Chapter (7)
Sampling Distributions
Examples

Example (1)

Random samples of size 3 were selected from populations' size 6 with the mean 10 and variance 9. Find the number of samples, the mean and standard deviation of the sampling distribution of the sample mean (*with replacement*)

Solution:

a. $N = 6$ $n = 3$ $\mu = 10$ $\sigma^2 = 9$ (*with replacement*)

The number of samples $= k = N^n = 6^3 = 216$

$\mu_{\bar{x}} = \mu = 10$

$\sigma_{\bar{x}}^2 = \dfrac{\sigma^2}{n} = \dfrac{9}{3} = 3$

$\sigma_{\bar{x}} = \sqrt{\sigma_{\bar{x}}^2} = \sqrt{3} = 1.73$

Example (2)

Given a normal distributed with $\mu = 30$, $\sigma = 12$, and $n = 25$. What is the probability that $\bar{X}$ is:

1) Less than 36?

2) Between 27 and 34?

3) Less than 27?

4) The probability is 95% that the sample mean will be between what two values symmetrically distributed around the population means?

Probability that $\bar{X}$ is?

5) Greater than 27?

6) Between 23 and 27?

7) There is a 72% chance that $\bar{X}$ is above what value?

8) There is a 72% chance that $\bar{X}$ is below what value?

9) Greater than 34?

10) Between 33 and 34?

11) There is a 43% chance that $\bar{X}$ is above what value?

12) There is a 43% chance that $\bar{X}$ is below what value?

13) Between 30 and 34?

14) Between 27 and 30?

Solution:
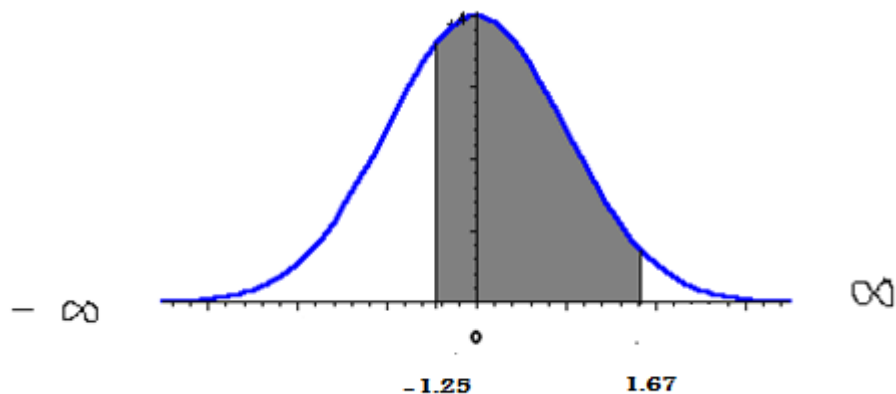1) Less than 36?

$$P(\bar{X} < 36) = P\left(Z < \dfrac{36-30}{\dfrac{12}{\sqrt{25}}} \right)$$

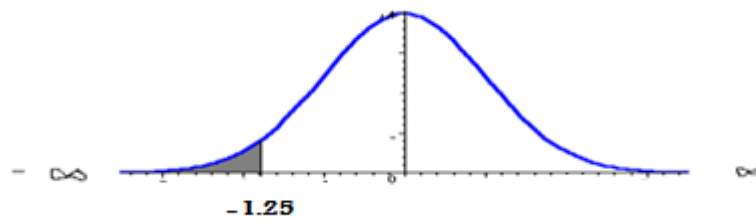$$= P\left(Z < \dfrac{6}{2.4} \right) = p(Z < 2.5) = \Phi(2.5) = 0.9938$$

2.5

2) Between 27 and 34?

$$P(27 < \bar{X} < 34) = P\left(\dfrac{27-30}{\dfrac{12}{\sqrt{25}}} < Z < \dfrac{34-30}{\dfrac{12}{\sqrt{25}}} \right)$$

$$= P\left(\dfrac{-3}{2.4} < Z < \dfrac{4}{2.4} \right) = p(-1.25 < Z < 1.67) = \Phi(1.67) - \Phi(-1.25)$$

$$= 0.9525 - 0.1056 = 0.8469$$

$-\infty$ 0 ∞

-1.25 1.67

3) Less than 27?

$$P(\bar{X} < 27) = P\left(Z < \dfrac{27-30}{\dfrac{12}{\sqrt{25}}} \right)$$

$$= P\left(Z < \dfrac{-3}{2.4} \right) = p(Z < -1.25) = \Phi(-1.25) = 0.1056$$

-1.25

4) The probability is 95% that the sample mean will be between what two values symmetrically distributed around the population means?

    Since the interval contains 95% of the sample means (1-0.95=0.05) 0.05 of the sample means will be outside the interval

    Since the interval is symmetric 0.05 will be above the upper limit and 0.025 will be below the lower limit.
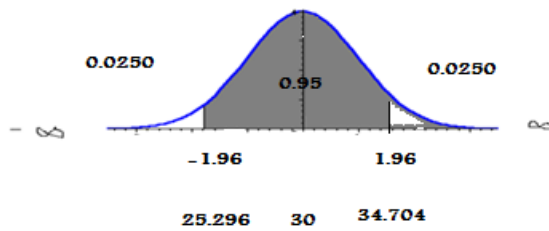
    From the standardized normal table, the Z score with 2.5% (0.0250) below it is -1.96 and the Z score with 2.5% (0.95+0.0250=0.9750) above it is 1.96.

Calculating the upper limit of the interval

$$\bar{X}_U = \mu + Z\dfrac{\sigma}{\sqrt{n}} = 30 + (1.96)\dfrac{12}{\sqrt{25}} = 30 + 1.96(2.4) = 30 + 4.704 = 34.704$$

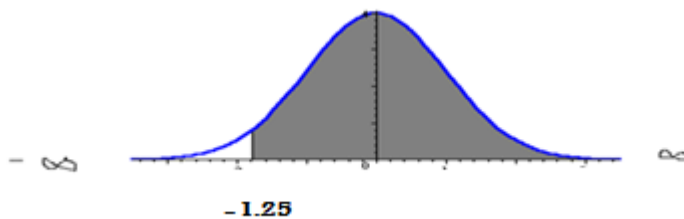Calculating the lower limit of the interval

$$\bar{X}_L = \mu + Z\dfrac{\sigma}{\sqrt{n}} = 30 + (-1.96)\dfrac{12}{\sqrt{25}} = 30 - 1.96(2.4) = 30 - 4.704 = 25.296$$

5) Greater than 27?

$$P(\bar{X} > 27) = P\left(Z > \dfrac{27-30}{\dfrac{12}{\sqrt{25}}}\right)$$

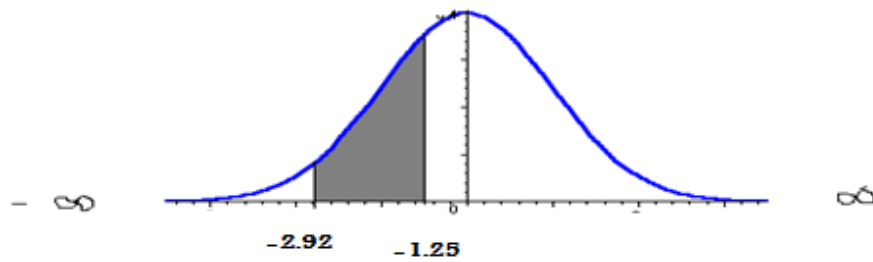$$= P\left(Z > \dfrac{-3}{2.4}\right) = p(Z > -1.25) = 1 - \Phi(-1.25) = 1 - 0.1056 = 0.8944$$

6) Between 23 and 27?

$$P(23 < \bar{X} < 27) = P\left(\dfrac{23-30}{\dfrac{12}{\sqrt{25}}} < Z < \dfrac{27-30}{\dfrac{12}{\sqrt{25}}}\right)$$

$$= P\left(\dfrac{-7}{2.4} < Z < \dfrac{-3}{2.4}\right) = p(-2.92 < Z < -1.25) = \Phi(-1.25) - \Phi(-2.92)$$
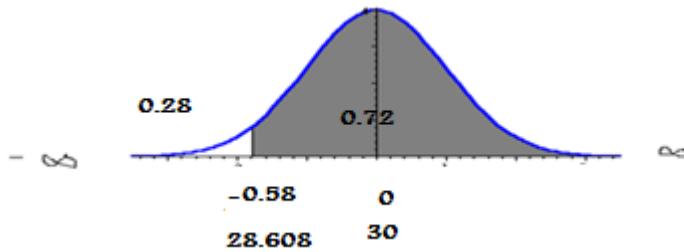
$$= 0.1056 - 0.0018 = 0.1038$$

-2.92   -1.25

7) There is a 72% chance that $\overline{X}$ is above what value?

1-0.7200=0.2800

Z = -0.58

$$\overline{X} = \mu + Z\sigma_{\overline{X}} = \mu + Z\frac{\sigma}{\sqrt{n}}$$

$$= 30 + (-0.58)\left(\frac{12}{\sqrt{25}}\right) = 30 - 0.58(2.4) = 30 - 1.392 = 28.608$$

0.28   0.72

-0.58   0

28.608   30

8) There is a 72% chance that $\overline{X}$ is below what value?

0.7200

Z = 0.58

$$\overline{X} = \mu + Z\sigma_{\overline{X}} = \mu + Z\frac{\sigma}{\sqrt{n}}$$

$$= 30 + (0.58)\left(\frac{12}{\sqrt{25}}\right) = 30 + 0.58(2.4) = 30 + 1.392 = 31.392$$

0.72

0   0.58

30   31.392

5

9) Greater than 34?

$$P(\bar{X} > 34) = P\left(Z > \frac{34 - 30}{\frac{12}{\sqrt{25}}} \right)$$

$$= P\left(Z > \frac{4}{2.4} \right) = p(Z > 1.67) = 1 - \Phi(1.67) = 1 - 0.0.9525 = 0.0475$$

1.67

10) Between 33 and 34?

$$P(33 < \bar{X} < 34) = P\left(\frac{33 - 30}{\frac{12}{\sqrt{25}}} < Z < \frac{34 - 30}{\frac{12}{\sqrt{25}}} \right)$$

$$= P\left(\frac{3}{2.4} < Z < \frac{4}{2.4} \right) = p(1.25 < Z < 1.67) = \Phi(1.67) - \Phi(1.25)$$

$$= 0.9525 - 0.8944 = 0.0581$$

1.25   1.67

11) There is a 43% chance that $\overline{X}$ is above what value?

1-0.4300=0.5700
Z = 0.18

$$\overline{X} = \mu + Z\sigma_{\overline{x}} = \mu + Z\frac{\sigma}{\sqrt{n}}$$

$$= 30 + (0.18)\left(\frac{12}{\sqrt{25}}\right) = 30 + 0.18(2.4) = 30 + 0.432 = 30.432$$
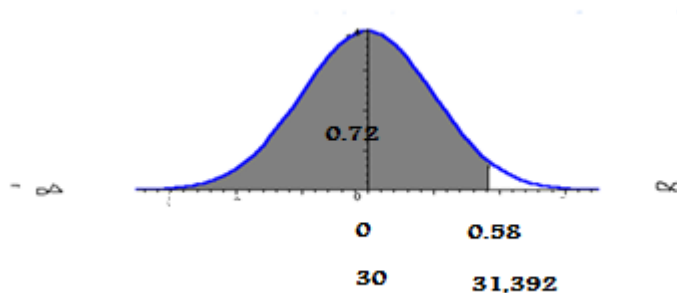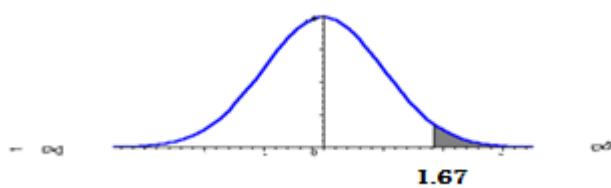

0.43
0.57

0 0.18

30   30.432
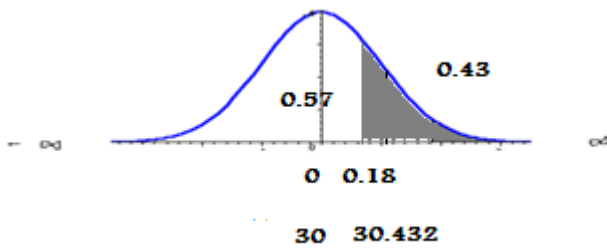
12) There is a 43% chance that $\overline{X}$ is below what value?
0.43
Z =- 0.18

$$\overline{X} = \mu + Z\sigma_{\overline{x}} = \mu + Z\frac{\sigma}{\sqrt{n}}$$

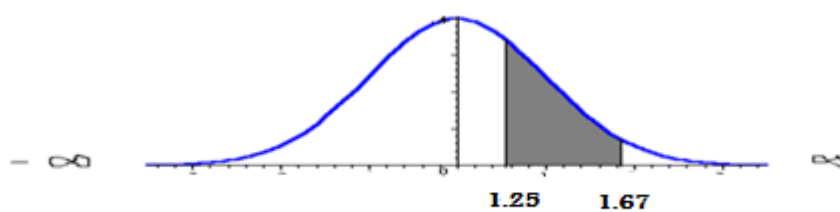$$= 30 + (-0.18)\left(\frac{12}{\sqrt{25}}\right) = 30 - 0.18(2.4) = 30 - 0.432 = 29.568$$

0.43

−0.18 0

29.568
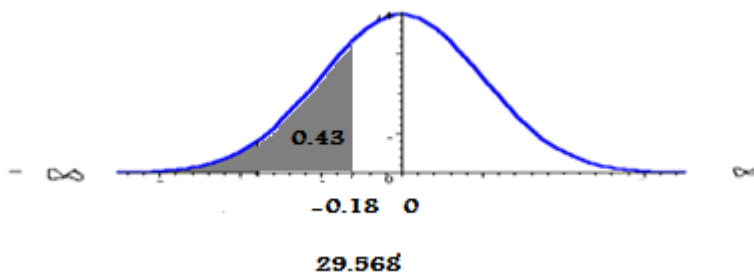
13) Between 30 and 34?

$$P(30 < \bar{X} < 34) = P\left(\frac{30-30}{\frac{12}{\sqrt{25}}} < Z < \frac{34-30}{\frac{12}{\sqrt{25}}} \right)$$

$$= P\left(\frac{0}{2.4} < Z < \frac{4}{2.4} \right) = p(0 < Z < 1.67) = \Phi(1.67) - 0.5000$$

$$= 0.9525 - 0.5000 = 0.4525$$
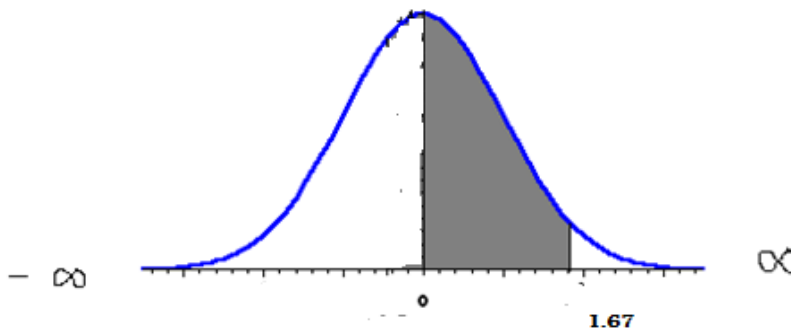
14) Between 27 and 30?

$$P(27 < \bar{X} < 30) = P\left(\frac{27-30}{\frac{12}{\sqrt{25}}} < Z < \frac{30-30}{\frac{12}{\sqrt{25}}} \right)$$

$$= P\left(\frac{-3}{2.4} < Z < \frac{0}{2.4} \right) = p(-1.25 < Z < 0) = 0.5000 - \Phi(-1.25)$$

$$= 0.5000 - 0.1056 = 0.3944$$

For a normal population distribution, the sampling distribution of the mean is always normally distributed

Suppose that $X \sim N(\mu, \sigma^2)$ and let $\bar{X}$ be the mean of a sample of size n then $\bar{X} \sim N(\mu, \sigma_{\bar{X}}^2)$ where;

$$\sigma_{\bar{X}}^2 = \frac{\sigma_X^2}{n} \qquad \qquad \text{\textit{in case of with replacment}}$$

For most distributions, n > 30 will give a sampling distribution that is nearly normal

Suppose that X *not normal* and let $\bar{X}$ be the mean of a sample of size n then $\bar{X} \dot\sim N(\mu, \sigma_{\bar{X}}^2)$ where;
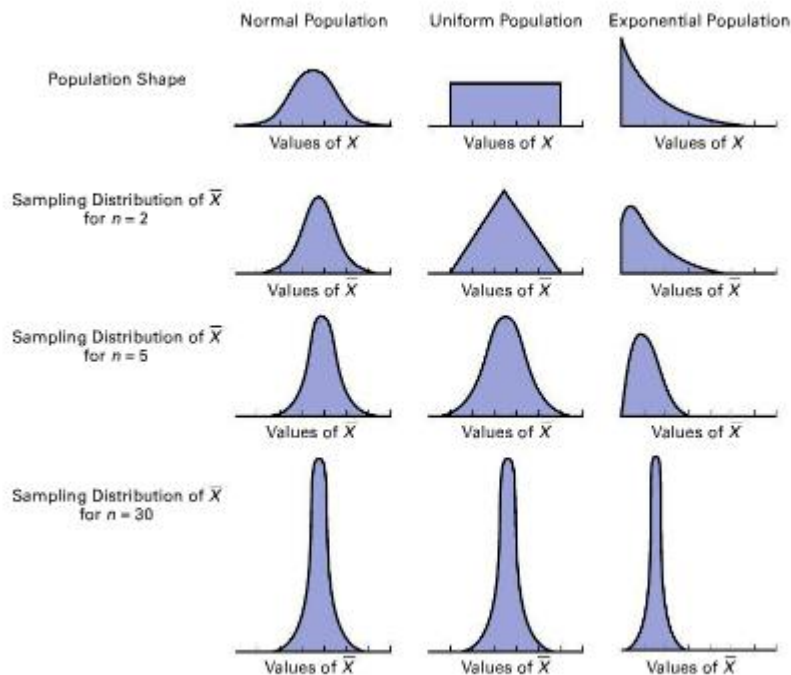
$$\sigma_{\bar{X}}^2 = \frac{\sigma_X^2}{n} \qquad \qquad \text{\textit{in case of with replacment}}$$

Example (3)

Suppose that $n = 100$, $\pi = 0.3$. Using the normal approximation for the binomial probabilities find:

1) The sample distribution of the proportion
2) The standard error of the proportion
3) $P(P \geq 0.25)$
4) P(0.25 ≤ p ≤ 0.30)
5) $P(P \leq 0.25)$

Solution:

1) The mean sample distribution of the proportion $= \mu_p = p = \dfrac{x}{n} = \pi = 0.3$

2) The standard error of the proportion $= \sigma_p = \sqrt{\dfrac{\pi(1-\pi)}{n}} = \sqrt{\dfrac{0.3(1-0.3)}{100}} = 0.0458$

3) $P(P \geq 0.25) = P\left(Z \geq \dfrac{0.25 - 0.3}{0.0458}\right) = P(Z \geq -1.09)$

$= 1 - \phi(-1.09) = 1 - 0.1379 = 0.8621$

-1.09
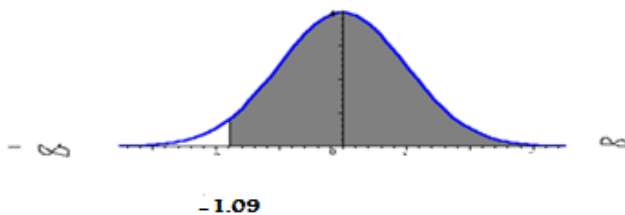
4) $P(0.25 \leq P \leq 0.30) = P\left(\dfrac{0.25 - 0.30}{0.0458} \leq Z \leq \dfrac{0.30 - 0.3}{0.0458}\right) = P(-1.09 \leq Z \leq 0)$

$= 0.5 - \phi(-1.09) = 0.5 - 0.1379 = 0.3621$

-1.09
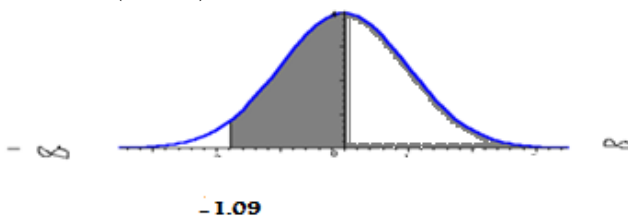
5) $P(P \leq 0.25) = P\left(Z \leq \dfrac{0.25 - 0.3}{0.0458}\right) = P(Z \leq -1.09)$

$= \phi(-1.09) = 0.1379$

-1.09
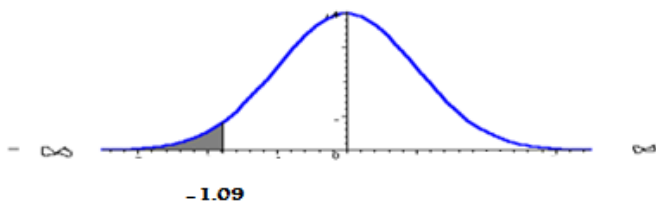
Z and Tables

Z Table: Negative Values

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.80	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
−3.70	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
−3.60	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
−3.50	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
−3.40	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.30	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.20	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.10	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.00	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.90	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.80	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.70	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.60	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.50	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.40	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.30	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.20	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.10	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.00	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.90	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.80	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.70	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.60	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.50	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.40	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.30	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.20	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.10	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.00	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.90	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.80	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.70	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.60	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.50	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.40	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.30	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.20	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.10	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.00	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Z Table: Positive Values

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.00	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.10	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.20	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.30	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.40	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.50	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.60	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.70	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.80	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.90	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.00	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.10	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.20	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.30	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.40	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.50	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.60	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.70	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.80	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.90	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.00	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.10	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.20	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.30	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.40	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.50	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.60	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.70	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.80	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.90	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.00	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.10	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.20	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.30	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.40	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.50	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.60	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.70	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.80	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999