)^5,\ 0 < y < 2,$ / $g_4(y) = \frac{15}{32}y^7\left(1-\frac{y^2}{4}\right)^2,\ 0 < y < 2,$ / $g_6(y) = \frac{3}{1024}y^{11},\ 0 < y < 2,$

Q1: $\mu = 33$, $\sigma^2 = 16$

a- Lower bound $p(23 < X < 43)$.

$$p(23 < X < 43) = p(23-\mu < X-\mu < 43-\mu)$$
$$= p(23-33 < X-33 < 43-33)$$
$$= p(-10 < X-33 < 10)$$
$$= p(|X-33| \leq 10)$$

$$\therefore\ p(|X-\underset{\mu}{33}| \leq \underset{\alpha}{10}) \geq 1-\frac{V(x)}{\alpha^2} = 1-\frac{16}{10^2} = 0.84$$

$\Leftarrow$ $p(|X-E(x)| \leq \alpha) \geq 1-\frac{V(x)}{\alpha^2}$

b- upper bound $p(|X-33| \geq 14)$.

$$p(|X-33| \geq \underset{\alpha}{14}) \leq \frac{V(x)}{\alpha^2} = \frac{16}{14^2} = 0.0816$$

$\Leftarrow$ $p(|X-E(x)| \geq \alpha) \leq \frac{V(x)}{\alpha^2}$

Q2: $E(x) = 17$, $E[x^2] = 298$

a- Lower bound $p(10 < X < 24)$.

$$p(10 < X < 24) = p(10-\mu < X-\mu < 24-\mu)$$
$$= p(10-17 < X-17 < 24-17)$$
$$= p(-7 < X-17 < 7)$$
$$= p(|X-17| \leq 7)$$

$p(|X-E(x)| \leq \alpha) \geq 1-\frac{V(x)}{\alpha^2}$

$$\therefore\ p(|X-17| \leq \underset{\alpha}{7}) \geq 1-\frac{V(x)}{\alpha^2} = 1-\frac{9}{7^2} = 0.8163$$

$* \ V(x) = E[x^2] - E[x]^2 = 298 - (17)^2 = 9$

b- Upper bound $p(|X-17| \geq 16)$.

$p(|X-E(x)| \geq \alpha) \leq \frac{V(x)}{\alpha^2}$

$$p(|X-17| \geq \underset{\alpha}{16}) \leq \frac{V(x)}{\alpha^2} = \frac{9}{16^2} = 0.035156$$

Q_3: $X \sim poisson(\lambda = 20)$ mean $= \lambda = 20$ Variance $= \lambda = 20$

- upper bound $p(X \geq 26)$

$$p(\underbrace{X \geq 26}_{\mu+\alpha \\ 20+\alpha}) = p(X \geq 20+6) \leq \frac{\sigma^2}{\alpha^2+\sigma^2} = \frac{20}{6^2+20} = 0.3571$$

$\Rightarrow \boxed{\alpha = 6}$

Q_4: mean $= 100$ Variance $= 400$

upper bound $p(X \geq 120)$

$$p(\underbrace{X \geq 120}_{\mu+\alpha \\ 100+\alpha}) = p(X \geq 100+20) \leq \frac{\sigma^2}{\alpha^2+\sigma^2} = \frac{400}{20^2+400} = 0.5$$

$\Rightarrow \boxed{\alpha = 20}$

Q_5: $Y_{(1)} < Y_{(2)} < Y_{(3)} < Y_{(4)} < Y_{(5)}$ Order Statistics

Exp. dis: $f(y) = \lambda e^{-\lambda y}$; $y > 0$ / mean $= \frac{1}{\lambda} = 3 \Rightarrow \lambda = \frac{1}{3}$ / $F(y) = 1 - e^{-\lambda y}$

$f(y) = \frac{1}{3} e^{-\frac{y}{3}}$; $y > 0$ and $F(y) = 1 - e^{-\frac{y}{3}}$

a- pdf $Y_{(3)}$:

$$f_{(k)}(y) = n f(y) \binom{n-1}{k-1} F(y)^{k-1} (1 - F(y))^{n-k}$$

$\boxed{k=3 \mid n=5}$

$$f_{(3)}(y) = 5 \, \frac{1}{3} e^{-\frac{y}{3}} \binom{5-1}{3-1} (1 - e^{-\frac{y}{3}})^{3-1} (1 - (1 - e^{-\frac{y}{3}}))^{5-3}$$

$$= 10 \, e^{-\frac{y}{3}} (1 - e^{-\frac{y}{3}})^2 (e^{-\frac{y}{3}})^2$$

$$= 10 \, e^{-y} (1 - e^{-\frac{y}{3}})^2 \quad ; \; y > 0$$

b- $P(Y_{(4)} < 5)$

· find pdf $Y_{(4)}$: $\boxed{k=4 \quad n=5}$

$$f_{(k)}(y) = n\, f(y) \binom{n-1}{k-1} F(y)^{k-1} (1-F(y))^{n-k}$$

$$f_{(4)}(y) = 5 \cdot \frac{1}{3} e^{-\frac{y}{3}} \binom{5-1}{4-1} (1-e^{-\frac{y}{3}})^{4-1} (1-(1-e^{-\frac{y}{3}}))^{5-4}$$

$$= \frac{20}{3} (e^{-\frac{y}{3}})^2 (1-e^{-\frac{y}{3}})^3 \quad ; \quad y>0$$

$$P(Y_{(4)} < 5) = \int_0^5 f_{(4)}(y)\, dy$$

$$= \int_0^5 \frac{20}{3} \cdot (e^{-\frac{y}{3}})^2 (1-e^{-\frac{y}{3}})^3 \, dy$$

let $u = 1-e^{-\frac{y}{3}} \Rightarrow e^{-\frac{y}{3}} = 1-u$

$du = \frac{1}{3} e^{-\frac{y}{3}} dy$

$$= \int_0^{20} 20 \, (1-u)\, u^3 \, du$$

$$= \int_0^{20} 20 (u^3 - u^4) \, du$$

$$= 20 \left[\frac{u^4}{4} - \frac{u^5}{5} \right]$$

$$= 20 \left[\frac{(1-e^{-\frac{y}{3}})^4}{4} - \frac{(1-e^{-\frac{y}{3}})^5}{5} \right]_0^5$$

$$= 20 \, [0.037995]$$

$$= 0.7599$$

c- $P(1 < Y_{(1)})$

find pdf of $Y_{(1)}$: $\boxed{n=5}$

$$f_{(1)}(y) = n\, f(y)\, (1-F(y))^{n-1}$$

$$= 5 \cdot \frac{1}{3} e^{-\frac{y}{3}} \cdot \left(e^{-\frac{y}{3}}\right)^4$$

$$= \frac{5}{3}\left(e^{-\frac{y}{3}}\right)^5 \quad ; \quad y>0$$

$$P(Y_{(1)} > 1) = \int_1^\infty \frac{5}{3} e^{-\frac{5y}{3}}\, dy = -e^{-\frac{5y}{3}} \Big|_1^\infty = 0 + e^{-\frac{5}{3}} = e^{-\frac{5}{3}} = 0.1889$$

another way to Solve :

$$P(1 < Y_{(1)}) = 1 - P(Y_{(1)} \leq 1)$$

$$= 1 - F_{Y_{(1)}}(1)$$

$$= 1 - \left[1 - (1 - F(1))^n\right]$$

$$= 1 - \left[1 - \left(1 - (1 - e^{-\frac{1}{3}})\right)^5\right]$$

$$1 - (1 - 1 + e^{-\frac{1}{3}})$$

$$= 1 - \left[1 - (e^{-\frac{1}{3}})^5\right]$$

$$= e^{-\frac{5}{3}} = 0.1889$$

Cumulative Distribution of min :

$$F_{(1)}(x) = 1 - (1 - F(x))^n$$

$$F(y) = 1 - e^{-\frac{y}{3}}$$

Q6: pdf of $Y_{(1)}$:

$n = 19$

exp. dis: $f(y) = \frac{1}{\theta} e^{-\frac{y}{\theta}}$, $y > 0$

$F(y) = 1 - e^{-\frac{y}{\theta}}$

$$f_{(1)}(y) = n\, f(y)\, (1 - F(y))^{n-1}$$

$$= 19\ \frac{1}{\theta} e^{-\frac{y}{\theta}} \left[1 - 1 + e^{-\frac{y}{\theta}}\right]^{18}$$

$$= 19\ \frac{1}{\theta} e^{-\frac{y}{\theta}} \left(e^{-\frac{y}{\theta}}\right)^{18} \quad ; \quad y > 0$$

Q7: $f(x) = \frac{1}{x^2}$; $1 \leqslant x < \infty$

Find cdf of X: $F(x) = \int_1^x \frac{1}{t^2}\, dt = \left[-t^{-1}\right]_1^x = -\frac{1}{x} + 1 = 1 - \frac{1}{x}$

a- find pdf of $Y_{(1)}$:

$$f_{(1)}(y) = n\, f(y)\, (1 - F(y))^{n-1}$$

$$= n\ \frac{1}{y^2} \left(\cancel{1} - \cancel{1} + \frac{1}{y}\right)^{n-1}$$

$$= n\ \frac{1}{y^2}\ \frac{1}{y^{n-1}}$$

$$= \frac{n}{y^{n+1}} \quad , \quad y > 1$$

b- find pdf of $Y_{(n)}$:

$1 < y_1 < y_2 < \cdots y_n$

$$f_{(n)}(y) = n\, f(y)\, F(y)^{n-1}$$

$$= n\ \frac{1}{y^2} \left(1 - \frac{1}{y}\right)^{n-1}$$

توحيد مقامات وتبسيط فقط :

$$\frac{n}{y^2} \left(\frac{y}{y} - \frac{1}{y}\right)^{n-1} = \frac{n}{y^2}\ \frac{(y-1)^{n-1}}{y^{n-1}}$$

$$= \frac{n\,(y-1)^{n-1}}{y^{n+1}} \quad ; \quad y > 1$$