Chapter 7: Multivariate Random Variables

Q1) The joint probability function of two discrete random variables X and Y is given by $f(x,y) = cxy$ for $x = 1, 2, 3$ and $y = 1, 2, 3$ and equals zero otherwise. Find:

a. The constant c.
b. $P(X = 2, Y = 3)$.
c. $P(1 \le X \le 2, Y \le 2)$.
d. $P(X \ge 2)$.
e. $P(Y < 2)$.
f. $P(X = 1)$.
g. $P(Y = 3)$.

Solution :

$$f(x,y) = cxy \quad ; x = 1,2,3 \ ; y = 1,2,3$$

a) We know that $\sum_x \sum_y f(x,y) = 1$

$\sum_{i=1}^{3} \sum_{j=1}^{3} f(x_i, y_j) = 1$

$C + 2C + 3C + 2C + 4C + 6C + 3C + 6C + 9C = 1 \Rightarrow \quad C = \frac{1}{36}$

Or $\quad 6C + 12C + 18C = 1 \Rightarrow C = \frac{1}{36}$

$$f(x,y) = \frac{1}{36}xy \quad ; x = 1,2,3 \ ; y = 1,2,3$$

$Y \backslash X$	1	2	3	$f_Y(y) = \sum_x f(x,y)$
1	1C	2C	3C	6C
2	2C	4C	6C	12C
3	3C	6C	9C	18C
$f_X(x) = \sum_y f(x,y)$	6C	12C	18C	1

b) $P(X = 2\,, Y = 3) = f(2,3) = 6C = \frac{6}{36} = \frac{1}{6}$

c) $P(1 \le X \le 2 \ \ , Y \le 2) \qquad ; x = 1,2 \quad , y = 1,2$

$P(1 \le X \le 2 \ \ , Y \le 2) = P(X = 1, Y = 1) + P(X = 1, Y = 2)$

$+P(X = 2, Y = 1) + P(X = 2, Y = 2)$

$= f(1,1) + f(1,2) + f(2,1) + f(2,2)$

$= \frac{1}{36}[1 + 2 + 2 + 4] = \frac{9}{36}$

d) $P(X \geq 2)$; $x = 2,3$

$$P(X \geq 2) = f_X(2) + f_X(3) = \frac{12}{36} + \frac{18}{36} = \frac{30}{36}$$

e) $P(Y < 2)$; $y = 1$

$$P(Y < 2) = f_Y(1) = \frac{6}{36} = \frac{1}{6}$$

f) $P(X = 1) = f_X(1) = \frac{6}{36} = \frac{1}{6}$

g) $P(Y = 3) = f_Y(3) = \frac{18}{36}$

Q2) For the random variables of **Problem 1**, find the marginal probability function of X and Y. Determine whether X and Y are independent.

Solution :

Marginal dis of X :

x	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	total
$f_X(x) = P(X = x) = \sum_y f(x,y)$	6/36 = 1/6	12/36 =2/6	18/36 =3/6	1

Marginal dis of Y :

y	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	total
$f_Y(y)$	6/36	12/36	18/36	1

Are X and Y independent ?

If $f_{XY}(x,y) = f_X(x)\, f_Y(y)\ \forall\, x = 1,2,3\ ; y = 1,2,3$
Then X and Y are independent. But if there some values of (x) and (y) which make that $f_{XY}(x,y) \neq f_X(x)\, f_Y(y)$ then X and Y are **not** independent .

In this example, we have :

$$f(1,1) = f_X(1)f_Y(1) \Rightarrow \frac{1}{36} = \left(\frac{6}{36}\right)\left(\frac{6}{36}\right)$$

$$f(1,2) = f_X(1)f_Y(2) \Rightarrow \frac{2}{36} = \left(\frac{6}{36}\right)\left(\frac{12}{36}\right)$$

$$\vdots$$

$$f(3,3) = f_X(3)f_Y(3) \Rightarrow \frac{9}{36} = \left(\frac{18}{36}\right)\left(\frac{18}{36}\right)$$

So as $f(x,y) = f_X(x)f_Y(y)\ \forall\, x = 1,2,3\ ; y = 1,2,3$, then X and Y are independent.

Q3) For the distribution of **Problem 1**, find the **conditional** probability function of X given Y, Y given X.

Solution :

Distribution of $X|Y: f_{X|Y}(x) = \frac{f(x,y)}{f_y(y)}$; $x = 1, 2, 3$

If $Y = 1$: $f_{X|Y=1}(x) = \frac{f(x,1)}{f_y(1)} = \frac{f(x,1)}{6/36}$

x	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	total
$f_{X\|Y=1}(y)$	$\frac{f(1,1)}{6/36} = \frac{1/36}{6/36} = 1/6$	$\frac{f(2,1)}{6/36} = \frac{2/36}{6/36} = \frac{2}{6} = 1/3$	$\frac{f(3,1)}{6/36} = \frac{3}{6} = 1/2$	1

If $Y = 2$: $f_{X|Y=2}(x) = \frac{f(x,2)}{f_y(2)} = \frac{f(x,2)}{12/36}$

x	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	total
$f_{X\|Y=2}(y)$	$\frac{f(1,2)}{12/36} = \frac{2/36}{12/36} = 1/6$	$\frac{f(2,2)}{12/36} = \frac{4/36}{12/36} = 2/6$	$\frac{f(3,2)}{12/36} = \frac{6/36}{12/36} = 3/6$	1

If $Y = 3$: $f_{X|Y=2}(x) = \frac{f(x,3)}{f_y(3)} = \frac{f(x,3)}{18/36}$

x	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	total
$f_{X\|Y=3}(y)$	$\frac{f(1,3)}{18/36} = \frac{3/36}{18/36} = 1/6$	$\frac{f(2,3)}{18/36} = \frac{6/36}{18/36} = 2/6$	$\frac{f(3,3)}{18/36} = \frac{9/36}{18/36} = 3/6$	1

Note: Since X and Y are independent so $f_{X|Y}(x) = \frac{f(x,y)}{f_y(y)} = \frac{f_X(x) f_Y(y)}{f_y(y)} = f_X(x) = \frac{x}{6}$

Distribution of $Y|X$: $f_{Y|X}(y) = \frac{f(x,y)}{f_X(x)}$; $y = 1, 2, 3$

If $X = 1$: $f_{Y|X=1}(y) = \frac{f(1,y)}{f_X(1)} = \frac{f(1,y)}{6/36} \Longrightarrow$

y	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	total
$f_{Y\|X=1}(y)$	$\frac{f(1,1)}{6/36} = \frac{1/36}{6/36} = \mathbf{1/6}$	$\frac{f(1,2)}{6/36} = \frac{2/36}{6/36} = \mathbf{2/6}$	$\frac{f(1,3)}{6/36} = \frac{3/36}{6/36} = \mathbf{3/6}$	1

If $X = 2$: $f_{Y|X=2}(y) = \frac{f(2,y)}{f_X(2)} = \frac{f(2,y)}{12/36} \Longrightarrow$

y	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	total
$f_{Y\|X=2}(y)$	$\frac{f(2,1)}{12/36} = \frac{2/36}{12/36} = \mathbf{2/12}$	$\frac{f(2,2)}{12/36} = \frac{4/36}{12/36} = \mathbf{4/12}$	$\frac{f(2,3)}{12/36} = \frac{6/36}{12/36} = \mathbf{6/12}$	1

If $X = 3$: $f_{Y|X=3}(y) = \frac{f(3,y)}{f_X(3)} = \frac{f(3,y)}{18/36} \Longrightarrow$

y	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	total
$f_{Y\|X=2}(y)$	$\frac{f(3,1)}{18/36} = \frac{2/36}{18/36} = \mathbf{3/18}$	$\frac{f(3,2)}{18/36} = \frac{4/36}{18/36} = \mathbf{6/18}$	$\frac{f(3,3)}{18/36} = \frac{6/36}{18/36} = \mathbf{9/18}$	1

Note: Since X and Y are independent so $f_{Y|X}(y) = \frac{f(x,y)}{f_X(x)} = \frac{f_X(x)f_Y(y)}{f_X(x)} = f_Y(y) = \frac{y}{6}$

Q4) Let X and Y be continuous random variables having joint density function

$$f(x,y) = \begin{cases} c(x^2 + y^2) & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

Determine:

a. The constant c.

b. $P\left(X < \frac{1}{2}, Y > \frac{1}{2}\right)$

c. $P\left(\frac{1}{4} < X < \frac{3}{4}\right)$.

d. $P\left(Y < \frac{1}{2}\right)$.

e. Whether X and Y are independent.

Solution :

$$f(x,y) = C(x^2 + y^2)\ , \qquad 0 < x < 1\ ,\ 0 < y < 1$$

a) We know that $\iint_{(x,y)\epsilon\, C} f(x,y)\, dx\, dy = 1$

$$\int_0^1 \int_0^1 C\,(x^2 + y^2) dx\, dy = 1 \ \Rightarrow\ C \int_0^1 \left[\int_0^1 (x^2 + y^2)\, dx\right] dy = 1$$

$$\Rightarrow\ C \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_0^1 dy = 1 \ \Rightarrow\ C \int_0^1 \left(\frac{1}{3} + y^2\right) = 1 \ \Rightarrow\ C\left[\frac{1}{3}y + \frac{y^3}{3}\right]_0^1 = 1$$

$$\Rightarrow C\left[\frac{1}{3} + \frac{1}{3}\right] = 1 \ \Rightarrow\ \boldsymbol{C = \frac{3}{2}}$$

b)

$$P\left(X < \frac{1}{2}, Y > \frac{1}{2}\right) = \frac{3}{2} \int_{1/2}^1 \left[\int_0^{1/2} (x^2 + y^2) dx\right] dy = \frac{3}{2} \int_{1/2}^1 \left[\frac{x^3}{3} + xy^2\right]_0^{1/2} dy$$

$$= \frac{3}{2} \int_{1/2}^1 \left(\frac{1}{24} + \frac{1}{2}y^2\right) dy = \frac{3}{2}\left[\frac{y}{24} + \frac{y^3}{6}\right]_{\frac{1}{2}}^1 = \frac{3}{2}\left[\left(\frac{1}{24} + \frac{1}{6}\right) - \left(\frac{1}{48} + \frac{1}{48}\right)\right] = \frac{1}{4}$$

c)

$$P\left(\frac{1}{4} < X < \frac{3}{4}\right) = \frac{3}{2} \int_0^1 \left[\int_{\frac{1}{4}}^{\frac{3}{4}} (x^2 + y^2) dx\right] dy = \frac{3}{2} \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_{\frac{1}{4}}^{\frac{3}{4}} dy$$

$$= \frac{3}{2} \int_0^1 \left[\left(\frac{9}{64} + \frac{3}{4}y^2\right) - \left(\frac{1}{192} + \frac{1}{4}y^2\right)\right] dy = \frac{3}{2} \int_0^1 \left(\frac{13}{96} + \frac{1}{2}y^2\right) dy$$

$$= \frac{3}{2}\left[\frac{13}{96}y + \frac{1}{6}y^3\right]_0^1 = \frac{3}{2}\left(\frac{13}{96} + \frac{1}{6}\right) = \frac{29}{64}$$

d)

$$P\left(Y < \frac{1}{2}\right) = \frac{3}{2} \int_0^{\frac{1}{2}} \left[\int_0^1 (x^2 + y^2) dx\right] dy = \frac{3}{2} \int_0^{\frac{1}{2}} \left[\frac{x^3}{3} + xy^2\right]_0^1 dy$$

$$= \frac{3}{2} \int_0^{1/2} \left(\frac{1}{3} + y^2\right) dy = \frac{3}{2}\left[\frac{y}{3} + \frac{y^3}{3}\right]_0^{\frac{1}{2}} = \frac{3}{2}\left[\frac{1}{6} + \frac{1}{24}\right] = \frac{5}{16}$$

e) X and Y are independent if satisfy :

1) $f(x,y) = f_X(x) f_Y(y)\ \forall\, x, y$.
2) the ranges of X and Y are independent.

In this example , we can see that

$$f(x,y) \neq f_X(x)f_Y(y) \quad \therefore X \text{ and } Y \text{ are not independent.}$$

$$f_Y(y) = \frac{3}{2}\int_0^1 (x^2+y^2)dx = \frac{3}{2}\left[\frac{x^3}{3} + xy^2\right]_0^1 = \frac{3}{2}\left(\frac{1}{3} + y^2\right) = \frac{1}{2} + \frac{3}{2}y^2$$

$$f_X(x) = \frac{3}{2}\int_0^1 (x^2+y^2)dy = \frac{3}{2}\left[x^2y + \frac{y^3}{3}\right]_0^1 = \frac{3}{2}\left(x^2 + \frac{1}{3}\right) = \frac{3}{2}x^2 + \frac{1}{2}$$

$$f_X(x)f_Y(y) = \left(\frac{3}{2}x^2 + \frac{1}{2}\right)\left(\frac{3}{2}y^2 + \frac{1}{2}\right) = \frac{9}{4}x^2y^2 + \frac{3}{4}x^2 + \frac{3}{4}y^2 + \frac{1}{4} \neq f(x,y)$$

Q5) For the random variables of **Problem 4**, find the marginal probability function of X and Y.

Solution :

Marginal distribution of X : $f(x) = \int_0^1 f(x,y)dy$

$$f(x) = \frac{3}{2}\int_0^1 (x^2+y^2)dy = \frac{3}{2}\left[x^2y + \frac{y^3}{3}\right]_0^1 = \frac{3}{2}\left[x^2 + \frac{1}{3}\right] = \frac{3}{2}x^2 + \frac{1}{2}$$

$$\therefore f_X(x) = \frac{3}{2}x^2 + \frac{1}{2} \quad ; 0 < x < 1$$

Marginal distribution of Y : $f(y) = \int_0^1 f(x,y)dx$

$$f(y) = \frac{3}{2}\int_0^1 (x^2+y^2)dx = \frac{3}{2}\left[\frac{x^3}{3} + y^2x\right]_0^1 = \frac{3}{2}\left(\frac{1}{3} + y^2\right) = \frac{3}{2}y^2 + \frac{1}{2}$$

$$\therefore f_Y(y) = \frac{3}{2}y^2 + \frac{1}{2} \quad ; \; 0 < y < 1$$

Q6) For the distribution of **Problem 4**, find the conditional probability function of X given Y, Y given X.

Solution :

Conditional distribution $X|Y$:

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}y^2 + \frac{1}{2}} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}\left(y^2 + \frac{1}{3}\right)} = \frac{x^2+y^2}{y^2 + \frac{1}{3}}$$

for $0 < x < 1$ *wher* $0 < y < 1$ fixed value .

Conditional distribution $Y|X$:

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}x^2+\frac{1}{2}} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}\left(x^2+\frac{1}{3}\right)} = \frac{x^2+y^2}{x^2+\frac{1}{3}}$$

for $0 < y < 1$ $wher$ $0 < x < 1$ fixed value .

Q7) Let $f(x,y) = \begin{cases} x+y & 0 \le x \le 1\,,\ 0 \le y \le 1 \\ 0 & otherwise \end{cases}$

Find the conditional probability function of X given Y, Y given X.

Solution : H.W

$$f(x,y) = x+y \quad ; 0 \le x \le 1\ , 0 \le y \le 1$$

Marginal pdf of X : $f(x) = \int_0^1 f(x,y)dy = \int_0^1 (x+y)dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}$

Marginal pdf of Y : $f(y) = y + \frac{1}{2}$

Conditional distribution $X|Y$:

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{x+y}{y+\frac{1}{2}} = \frac{2(x+y)}{(2y+1)}$$

Conditional distribution $Y|X$:

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{x+y}{x+\frac{1}{2}} = \frac{2(x+y)}{(2x+1)}$$

Q8) Let $f(x,y) = \begin{cases} e^{-(x+y)} & x \ge 0\,, y \ge 0 \\ 0 & otherwise \end{cases}$

be the joint density function of X and Y. Find the conditional probability function of X given Y, Y given X.

Solution :

Marginal pdf of X : $f(x) = \int_0^{\infty} e^{-(x+y)}\,dy = e^{-x}[-e^{-y}]_0^{\infty} = e^{-x}[0+1] = e^{-x}$

Marginal pdf of Y : $f(y) = \int_0^{\infty} e^{-(x-y)}\,dx = e^{-y}$

Conditional distribution $X|Y$: $f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-(x+y)}}{e^{-y}} = e^{-x}$

Conditional distribution $Y|X$: $f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-(x-y)}}{e^{-x}} = e^{-y}$

Q9) Let X and Y be random variables having joint density function

$$f(x,y) = \begin{cases} c(2x+y) & 0 < x < 1\,, \quad 0 < y < 2 \\ 0 & otherwise \end{cases}$$

Find:

a. The constant c.
b. $P\left(X > \frac{1}{2}, Y < \frac{3}{2}\right)$.
c. The (marginal) density function of X.
d. The (marginal) density function of Y.

Solution : H.W

Q10) The joint probability function for the random variables X and Y is given in following table, then find:

X \ Y	0	1	2
0	1/18	1/9	1/6
1	1/9	1/18	1/9
2	1/6	1/6	1/18

a. The marginal probability functions of X and Y.
b. $P(1 \le X < 3, Y \ge 1)$.
c. Determine whether X and Y are independent.

Solution : H.W

Q11) Let X and Y be random variables having joint density function

$$f(x,y) = \begin{cases} x+y & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

Find: a. Var(X). b. Var(Y). c. σ_X. d. σ_Y. e. σ_{XY}. f. ρ.

Solution :

$f(x) = \int_0^1 f(x,y)dy = \int_0^1 (x+y)dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}$

$f(x) = x + \frac{1}{2} \quad ,0 < x < 1$

$f(y) = y + \frac{1}{2} \quad ,0 < y < 1$

a, b)

$$E(X) = \int_0^1 x\, f(x)dx = \int_0^1 x\left(x + \frac{1}{2}\right)dx$$

$$= \int_0^1 \left(x^2 + \frac{x}{2}\right)dx = \left[\frac{x^3}{3} + \frac{1}{4}x^2\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$E(Y) = \int_0^1 y\, f(y)dy = \frac{7}{12}$$

$$E(X^2) = \int_0^1 x^2\, f(x)dx = \int_0^1 x^2\left(x + \frac{1}{2}\right)dx$$

$$= \int_0^1 \left(x^3 + \frac{x^2}{2}\right)dx = \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

$$E(Y^2) = \int_0^1 y^2\, f(y)dy = \frac{5}{12}$$

$$var(X) = E(X^2) - [E(X)]^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$

$$var(Y) = E(Y^2) - [E(Y)]^2 = \frac{11}{144}$$

c, d)

$$\sigma_X = \sqrt{var(X)} = \sqrt{11/144} = 0.2764$$

$$\sigma_Y = \sqrt{var(Y)} = \sqrt{11/144} = 0.2764$$

e) $\sigma_{XY} = cov(x,y) = E(XY) - E(X)E(Y)$

$$E(XY) = \int_0^1 \int_0^1 xy\,(x+y)dx\,dy \;=\; \int_0^1 \left[\frac{x^3}{3}y + \frac{x^2}{2}y^2\right]_0^1 dy$$

$$= \int_0^1 \left(\frac{1}{3}y + \frac{y^2}{2}\right) dy \;=\; \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

$$cov(x,y) = \frac{2}{6} - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right) = -\frac{1}{144}$$

f) $\rho = cor(x,y) = \frac{cov(x,y)}{\sqrt{var(x)var(y)}} = \frac{-1/144}{\sqrt{\left(\frac{11}{144}\right)\left(\frac{11}{144}\right)}} = \frac{-1}{11} = -\,0.091$

(Weak negative correlation)

Q12) The joint density function is

$$f(x,y) = \begin{cases} e^{-(x+y)} & x \geq 0\,, y \geq 0 \\ 0 & otherwise \end{cases}$$

Find: a. Var(X). b. Var(Y). c. σ_X. d. σ_Y. e. σ_{XY}. f. ρ.

Solution : H.W

Q13) Find a. The covariance. b. The correlation coefficient of two random variables X and Y. If $E(X) = 2,\; E(Y) = 3,\; E(XY) = 10,\; E(X^2) = 9,\; E(Y^2) = 16$.

Solution :

a) $cov(x,y) = E(XY) - E(X)E(Y) = 10 - (2)(3) = 4$

b) $var(x) = E(x^2) - [E(x)]^2 = 9 - 2^2 = 5$

$$var(y) = E(y^2) - [E(y)]^2 = 16 - 3^2 = 7$$

$\rho = \frac{cov(x,y)}{\sqrt{var(x)var(y)}} = \frac{4}{\sqrt{(5)(7)}} = \frac{4}{\sqrt{35}} = 0.676123$ (Moderate positive correlation)

Q14) The correlation coefficient of two random variables **X** and **Y** is **(-1/4)** while their variances are **3** and **5**. Find the covariance.

Solution : H.W

$$\rho = -\frac{1}{4} \quad ; var(x) = 3\,, var(y) = 5$$

$$\rho = cor(x,y) = \frac{cov(x,y)}{\sqrt{var(x)var(y)}}$$

$$-\frac{1}{4} = \frac{cov(x,y)}{\sqrt{(3)(5)}} \;\Longrightarrow\; -\frac{1}{4}\sqrt{15} = cov(x,y) \Longrightarrow cov(x,y) = -0.9682$$

(Strong negative correlation)

Q15) The joint probability function of two **discrete** random variables X and Y is given by $f(x,y) = c(2x + y)$, where x and y can assume all integers such that $0 < x < 1,\ 0 < y < 3$, and $f(x,y) = 0$ otherwise. Find:

a. The value of the constant c.

b. $P(X = 2, Y = 1)$.

c. $P(X \geq 1, Y \leq 2)$

Solution : H.W

$$f(x,y) = c(2x + y) \ \ ;x = 0,1 \ ; \ \ y = 0,1,2,3$$

a) $\sum_{i=0}^{3} \sum_{j=0}^{1} f(x_i, y_j) \ = 1$

$$6c + 14c = 1 \ \Rightarrow \ 20c = 1 \Rightarrow c = \frac{1}{20}$$

y \ x	0	1	$f_Y(y)$
0	0	2c	2c
1	c	3c	4c
2	2c	4c	6c
3	3c	5c	8c
$f_X(x)$	6c	14c	1

b) $P(X = 2, Y = 1) = 0$

c) $P(X \geq 1, Y \leq 2) = P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 1, Y = 2)$

$$= \frac{1}{20}(2 + 3 + 4) = \frac{9}{20}$$

Q16) For the **Problem 15**, find a. E(X). b. E(Y). c. E(XY). d. E(X^2). e. E(Y^2). f. Var(X). g. Var(Y). h. Cov(X,Y). i. ρ.

Solution :

$$E(X) = \sum_{x=0}^{1} x\, f(x) = \frac{1}{20}[0 * 6 + 1 * 14] = \frac{14}{20} = \frac{7}{10}$$

$$E(X^2) = \sum_{x=0}^{1} x^2\, f(x) = \frac{1}{20}[0 * 6 + 1 * 14] = \frac{14}{20} = \frac{7}{10}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{7}{10} - \left(\frac{7}{10}\right)^2 = \frac{21}{100}$$

$$E(Y) = \sum_{y=0}^{3} y\, f(y) = \frac{1}{20}[0*2+1*4+2*6+3*8] = \frac{40}{20} = 2$$

$$E(Y^2) = \sum_{y=0}^{3} y^2\, f(y) = \frac{1}{20}[0*2+1*4+4*6+9*8] = \frac{100}{20} = 5$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = 5 - (2)^2 = 1$$

$$E(XY) = \sum_{y=0}^{3}\sum_{x=0}^{1} yx\, f(x,y) = \frac{1}{20}(1*1*3+1*2*4+1*3*5) = \frac{26}{20} = \frac{13}{10}$$

$$\sigma_{XY} = cov(x,y) = E(XY) - E(X)E(Y) = \frac{13}{10} - \left(\frac{7}{10}\right)(2) = -0.1$$

$$\rho = \frac{cov(x,y)}{\sqrt{var(x)var(y)}} = \frac{-0.1}{\sqrt{\frac{21}{100}*1}} = -0.2182$$

weak negative correlation.

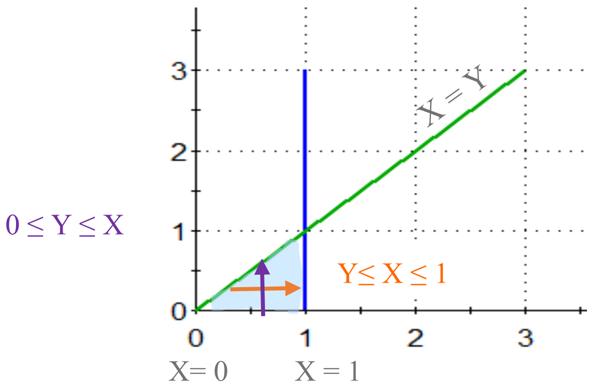
Q17) The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} 8xy & 0 \le x \le 1\ ,0 \le y \le x \\ 0 & otherwise \end{cases}$$

Find:

a. The marginal density of X.
b. The marginal density of Y.
c. The conditional density of X.
d. The conditional density of Y.

Solution :



a) $f_X(x) = \int_0^x f(x,y)dy$

$$= 8\int_0^x xy\,dy = 8x\left[\frac{y^2}{2}\right]_0^x = 8x\left[\frac{x^2}{2}\right] = 4x^3 \quad for\;\; 0 < x < 1$$

b) $f_Y(y) = \int_y^1 f(x,y)dx$

$$= 8y\int_y^1 x\,dx = 8y\left[\frac{x^2}{2}\right]_y^1 = 4y[1-y^2] = 4y - 4y^3$$

$$f_Y(y) = 4(y - y^3) \quad for\;\; 0 < y < 1$$

c) Conditional distribution $X|Y$: $f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{8\,x\,y}{4(y-y^3)} = \frac{2xy}{y(1-y^2)} = \frac{2x}{1-y^2}$

$for\;\; y < x < 1\; wher\; 0 < y < 1\; fixed\; value\,.$

d) Conditional distribution $Y|X$: $f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{8\,x\,y}{4\,x^3} = \frac{2y}{x^2}$

$for\;\; 0 < y < x\;\; wher\; 0 < x < 1\; fixed\; value\,.$

Q18) Find the conditional expectation of X given Y and Y given X in **Problem 17**.

Solution :

$$E(X|Y=y) = \int_y^1 x\, f_{X|Y=Y}(x)dx = \frac{2}{1-y^2}\int_y^1 x^2\,dx$$

$$= \frac{2}{1-y^2}\left[\frac{x^3}{3}\right]_y^1 = \frac{2}{3}\frac{1}{1-y^2}(1-y^3) = \frac{2}{3}\frac{(1-y^3)}{(1-y^2)}$$

$$E(Y|X=x) = \int_0^x y\, f_{Y|X=x}(y)dy = \frac{2}{x^2}\int_0^x y^2\,dy = \frac{2}{x^2}\left[\frac{y^3}{3}\right]_0^x = \frac{2}{3}\frac{1}{x^2}[x^3] = \frac{2}{3}\,x$$

Q19) Find the conditional variance of Y given X for **Problem 17**.

Solution :

$$var(Y|X) = E(Y^2|X) - E(Y|X)^2$$

$$E(Y^2|X) = \int_0^x y^2 f_{Y|X=x}(y)dy = \frac{2}{x^2}\int_0^x y^3 dy = \frac{2}{x^2}\left[\frac{y^4}{4}\right]_0^x = \frac{2}{4}\frac{1}{x^2}[x^4] = \frac{x^2}{2}$$

$$var(Y|X) = \frac{x^2}{2} - \left[\frac{2}{3}x\right]^2 = \left(\frac{1}{2} - \frac{2^2}{3^2}\right)x^2 = \frac{1}{18}x^2$$

Q20) The joint pdf of (X,Y) is given by $f(x,y) = \frac{e^{-y}}{y}$; $0 < x < y,\ 0 < y < \infty$.
Find $E(X), E(Y), V(X), V(Y)$ and $Cov(X,Y)$.

Solution : H.W

$$\boldsymbol{E(X)} = \int_0^\infty \int_0^y x\frac{1}{y}e^{-y}\ dx\, dy = \int_0^\infty \frac{1}{y}e^{-y}\left[\frac{x^2}{2}\right]_0^y dy \ = \int_0^\infty \frac{1}{2}y\, e^{-y}dy$$

Note: $X\sim \text{exponential}(\lambda) \rightarrow f(x) = \lambda e^{-\lambda x}\ \ x > 0\,;\ \ E(x) = \frac{1}{\lambda};\ V(x) = \frac{1}{\lambda^2}$

$$let\ W = e^{-y}\sim Exp(1)\quad ; E(W) = V(W) = 1$$

$$\therefore E(X) = \frac{1}{2}E(W) = \frac{1}{2}$$

$$\boldsymbol{or}\ use\ \int_0^\infty x^a\, e^{-b\,x}\ dx = \frac{\Gamma(a+1)}{b^{a+1}}\quad ,\qquad \Gamma(a) = (a-1)!$$

$$E(X^2) = \int_0^\infty \int_0^y x^2\, \frac{1}{y}e^{-y}\ dx\, dy = \int_0^\infty \frac{1}{y}e^{-y}\left[\frac{x^3}{3}\right]_0^y\ dy$$

$$= \int_0^\infty \frac{1}{3}y^2e^{-y}\ \ dy = \frac{1}{3}\ E(W^2) = \frac{1}{3}(2) = \frac{2}{3}$$

$$\boldsymbol{V(X)} = \frac{2}{3} - \left(\frac{1}{2}\right)^2 = \frac{5}{12}$$

$$\boldsymbol{E(Y)} = \int_0^\infty \int_0^y y\ \frac{1}{y}e^{-y}\ dx\, dy = \int_0^\infty e^{-y}\, dy = 1$$

$$E(Y^2) = \int_0^\infty y^2\, e^{-y}\ dy = 2\qquad ;\qquad V(Y) = 2 - 1 = 1$$

$$E(XY) = \int_0^\infty \int_0^y xy\ \frac{1}{y}\ e^{-y}\ dx\, dy = \int_0^\infty \int_0^y xe^{-y}\ dx\, dy$$

$$= \int_0^\infty e^{-y}\left[\frac{x^2}{2}\right]_0^y\ dy = \int_0^\infty \frac{1}{2}\ y^2e^{-y}\, dy = \frac{1}{2}\frac{\Gamma(3)}{1^3} = \frac{1}{2}(2!) = 1$$

$$cov(x,y) = E(XY) - E(X)E(Y) = 1 - \frac{1}{2}(1) = \frac{1}{2}$$

Q21) Let (X,Y) have joint density given by

$f(x,y) = 24xy \qquad ; 0 < x < 1, \quad 0 < y < 1, \quad x + y < 1$

Find:

A. The marginal pdf's.

B. The following expectations:

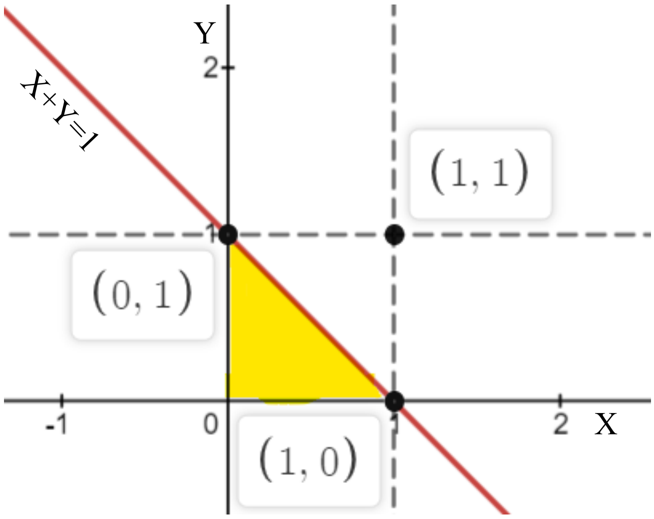
i. E(X) and E(X^2).

ii. E(Y) and E(Y^2).

iii. E(XY) and E(X^2 Y^3).

iv. V(X), V(Y), Cov(X,Y). Do X and Y have a positive or negative relationship?

Solution : H.W



a) $f(x) = \int_0^{1-x} 24xy\ dy = 24x\left[\frac{y^2}{2}\right]_0^{1-x} = 12\,x(1-x)^2\ ;\ 0 < x < 1$

$f(y) = \int_0^{1-y} 24xy\ dx = 24y\left[\frac{x^2}{2}\right]_0^{1-y} = 12\,y(1-y)^2\ ;\ 0 < y < 1$

b) Expectations $E(X) = \int_{-\infty}^{\infty} xf(x)\ dx$

$$E(X) = \int_0^1 12x^2(1-x)^2\ dx = \int_0^1 12x^2(1-2x+x^2)\ dx$$

$$= 12\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1 = 12\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$$

$$\boldsymbol{E(X) = \frac{2}{5}}$$

Or you can solve it as $E(X) = \int_0^1 \int_0^{1-y} 24\,x^2\,y\,dx\ dy = \int_0^1 24y\left[\frac{x^3}{3}\right]_0^{1-y} dy$

$= \int_0^1 8\,y(1-y)^3\ \ dy = \int_0^1 8\,y(1-3y+3y^2-y^3)\ dy$

$= 8\left[\frac{y^2}{2} - \frac{3y^3}{3} + \frac{3y^4}{4} - \frac{y^5}{5}\right]_0^1 = 8\left[\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right] = \frac{8}{20} = \frac{2}{5}$

$$E(X^2) = \int_0^1 12x^3(1-x)^2 \; dx = \int_0^1 12x^3(1-2x+x^2) \; dx = 12\left[\frac{x^4}{4} - \frac{2x^5}{5} + \frac{x^6}{6}\right]_0^1$$

$$\boldsymbol{E(X^2) = \frac{1}{5}}$$

$$V(X) = E(X^2) - \left(E(X)\right)^2 = \frac{1}{5} - \left(\frac{2}{5}\right)^2 = \frac{1}{25} = 0.04$$

By similarly, $E(Y) = \frac{2}{5}$; $E(Y^2) = \frac{1}{5}$; $V(Y) = \frac{1}{25}$

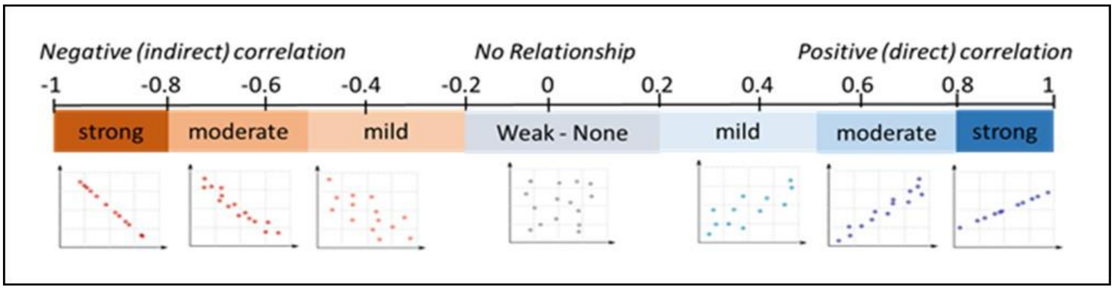
$$E(XY) = \int_0^1 \int_0^1 24\, x^2 y^2 \, dx \; dy = \int_0^1 24\, y^2 \left[\frac{x^3}{3}\right]_0^1 dy \; = \int_0^1 8\, y^2 \; dy \; = 8\left[\frac{y^3}{3}\right]_0^1$$

$$\boldsymbol{E(XY) = 8}$$

$$cov(x, y) = E(XY) - E(X)E(Y) = 8 - \left(\frac{1}{5}\right)^2 = 7.96$$

Since $cov(x, y) > 0$, then X and Y have a positive relationship.

Note:



Q22) Let X and Y be two random variables with joint distribution function

$$f_{XY}(x,y) = \begin{cases} e^{-x-y} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$X \sim Exp(1)$

$Y \sim Exp(1)$.

Find $E(XY), E(X), E(Y)$ and $\text{Cov}(X,Y)$. Using MGF.

Solution.
We note first that $f_{XY}(x,y) = f_X(x)f_Y(y)$ so that X and Y are independent.
Thus, the moment generating function is given by

$$M_{X,Y}(t_1,t_2) = E(e^{t_1X+t_2Y}) = E(e^{t_1X})E(e^{t_2Y}) = \frac{1}{1-t_1}\frac{1}{1-t_2}.$$

Thus,

$$E(XY) = \frac{\partial^2}{\partial t_2 \partial t_1} M(t_1,t_2)\bigg|_{(0,0)} = \frac{1}{(1-t_1)^2(1-t_2)^2}\bigg|_{(0,0)} = 1$$

$$E(X) = \frac{\partial}{\partial t_1} M(t_1,t_2)\bigg|_{(0,0)} = \frac{1}{(1-t_1)^2(1-t_2)}\bigg|_{(0,0)} = 1$$

$$E(Y) = \frac{\partial}{\partial t_2} M(t_1,t_2)\bigg|_{(0,0)} = \frac{1}{(1-t_1)(1-t_2)^2}\bigg|_{(0,0)} = 1$$

and

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0 \blacksquare$$

Q23) Let X and Y have joint pdf

$$f(x,y) = \frac{2}{5}(x + 4y) \; ; \; 0 < x < 1, \quad 0 < y < 1$$

a. Find the conditional pdf of $Y|X = x$.

b. Find $P\left(Y < \frac{1}{3} \middle| X = \frac{1}{2}\right)$.

Solution :

a) $f(Y|X = x) = \frac{f(x,y)}{f(x)}$

$$f(x) = \int_y f(x,y)\,dy = \int_0^1 \frac{2}{5}(x + 4y)\,dy = \frac{2}{5}\left[xy + \frac{4y^2}{2}\right]_0^1 = \frac{2}{5}(x+2)$$

$$\therefore f(Y|X=x) = \frac{\frac{2}{5}(x + 4y)}{\frac{2}{5}(x+2)} = \frac{x+4y}{x+2}$$

b) $f\left(Y \middle| X = \frac{1}{2}\right) = \frac{\frac{1}{2} + 4y}{\frac{1}{2} + 2} = \frac{1+8y}{5}$

$$P\left(Y < \frac{1}{3} \middle| X = \frac{1}{2}\right) = \int_0^{\frac{1}{3}} f\left(Y \middle| X = \frac{1}{2}\right) dy$$

$$= \int_0^{1/3} \frac{1+8y}{5} dy = \frac{1}{5}\left[y + \frac{8y^2}{2}\right]_0^{\frac{1}{3}} = \frac{1}{5}\left(\frac{1}{3} + \frac{4}{9}\right) = \frac{7}{45}$$

Q24) Let $X_1 \sim Exp(\lambda_1)$ independent of $X_2 \sim Exp(\lambda_2)$ r.v.. Find: $P(X_1 < X_2)$.

Solution :

HW

$X_1 \sim Exp(\lambda_1) \Rightarrow f(x_1) = \lambda_1 e^{-\lambda_1 x_1} \quad , x_1 > 0$

$X_2 \sim Exp(\lambda_2) \Rightarrow f(x_2) = \lambda_2 e^{-\lambda_2 x_2} \quad , x_2 > 0$

$$P(X_1 < X_2) = \int_0^\infty \int_0^{x_2} f(x_1, x_2) \; dx_1 \, dx_2 =$$

$$\int_0^\infty \int_0^{x_2} \lambda_2 e^{-\lambda_2 x_2} \lambda_1 e^{-\lambda_1 x_1} \; dx_1 \, dx_2$$

$$= \int_0^\infty \lambda_2 e^{-(\lambda_2 x_2)} \left[-e^{-(\lambda_1 x_1)}\right]_0^{x_2} dx_2$$

$$= \int_0^\infty \lambda_2 e^{-(\lambda_2 x_2)} \left(1 - e^{-\lambda_1 x_2}\right) dx_2$$

$$= \int_0^\infty \lambda_2 e^{-(\lambda_2 x_2)} \; dx_2 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \int_0^\infty (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2) x_2} \; dx_2$$

$$= \left[-e^{-(\lambda_2 x_2)}\right]_0^\infty - \frac{\lambda_2}{\lambda_1 + \lambda_2}\left[-e^{-(\lambda_2 x_2)}\right]_0^\infty = 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2} \; ; \quad \{e^{-\infty} = 0, e^0 = 1\}$$

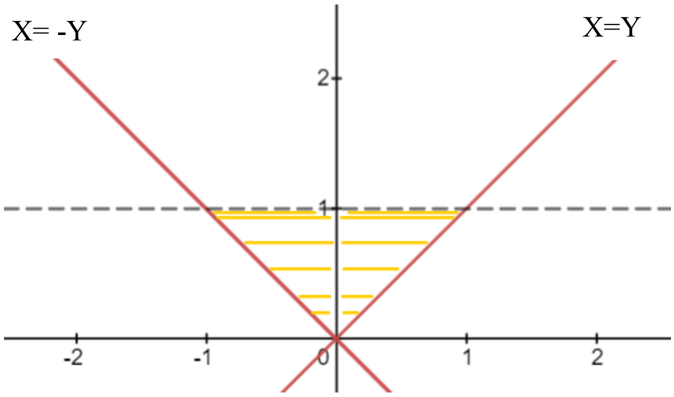
Q25) Let X and Y have joint pdf $f(x, y) = 1\ ;\ \ -y < x < y\ ,\ 0 < y < 1.$

a. Find the conditional pdf of $X|Y = y$.

b. Find $P(X < 0|Y = y)$.

c. Find $P(X > \frac{1}{4}\ |Y = \frac{1}{3})$.

d. Find $P(0 < X < \frac{1}{4}\ |Y = \frac{1}{2})$.



Solution : H.W

Summary of Differentiation Rules:

Power rule:
if $f(x) = x^n$ then $f'(x) = nx^{n-1}$

Sum rule:
if $f(x) = g(x) + h(x)$ then $f'(x) = g'(x) + h'(x)$

Product rule:
if $f(x) = g(x)h(x)$ then $f'(x) = g'(x)h(x) + g(x)h'(x)$

Quotient rule:
if $f(x) = \frac{g(x)}{h(x)}$ then $f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}$

if $f(x) = \sqrt{u}$ *then* $f'(x) = \frac{u'}{2\sqrt{u}}$

$$example\ f(x)\sqrt{x} = x^{\frac{1}{2}}\ then\ f'(x) = \frac{1}{2}\ x^{\frac{1}{2}-1}\ = \frac{1}{2}\ x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Exponential and Logarithm Functions:

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln(a) \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Summary of Integration Rules

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C; \quad n \neq -1$$

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C; \quad n \neq -1$$

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln|ax+b| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$$

Natural Log Rules

$\ln 1 = 0$
$\ln 0 = -\infty$
$\ln \infty = 0$
$\ln e = 1 \quad , \ln e^x = x$